Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1573 – 1584
© TÜBİTAK
doi:10.55730/1300-0098.3449

Neumann boundary value problem for the Beltrami equation in a ring domain*

İlker GENÇTÜRK†

Department of Mathematics, Faculty of Science and Art, Kırıkkale University, Kırıkkale, Turkey

Received: 08.04.2021 • **Accepted/Published Online:** 15.06.2023 • **Final Version:** 18.07.2023

Abstract: In this paper, the Neumann boundary value problem for the Beltrami operator is explicitly solved in a circular ring domain, solvability conditions for this problem are also given in explicit forms. Moreover, the Neumann problem for second-order operators with the Bitsadze/Laplace operator as the main part as combinations of the Cauchy-Riemann and the Beltrami operators is investigated.

Key words: Neumann problem, Beltrami Equation, singular integral equation

1. Introduction

Explicit solutions to the boundary value problems for some model complex partial differential equations are given in different domains [1, 4, 6, 7, 9, 20, 28]. As one of the classical boundary value problems in complex analysis, in a domain $D \subset \mathbb{C}$ the Neumann problem consists of prescribing the normal derivative of a function on the boundary of D. The Neumann problem, also known as a flux boundary condition, has been the subject of many classic studies in the physics and applied sciences [10, 11, 16, 19, 21–23, 29].

The Beltrami equation, a notable generalization of the Cauchy-Riemann equation, in a domain $D \subset \mathbb{C}$ takes the form with the complex notation $w = u + iv$ and $z = x + iy$

$$w_{\overline{z}} = \rho(z) w_z, \tag{1.1}$$

where $\rho : D \to \mathbb{C}$ is a measurable function satisfying $|\rho(z)| \leq \rho_0 < 1$ and

$$w_{\overline{z}} = \partial_{\overline{z}} w = \frac{1}{2}(w_x + i w_y) \quad \text{and} \quad w_z = \partial_z w = \frac{1}{2}(w_x - i w_y)$$

are formal derivatives of w in $\overline{z}$ and z, while w_x and w_y are partial derivatives of w in the variables x and y, respectively.

In [17], the author obtains the general solution for the inhomogeneous Beltrami equation and gives the solvability conditions and solutions of the Schwarz and Dirichlet boundary value problems for the Beltrami operator in the unit disc. As a natural continuation of the latter, in [8], the Neumann boundary value problem is considered for the Beltrami operator with constant coefficient in the unit disc and authors examine second order operators with the Poisson/Bitsadze operator as the main part as combinations of the Cauchy–Riemann and

*This paper is dedicated to the memory of Prof. Dr. Recep Şahin (1978-2023) and Prof. Dr. İlker Akkuş (1978-2023).

†Correspondence: ilkergencturk@gmail.com

2010 *AMS Mathematics Subject Classification:* Primary 30E25, 31A10; Secondary 35J25, 31A30 the Beltrami operators. Moreover, for the Beltrami operator with constant coefficient, the Neumann boundary value problem in the unit disk sector and the Dirichlet problem in a lens are discussed in [2] and [3], respectively. The common method in these studies is to transform the Beltrami equation into a singular integral equation and present the problem's solution through means of the Neumann series.

This work is intended as an attempt to motivate basic boundary value problems for complex partial differential equations in a circular ring domain $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$. For the studies in ring domain, see [5, 12–15, 18, 25, 26]. In this paper, our main goal is to give the solvability conditions of the Neumann boundary value problem for the Beltrami operator and for the Bitsadze/Laplace operator in R. We also obtain the integral representations for the solutions.

The rest of the paper is organized as follows. In the Preliminaries, we review several basic important representations and known results without proof. The important point to note in this section, we recall the Neumann problem for the inhomogeneous Cauchy–Riemann equation in R from [24]. In Section 3, we aim to solve the Neumann boundary value problems for the inhomogeneous Beltrami equation in the ring domain by the Neumann series method mentioned above. Section 4 concerns the study of the Neumann problem for the Bitsadze/Laplace operator by splitting it into two Neumann problems, for the inhomogeneous Cauchy–Riemann equation and for the inhomogeneous Beltrami equation.

