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On the Frobenius norm of commutator of Cauchy-Toeplitz matrix and exchange matrix

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Abstract: Matrix commutator and anticommutator play an important role in mathematics, mathematical physic, and quantum physic. The commutator and anticommutator of two $n \times n$ complex matrices A and B are defined by $[A, B] = AB - BA$ and $(A, B) = AB + BA$, respectively. Cauchy-Toeplitz matrix and exchange matrix are two of the special matrices and they have excellent properties. In this study, we mainly focus on Frobenius norm of the commutator of Cauchy-Toeplitz matrix and exchange matrix. Moreover, we give upper and lower bounds for the Frobenius norm of the commutator of Cauchy-Toeplitz matrix and exchange matrix.

Key words: Matrix commutator, Frobenius norm, Cauchy-Toeplitz matrix, exchange matrix

1. Introduction

A real $n \times n$ matrix $A = (a_{ij})$ is called symmetric if $a_{ij} = a_{ji}$, persymmetric if $a_{ij} = a_{n-j+1,n-i+1}$, centrosymmetric if $a_{ij} = a_{n-i+1,n-j+1}$, skew-centrosymmetric if $a_{ij} = -a_{n-i+1,n-j+1}$ and bisymmetric if $a_{ij} = a_{n-j+1,n-i+1} = a_{ji} = a_{n-i+1,n-j+1}$. Equivalently, a real $n \times n$ matrix A is called symmetric if $A = A^t$, persymmetric if $PAP = A^t$, centrosymmetric if $AP = PA$ and skew-centrosymmetric if $AP = -PA$, bisymmetric if $A = A^t$ and $PAP = A^t = A$, where A^t denotes the transpose of A and $P = (p_{ij})$ is exchange matrix defined by p_{ij} is 1 if $i + j = n + 1$ zero elsewhere. Thus, if any matrix A is symmetric and persymmetric or symmetric and centrosymmetric then A is bisymmetric. The matrix P satisfies the nice properties $P = P^{-1} = P^t$. For more information about these special matrices we refer to [6, 9] and their references.

One of the popular persymmetric matrices is Toeplitz matrix. Toeplitz matrices, $T_n = [t_{j-i}]_{i,j=1}^{n}$, are precisely those matrices that one constant along all diagonals parallel to the main diagonal, and thus a Toeplitz matrix is determined by its first row and column. A Cauchy matrix is $m \times n$ matrix with elements a_{ij} in the form $a_{ij} = \frac{1}{x_i - y_j}$ $(x_i - y_j \neq 0)$. In generally, Cauchy-Toeplitz matrix is defined by;

$$T_n = (t_{ij}) = \left[\frac{1}{g + (i-j)h}\right]_{i,j=1}^{n} \tag{1.1}$$

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where $h \neq 0$, h and g are any number and g/h is not to be integer. In recent times, there have been several studies on the norms of Cauchy-Toeplitz matrix [2, 5, 7, 8]. Solak and Bozkurt [2, 5] have found the bounds belong to l_p norms of Cauchy-Toeplitz matrix with different values h and g. A lower bound for the spectral norm of Cauchy-Toeplitz matrix was obtained by Tyrtyshnikov [8] taking $g = 1/2$ and $h = 1$. Moreover, Türkmen and Bozkurt [7] have established bounds for the spectral norms of general Cauchy-Toeplitz matrices by taking $g = 1/2$ and $h = 1$.