2. Preliminaries

For the convenience of the reader, some relevant essential theorems for complex boundary problems are indicated.

Theorem 2.1 (The complex form of Gauss Theorem) *[7] Let $D \subset \mathbb{C}$ be a bounded domain with smooth boundary ∂D, and the closure $\overline{D} = D \cup \partial D$. Assume that $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then for $z = x+iy$, $x, y \in \mathbb{R}$*

$$\int_D w_{\overline{z}}(z)dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz, \quad \int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\overline{z}.$$

Theorem 2.2 (Cauchy integral formula) *Let γ be a simply closed smooth curve and D be the inner domain, bounded by γ. If w is an analytic function in D, continuous in $\overline{D}$ and $z \in D$, then*

$$w(z) = \frac{1}{2\pi i}\int_\gamma w(\zeta)\frac{d\zeta}{\zeta - z}. \tag{2.1}$$

Theorem 2.3 (Cauchy-Pompeiu representation) *[6] Under the assumptions of Theorem 2.1, we have for $z \in D$ that*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_\zeta(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}$$

where $\zeta = \xi + i\eta$.

Definition 2.4 *[27] For $f \in L_1(D;\mathbb{C})$ the integral operator*

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z} \tag{2.2}$$

is called the Pompeiu operator and differentiable in distributional sense with respect to $\bar{z}$ if $f \in L_1(D;\mathbb{C})$, furthermore

$$\partial_{\overline{z}} Tf = f \tag{2.3}$$

in D. If $f \in L_p(D;\mathbb{C}), p > 1$, the Pompeiu operator Tf is differentiable in distributional sense with respect to z and

$$\partial_z Tf(z) =: \Pi f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}. \tag{2.4}$$

It is worth pointing out that on the boundary of R, the normal derivative is given by the formula

$$\partial_{\nu_z} = \begin{cases} z\partial_z + \overline{z}\partial_{\overline{z}}, & |z| = 1, \\ -\frac{z}{r}\partial_z - \frac{\overline{z}}{r}\partial_{\overline{z}}, & |z| = r. \end{cases}$$

The following theorem is proved in [24]:

Theorem 2.5 *[24]The Neumann problem for the inhomogeneous Cauchy-Riemann equation in R,*

$$w_{\overline{z}} = f, \ \lambda|z|\partial_{\nu_z} w = \gamma \ on \ \partial R, \ w(z_0) = c, \ \lambda = \begin{cases} 1, & |z| = 1, \\ -1, & |z| = r, \end{cases}$$

for $f \in C^\alpha(\overline{R};\mathbb{C})$, $0 < \alpha < 1$, $\gamma \in C(\partial R;\mathbb{C})$, $c \in \mathbb{C}$, $z_0 \in R$ given is solvable by a function from $W^{1+\alpha}_{\overline{z}}(\overline{R};\mathbb{C})$ with continuous weak z-derivative on $\overline{R}$ if and only if for $z \in R$

$$\frac{1}{2\pi i}\int_{\partial R}\left[\gamma(\zeta) - \overline{\zeta} f(\zeta)\right]\frac{d\zeta}{1 - \overline{z}\zeta} + \frac{1}{\pi}\int_R f(\zeta)\frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} = 0,$$

$$\frac{1}{2\pi i}\int_{\partial R}\left[\gamma(\zeta) - \overline{\zeta} f(\zeta)\right]\frac{d\zeta}{r^2 - \overline{z}\zeta} + \frac{r^2}{\pi}\int_R f(\zeta)\frac{d\xi d\eta}{(r^2 - \overline{z}\zeta)^2} = 0.$$

Moreover if γ and f satisfy the condition

$$\frac{1}{2\pi i}\int_{\partial R}\left[\gamma(\zeta) - \overline{\zeta} f(\zeta)\right]\frac{d\zeta}{\zeta} = 0$$

then the solution is uniquely given by

$$
\begin{aligned}
w(z) =& c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \left[\gamma(\zeta) - \overline{\zeta} f(\zeta)\right] \log\left(\frac{1 - z\overline{\zeta}}{1 - z_0\overline{\zeta}}\right) \frac{d\zeta}{\zeta} \\
&+ \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \left[\gamma(\zeta) - \overline{\zeta} f(\zeta)\right] \log\left(\frac{r^2 - z\overline{\zeta}}{r^2 - z_0\overline{\zeta}}\right) \frac{d\zeta}{\zeta} \\
&- \frac{1}{\pi} \int\limits_{R} f(\zeta) \frac{z - z_0}{(\zeta - z_0)(\zeta - z)} d\xi d\eta.
\end{aligned}
\tag{2.5}
$$

3. Neumann problems for the Beltrami operator in the ring domain

In this section, we deal with the Neumann problem for the inhomogeneous Beltrami equation in R

$$w_{\overline{z}} + q w_z = f(z), \ z \in R, \tag{3.1}$$

$$\lambda |z| \partial_{\nu_z} w = \gamma \text{ on } \partial R, \ w(z_0) = c, \ \lambda = \begin{cases} 1, & |z| = 1, \\ -1, & |z| = r, \end{cases} \tag{3.2}$$

where $f \in L_p(R; \mathbb{C}), p > 2, \ \gamma \in C(\partial R; \mathbb{C}), \ c \in \mathbb{C}, \ z_0 \in R,$ and $q \in \mathbb{C}, \ |q| < 1$.

We can rewrite (3.1) and (3.2) in the form

$$w_{\overline{z}} = f(z) - q w_z, \ z \in R, \tag{3.3}$$

$$\lambda |\zeta| \partial_{\nu_\zeta} w = \gamma \text{ on } \partial R, \ w(z_0) = c, \ \lambda = \begin{cases} 1, & |\zeta| = 1, \\ -1, & |\zeta| = r. \end{cases} \tag{3.4}$$

Since

$$\lambda |\zeta| \partial_{\nu_\zeta} w = \zeta w_\zeta + \overline{\zeta} w_{\overline{\zeta}} = \gamma, \ \zeta \in \partial R,$$

it follows that

$$w_\zeta = \frac{\gamma(\zeta) - \overline{\zeta} f(\zeta)}{\zeta - q\overline{\zeta}}, \ \zeta \in \partial R. \tag{3.5}$$

According to Theorem 2.5, the above problem (3.3)-(3.4) is solvable if and only if for $z \in R$

$$
\begin{aligned}
&\frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{d\zeta}{1 - \overline{z}\zeta} - \frac{1}{2\pi i} \int\limits_{\partial R} \overline{\zeta} \left(f(\zeta) - q w_\zeta(\zeta)\right) \frac{d\zeta}{1 - \overline{z}\zeta} \\
&\qquad + \frac{1}{\pi} \int\limits_{R} \left(f(\zeta) - q w_\zeta(\zeta)\right) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} = 0,
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
&\frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{d\zeta}{r^2 - \overline{z}\zeta} - \frac{1}{2\pi i} \int\limits_{\partial R} \overline{\zeta} \left(f(\zeta) - q w_\zeta(\zeta)\right) \frac{d\zeta}{r^2 - \overline{z}\zeta} \\
&\qquad + \frac{r^2}{\pi} \int\limits_{R} \left(f(\zeta) - q w_\zeta(\zeta)\right) \frac{d\xi d\eta}{(r^2 - \overline{z}\zeta)^2} = 0.
\end{aligned}
\tag{3.7}
$$