Let $A, B \in M_n(\mathbb{C})$, the commutator and anticommutator of the matrices A and B are defined by $[A, B] = AB - BA$ and $(A, B) = AB + BA$, respectively. Matrix commutator and anticommutator play an important role in many branches of mathematics, mathematical physic, quantum physic, etc. Studies related to this topic have focused on norm inequalities in recent years and many good results have been carried out [1, 3]. For instance, Böttcher and Wenzel [3] showed that

$$\|[A, B]\|_F \leq \sqrt{2}\, \|A\|_F \, \|B\|_F \, . \tag{1.2}$$

The Frobenius norm of an $n \times n$ matrix A is computed by the formula

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2} . \tag{1.3}$$

Moreover, the Frobenius norm is the Schatten *2*-norm. For $p \geq 1$, the Schatten p-norms of an $n \times n$ matrix A are defined by

$$\|A\|_p = \left(\sum_{i=1}^{n} \sigma_i^p \right)^{1/p} \tag{1.4}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the singular values of A. The Schatten p-norms hold the following inequalities known as monotonicity property,

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|_\infty \tag{1.5}$$

for $1 \leq p \leq q \leq \infty$. A function ψ is called as psi (or digamma) function if $\psi(x) = \dfrac{d}{dx}\left[\ln \Gamma(x)\right]$ where $\Gamma(x) = \int\limits_0^\infty e^{-t} t^{x-1} dt$. It is called as polygamma function the nth derivatives of psi function [4] i.e.

$$\psi(n, x) = \frac{d}{dx^n} \psi(x) = \frac{d}{dx^n} \left[\frac{d}{dx} \ln\left[\Gamma(x)\right] \right]. \tag{1.6}$$

Polygamma function reduces to psi function if $n = 0$, that is $\psi(0, x) = \psi(x)$, and

$$\lim_{n \to \infty} \psi(a, n + b) = 0$$

for $a > 0$ and positive integer b [7].

In this study, we examine some properties of the commutator of Cauchy-Toeplitz matrix T_n and the exchange matrix P. We mainly deal with the Frobenius norm of the commutator of the matrices T_n and P.

2. Commutator of the matrices T_n and P

The commutator and anticommutator of the matrices T_n and P are

$$[T_n, P] = T_nP - PT_n = (c_{ij}) \quad \text{and} \quad (T_n, P) = T_nP + PT_n = (d_{ij})$$

respectively. Left (or right) multiplication by the matrix P against the matrix T_n reverses the row (or column) order of T_n, that is $T_nP = (t_{i,n+1-j})$ and $PT_n = (t_{n+1-i,j})$. Thus

$$c_{ij} = t_{i,n+1-j} - t_{n+1-i,j} = 2\frac{(n+1-j-i)h}{g^2 - (n+1-j-i)^2h^2}$$

and

$$d_{ij} = t_{i,n+1-j} + t_{n+1-i,j} = 2\frac{g}{g^2 - (n+1-j-i)^2h^2} .$$

Also, we observe that $c_{ij} = -c_{n+1-i,n+1-j} = c_{ji}$ and $d_{ij} = d_{n+1-i,n+1-j} = d_{n+1-j,n+1-i} = d_{ji}$. So, the commutator matrix $[T_n, P]$ is symmetric and skew-centrosymmetric, the anticommutator matrix (T_n, P) is symmetric, persymmetric, centrosymmetric, and bisymmetric. The determinant of an $n \times n$ skew-centrosymmetric matrix is zero for any odd number n. Thus, $\det([T_n, P]) = 0$ and $[T_n, P]$ has not an inverse for any odd number n. By using the fact that the matrix T_n is persymmetric (equivalently $PT_nP = T_n^t$) we have

$$P[T_n, P] = P(T_nP - PT_n) = PT_nP - T_n = T_n^t - T_n$$

and

$$[T_n, P]^{-1} = \left(T_n^t - T_n\right)^{-1} P$$

for the inverse matrix (if any) of the commutator matrix $[T_n, P]$. Next we give our main results.

Theorem 2.1 *The Frobenius norm of the commutator matrix $[T_n, P]$ holds*

$$\|[T_n, P]\|_F^2 = 2\|T_n\|_F^2 - \frac{1}{g}x_n \tag{2.1}$$

where x_n is the sum of the elements of the anticommutator matrix (T_n, P), that is

$$x_n = \sum_{i,j=1}^{n} d_{ij} = 2\left[\frac{n}{g} + \frac{g-hn}{h^2}\psi(n-\frac{g}{h}) + \frac{g+hn}{h^2}\psi(n+\frac{g}{h}) - \frac{g-hn}{h^2}\psi(1-\frac{g}{h}) - \frac{g+hn}{h^2}\psi(1+\frac{g}{h})\right].$$

Proof By considering the definition of the Frobenius norm, we have

$$\begin{aligned}
\|[T_n,P]\|_F^2 &= \sum_{i,j=1}^{n} c_{ij}^2 = \sum_{i,j=1}^{n} \left(t_{i,n+1-j} - t_{n+1-i,j}\right)^2 \\
&= \sum_{i,j=1}^{n} \left(\frac{1}{g+(i-n-1+j)h} - \frac{1}{g+(n+1-i-j)h}\right)^2 \\
&= \sum_{i,j=1}^{n} \left(\frac{1}{g+(i-n-1+j)h}\right)^2 + \sum_{i,j=1}^{n} \left(\frac{1}{g+(n+1-i-j)h}\right)^2 \\
&\qquad - 2\sum_{i,j=1}^{n} \left(\frac{1}{g^2-(n+1-i-j)^2 h^2}\right) \\
&= \|T_n\|_F^2 + \|T_n\|_F^2 - 2\sum_{i,j=1}^{n} \frac{d_{ij}}{2g} \\
&= 2\|T_n\|_F^2 - \frac{1}{g} x_n,
\end{aligned}$$

where $x_n = \sum_{i,j=1}^{n} d_{ij} = \sum_{i,j=1}^{n} \left(\frac{2g}{g^2-(n+1-i-j)^2h^2}\right)$

$$= 2\left[\frac{n}{g} + \frac{g-hn}{h^2}\psi(n-\frac{g}{h}) + \frac{g+hn}{h^2}\psi(n+\frac{g}{h}) - \frac{g-hn}{h^2}\psi(1-\frac{g}{h}) - \frac{g+hn}{h^2}\psi(1+\frac{g}{h})\right].$$ □

For the Frobenius norm of the matrix T_n, by means of the proof of Theorem 2.1 in [5], we write

$$\begin{aligned}
\|[T_n]\|_F^2 &= \frac{n}{g^2} - \frac{1}{h^2}\psi(n-\frac{g}{h}) + \frac{g-hn}{h^3}\psi(1,n-\frac{g}{h}) - \frac{1}{h^2}\psi(n+\frac{g}{h}) \\
&\quad - \frac{g+hn}{h^3}\psi(1,n+\frac{g}{h}) + \frac{1}{h^2}\psi(1-\frac{g}{h}) - \frac{g-hn}{h^3}\psi(1,1-\frac{g}{h}) \\
&\quad + \frac{1}{h^2}\psi(1+\frac{g}{h}) + \frac{g+hn}{h^3}\psi(1,1+\frac{g}{h}).
\end{aligned} \tag{2.2}$$

Thus, the Frobenius norm of the commutator matrix $[T_n,P]$ can also be given depending on psi and polygamma functions as follows:

Corollary 2.2 *Theorem 2.1 and equation (2.2) yield*

$$\begin{aligned}
\|[T_n,P]\|_F^2 &= 2\frac{g-hn}{h^3}\psi(1,n-\frac{g}{h}) - 2\frac{2g-hn}{gh^2}\psi(n-\frac{g}{h}) - 2\frac{g+hn}{h^3}\psi(1,n+\frac{g}{h}) \\
&\quad -2\frac{2g+hn}{gh^2}\psi(n+\frac{g}{h}) - 2\frac{g-hn}{h^3}\psi(1,1-\frac{g}{h}) + 2\frac{2g-hn}{gh^2}\psi(1-\frac{g}{h}) \\
&\quad + 2\frac{g+hn}{h^3}\psi(1,1+\frac{g}{h}) + 2\frac{2g+hn}{gh^2}\psi(1+\frac{g}{h}).
\end{aligned}$$