Moreover if γ satisfies the condition

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial R}\overline{\zeta}\left(f(\zeta)-qw_{\zeta}(\zeta)\right)\frac{d\zeta}{\zeta}, \tag{3.8}$$

then its solution is given by the form

$$\begin{aligned} w(z) =& c-\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta} \\ &+\frac{1}{2\pi i}\int\limits_{\partial R}\overline{\zeta}\left(f(\zeta)-qw_{\zeta}(\zeta)\right)\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R}\left(f(\zeta)-qw_{\zeta}(\zeta)\right)\frac{z-z_0}{(\zeta-z_0)(\zeta-z)}d\xi d\eta. \end{aligned} \tag{3.9}$$

Hence, by (3.5), we have

$$\frac{1}{2\pi i}\int\limits_{\partial R}\overline{\zeta}w_{\zeta}(\zeta)\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial R}|\zeta|^2\frac{\gamma(\zeta)-\overline{\zeta}f(\zeta)}{\zeta^2-q|\zeta|^2}\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}.$$

Therefore, the solution (3.9) can be written in the form

$$\begin{aligned} w(z) =& c-\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\left(\gamma(\zeta)-\overline{\zeta}f(\zeta)\right)}{\zeta-q\overline{\zeta}}\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)d\zeta \\ &-\frac{1}{\pi}\int\limits_{R}\left(f(\zeta)-qw_{\zeta}(\zeta)\right)\frac{z-z_0}{(\zeta-z_0)(\zeta-z)}d\xi d\eta \end{aligned} \tag{3.10}$$

with some simplifications.

The derivative of (3.10) with respect to z implies

$$w_z(z) = \psi(z) + \Pi[f-qw_z](z),\ \ z\in R, \tag{3.11}$$

where

$$\psi(z) = \frac{1}{2\pi i}\int\limits_{\partial R}\frac{\gamma(\zeta)-\overline{\zeta}f(\zeta)}{\zeta-q\overline{\zeta}}\frac{d\zeta}{\zeta-z}, \tag{3.12}$$

and

$$\Pi[k](z) = \frac{1}{\pi}\int\limits_{R}k(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2}. \tag{3.13}$$

(3.13) is the singular integral operator called as Π operator previously defined by (2.4). Also, we remark that Π operator is bounded on $L_p(R;\mathbb{C}),\ p>2$.

According to [27, Lemma 3.1], and under the conditions on q and p, it may be concluded that

$$|q|||\Pi||_{L_p(R;\mathbb{C})} < 1. \tag{3.14}$$

Therefore $I + q\Pi$ is an invertible operator, and by applying Fredholm alternative to (3.11), its solution can be given by the following Neumann series

$$f(z) - qw_z(z) = \sum_{m=0}^{\infty} (-1)^m (q\Pi)^m \left[f - q\psi\right](z), \ z \in R. \tag{3.15}$$

The condition (3.14) enables the convergence of the above series (3.15).

Inserting the representation (3.15) in (3.10), we have the solution of the problem (3.1) - (3.2) as

$$\begin{aligned} w(z) = & c - \frac{1}{2\pi i} \int\limits_{\partial R} \frac{\left(\gamma(\zeta) - \overline{\zeta} f(\zeta)\right)}{\zeta - q\overline{\zeta}} \log\left(\frac{|\zeta|^2 - z\overline{\zeta}}{|\zeta|^2 - z_0\overline{\zeta}}\right) d\zeta \\ & - \frac{1}{\pi} \int\limits_{R} \sum_{m=0}^{\infty} (-1)^m (q\Pi)^m \left[f - q\psi\right](\zeta) \frac{z - z_0}{(\zeta - z_0)(\zeta - z)} d\xi d\eta, \end{aligned} \tag{3.16}$$

where ψ and Π are defined by (3.12) - (3.13).