Now we give some bounds for the Frobenius norm of the commutator matrix $[T_n,P]$. According to upper bound of Böttcher and Wenzel in (1.2), we have

$$\|[T_n,P]\|_F \le \sqrt{2}\,\|T_n\|_F\,\|P\|_F = \sqrt{2n}\,\|T_n\|_F \tag{2.3}$$

where $\|P\|_F = \sqrt{n}$. We know that left (or right) multiplication by the matrix P against the matrix T_n reverses the row (or column) order of T_n. Thus, we get

$$\|[T_n,P]\|_F = \|T_nP - PT_n\|_F \le \|T_nP\|_F + \|PT_n\|_F$$

$$= \|T_n\|_F + \|T_n\|_F = 2\,\|T_n\|_F\,. \tag{2.4}$$

Next we derive a new bound for the commutator matrix $[T_n,P]$.

Theorem 2.3 *For the Frobenius norm of the commutator matrix $[T_n,P]$, we have*

$$i)\sqrt{2}\sqrt{\|T_n\|_F^2 - \frac{n}{g^2}} \le \|[T_n,P]\|_F \le \sqrt{2}\sqrt{\|T_n\|_F^2 - \frac{n\pi\cot\left(\frac{\pi g}{h}\right)}{gh}} \tag{2.5}$$

$$ii)\ \|[T_n,P]\|_F \le \sqrt{2n\pi}\sqrt{\frac{\pi g + \pi g\cot^2\left(\frac{\pi g}{h}\right) - h\cot\left(\frac{\pi g}{h}\right)}{gh^2}}. \tag{2.6}$$

Proof Consider the sequence $\{x_n\}_{n\ge1}$ where

$$x_n = \frac{2n}{g} + 2\frac{g-hn}{h^2}\psi(n-\frac{g}{h}) + 2\frac{g+hn}{h^2}\psi(n+\frac{g}{h}) - 2\frac{g-hn}{h^2}\psi(1-\frac{g}{h}) - 2\frac{g+hn}{h^2}\psi(1+\frac{g}{h}).$$

If we divide both sides by n of this equality and take the limit as $n\to\infty$, then we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}x_n &= \lim_{n\to\infty}\frac{1}{n}\left\{\begin{array}{c}\frac{2n}{g} + 2\frac{g-hn}{h^2}\psi(n-\frac{g}{h}) + 2\frac{g+hn}{h^2}\psi(n+\frac{g}{h})\\ -2\frac{g-hn}{h^2}\psi(1-\frac{g}{h}) - 2\frac{g+hn}{h^2}\psi(1+\frac{g}{h})\end{array}\right\}\\ &= 2\frac{\pi\cot\left(\frac{\pi g}{h}\right)}{h}.\end{aligned}$$

Also, the sequence $\left\{\frac{1}{n}x_n\right\}_{n\ge1}$ is strictly decreasing and $\frac{1}{n}x_n = \frac{2}{g}$ for $n=1$. Thus,

$$2\frac{\pi\cot\left(\frac{\pi g}{h}\right)}{h} \le \frac{1}{n}x_n \le \frac{2}{g}$$

and

$$2n\frac{\pi\cot\left(\frac{\pi g}{h}\right)}{h} \le x_n \le \frac{2n}{g}.$$

Therefore, from Theorem 2.1 we get

$$\sqrt{2}\sqrt{\|T_n\|_F^2-\frac{n}{g^2}} \leq \|[T_n,P]\|_F \leq \sqrt{2}\sqrt{\|T_n\|_F^2-\frac{n\pi\cot\left(\frac{\pi g}{h}\right)}{gh}}. \tag{2.7}$$

ii) By means of the proof of Theorem 2.1 in [5], we write

$$\frac{1}{n}\|T_n\|_F^2 \leq \frac{\pi^2}{h^2\sin^2\left(\frac{\pi g}{h}\right)} = \frac{\pi^2\left(1+\cot^2\left(\frac{\pi g}{h}\right)\right)}{h^2}. \tag{2.8}$$

Combining the inequalities (2.7) and (2.8), we have

$$\|[T_n,P]\|_F \leq \sqrt{2n\pi}\sqrt{\frac{\pi g+\pi g\cot^2\left(\frac{\pi g}{h}\right)-h\cot\left(\frac{\pi g}{h}\right)}{gh^2}}.$$

□

Corollary 2.4 *Let k be any integer and $\frac{g}{h}=k+\frac{1}{2}$. Then*

$$\|[T_n,P]\|_F \leq \frac{\sqrt{2n}}{|h|}\pi. \tag{2.9}$$

More specially

$$\|[T_n,P]\|_F \leq \sqrt{2n}\pi \tag{2.10}$$

for $g=k+\frac{1}{2}$ and $h=\pm 1$.

Now, we can establish some upper bounds for the Schatten p-norms of the commutator matrix $[T_n,P]$. For $p=2$, $\|.\|_2$ is the Frobenius norm. From the monotonicity property of Schatten p-norms, we have

$$\|[T_n,P]\|_p \leq \|[T_n,P]\|_2 = \|[T_n,P]\|_F \tag{2.11}$$

where $p>2$. So, the equality (2.1) and the inequalities (2.4), (2.5), and (2.6) yield the upper bounds for the Schatten p-norms of the commutator matrix $[T_n,P]$ as follows:

Corollary 2.5 *For the Schatten p-norms $(p>2)$ of the commutator matrix $[T_n,P]$, we have*

$$i)\ \|[T_n,P]\|_p^2 \leq 2\|T_n\|_F^2-\frac{1}{g}x_n \tag{2.12}$$

$$ii)\ \|[T_n,P]\|_p \leq 2\|T_n\|_F \tag{2.13}$$

$$iii)\ \|[T_n,P]\|_p \leq \sqrt{2}\sqrt{\|T_n\|_F^2-\frac{n\pi\cot\left(\frac{\pi g}{h}\right)}{gh}} \tag{2.14}$$

$$iv)\ \|[T_n,P]\|_p \leq \sqrt{2n\pi}\sqrt{\frac{\pi g+\pi g\cot^2\left(\frac{\pi g}{h}\right)-h\cot\left(\frac{\pi g}{h}\right)}{gh^2}}. \tag{2.15}$$

To illustrate our results we will give two tables related to bounds for the Frobenius norm of the commutator matrix $[T_n,P]$. With the tables, we aim to compare our bounds with the previous bounds. We compute Frobenius norms of $[T_n,P]$ and their bounds for different values of n. Obtained values are summarized in Tables 1 and 2. The values of the second and fourth columns in the tables denote our lower and upper bounds given in Theorem 2.3 and Corollary 2.4 while the values of sixth column denotes the upper bound of Böttcher and Wenzel, and also the values of the third column indicates the values of $\|[T_n,P]\|_F$.

Table 1. Bounds for Frobenius norm of $[T_n,P]$ for $g=3$ and $h=4$

n	$\sqrt{2}\sqrt{\|T_n\|_F^2-\frac{n}{g^2}}$	$\|[T_n,P]\|_F$	$\sqrt{2n\pi}\sqrt{\frac{\pi g+\pi g\cot^2\left(\frac{\pi g}{h}\right)-h\cot\left(\frac{\pi g}{h}\right)}{gh^2}}$	$2\|T_n\|_F$	$\sqrt{2n}\|T_n\|_F$
1	0	0	1.7294	0.6666	0.4714
2	1.4285	1.6162	2.4458	2.2294	2.2294
3	2.0440	2.3224	2.9954	3.1128	3.8124
4	2.5197	2.8702	3.4589	3.8047	5.3806
5	2.9218	3.3341	3.8671	4.3928	6.9456
10	4.4258	5.0718	5.4690	6.6046	14.7684
20	6.4704	7.4354	7.7343	9.6240	30.4338
30	8.0135	9.2192	9.4725	11.9066	46.1141
40	9.3057	10.7126	10.9380	13.8192	61.8015
50	10.4399	12.0232	12.2290	15.4986	77.4932
100	14.8687	17.1394	17.2945	22.0590	155.9809
200	21.1050	24.3416	24.4581	31.3007	313.0071

Table 1 shows that our upper bound (2.6) is more useful than the upper bound of Böttcher and Wenzel for $g=3$, $h=4$ and $n\geq 3$, and also our upper and lower bounds are quite sharp. Moreover, the value of $\|[T_n,P]\|_F$ increases and tends to $\sqrt{2n\pi}\sqrt{\frac{\pi g+\pi g\cot^2\left(\frac{\pi g}{h}\right)-h\cot\left(\frac{\pi g}{h}\right)}{gh^2}}$ as $n\to\infty$.

Table II shows that our bounds are sharp and our upper bound (2.8) is better than the upper bound of Böttcher and Wenzel for $g=1/2$, $h=1$ and $n\geq 2$. Also, the value of $\|[T_n,P]\|_F$ increases and tends to $\sqrt{2n\pi}$ as $n\to\infty$.

For the different values of g and h, one can see that our upper bound is better than the upper bound of Böttcher and Wenzel.

3. Conclusion

In this paper, we established upper and lower bounds for the Frobenius norm of the commutator of Cauchy-Toeplitz matrix and exchange matrix. Moreover, our numerical examples given in Tables I and II showed that our bounds are sharp.

Table 2. Bounds for Frobenius norm of $[T_n, P]$ for $g = 1/2$ and $h = 1$

n	$\sqrt{2}\sqrt{\|T_n\|_F^2 - \frac{n}{g^2}}$	$\|[T_n, P]\|_F$	$\sqrt{2n}\pi$	$2\|T_n\|_F$	$\sqrt{2n}\|T_n\|_F$
1	0	0	4.4428	4	2.8284
2	2.9814	3.7712	6.2831	7.0553	7.0553
3	4.3573	5.5425	7.6952	9.2721	11.3560
4	5.4376	6.9396	8.8857	11.0966	15.6930
5	6.3569	8.1309	9.9345	12.6815	20.0512
10	9.8147	12.6199	14.0496	18.7792	41.9915
20	14.5240	18.7379	19.8691	27.2377	86.1333
30	18.0753	23.3507	24.3346	33.6665	130.3899
40	21.0465	27.2093	28.0992	39.0629	174.6950
50	23.6528	30.5934	31.4159	43.8054	219.0272
100	33.8178	43.7878	44.4288	62.3481	440.8680
200	48.1123	62.3357	62.8318	88.4849	884.8496

References

[1] Audenaert KMR. Variance bounds, with an application to norm bounds for commutators. Linear Algebra and its Applications 2010; 432: 1126-1143.

[2] Bozkurt D. On ℓ_p norms of Cauchy-Toeplitz matrices. Linear and Multilinear Algebra. 1998; 44: 341-346.

[3] Böttcher A, Wenzel D. The Frobenius norm and the commutator. Linear Algebra and its Applications 2008; 429: 1864-1885.

[4] Moenck R. On computing closed forms for summations. Proceedings of the MACSYMA User's Conference. 1977; 225-236.

[5] Solak S, Bozkurt D. On the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices. Applied Mathematics and Computations 2003; 140: 231-238.

[6] Tao D, Yasuda M. A spectral characterization of generalized real symmetric centrosymmetric and generalized real symmetric skew-centrosymmetric matrices. SIAM Journal on Matrix Analysis and Applications 23 (3): 885-895.

[7] Türkmen R, Bozkurt D. On the bounds for the norms of Cauchy-Toeplitz and Cauchy-Hankel matrices. Applied Mathematics and Computations 2002; 132: 633-642.

[8] Tyrtysnikov EE. Cauchy-Toeplitz matrices and some applications. Linear Algebra and its Applications 1991; 149: 1-18.

[9] Zhao L, Hu X, Zhang L. Inverse eigenvalue problems for bisymmetric matrices under a central principal submatrix constraint. Linear and Multilinear Algebra. 2011; 59 (2): 117-128.