Reconsidering solvability conditions (3.6) - (3.7), by (3.5), we have

$$\frac{1}{2\pi i} \int\limits_{\partial R} \overline{\zeta} \left(f(\zeta) - qw_\zeta(\zeta)\right) \frac{d\zeta}{1 - \overline{z}\zeta} = \frac{1}{2\pi i} \int\limits_{\partial R} \left(f(\zeta) - q\overline{\zeta}\gamma(\zeta)\right) \frac{|\zeta|^2}{\zeta^2 - q|\zeta|^2} \frac{d\zeta}{1 - \overline{z}\zeta} \tag{3.17}$$

and

$$\frac{1}{2\pi i} \int\limits_{\partial R} \overline{\zeta} \left(f(\zeta) - qw_\zeta(\zeta)\right) \frac{d\zeta}{r^2 - \overline{z}\zeta} = \frac{1}{2\pi i} \int\limits_{\partial R} \left(f(\zeta) - q\overline{\zeta}\gamma(\zeta)\right) \frac{|\zeta|^2}{\zeta^2 - q|\zeta|^2} \frac{d\zeta}{r^2 - \overline{z}\zeta}. \tag{3.18}$$

By applying the Gauss theorem, we obtain

$$\frac{1}{\pi} \int\limits_{R} w_\zeta(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} = -\frac{1}{2\pi i} \int\limits_{\partial R} w(\zeta) \frac{d\overline{\zeta}}{(1 - \overline{z}\zeta)^2} - \frac{2\overline{z}}{\pi} \int\limits_{R} w(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^3} \tag{3.19}$$

and

$$\frac{1}{\pi} \int\limits_{R} w_\zeta(\zeta) \frac{d\xi d\eta}{(r^2 - \overline{z}\zeta)^2} = -\frac{1}{2\pi i} \int\limits_{\partial R} w(\zeta) \frac{d\overline{\zeta}}{(r^2 - \overline{z}\zeta)^2} - \frac{2\overline{z}}{\pi} \int\limits_{R} w(\zeta) \frac{d\xi d\eta}{(r^2 - \overline{z}\zeta)^3}. \tag{3.20}$$

Substituting (3.17)-(3.18) and with (3.19)-(3.20) into (3.6)-(3.7), respectively, it gives solvability conditions of the problem

$$\begin{aligned} & \frac{1}{2\pi i} \int\limits_{\partial R} \frac{\left(\zeta\gamma(\zeta) - \overline{\zeta} f(\zeta)\right)}{\zeta - q\overline{\zeta}} \frac{d\zeta}{1 - \overline{z}\zeta} + \frac{1}{\pi} \int\limits_{R} f(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} \\ & + \frac{q}{2\pi i} \int\limits_{\partial R} w(\zeta) \frac{d\overline{\zeta}}{(1 - \overline{z}\zeta)^2} + \frac{2\overline{z} q}{\pi} \int\limits_{R} w(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^3} = 0, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\left(\zeta\gamma(\zeta)-\overline{\zeta}f(\zeta)\right)}{\zeta-q\overline{\zeta}}\frac{d\zeta}{r^2-\overline{z}\zeta}+\frac{r^2}{\pi}\int\limits_{R}f(\zeta)\frac{d\xi d\eta}{\left(r^2-\overline{z}\zeta\right)^2}\\&\quad+\frac{q}{2\pi i}\int\limits_{\partial R}w(\zeta)\frac{d\overline{\zeta}}{\left(r^2-\overline{z}\zeta\right)^2}+\frac{2\overline{z}q}{\pi}\int\limits_{R}w(\zeta)\frac{d\xi d\eta}{\left(r^2-\overline{z}\zeta\right)^3}=0,\end{aligned}\tag{3.22}$$

where $w(z)$ is defined by (3.16).

Similarly, the Condition (3.8) can be deduced as

$$\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\gamma(\zeta)}{\zeta-q\overline{\zeta}}d\zeta=\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\overline{\zeta}f(\zeta)}{\zeta-q\overline{\zeta}}d\zeta,\tag{3.23}$$

since by aid of (3.5)

$$\frac{1}{2\pi i}\int\limits_{\partial R}\overline{\zeta}w_{\zeta}(\zeta)\frac{d\zeta}{\zeta}=\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\overline{\zeta}\left(\gamma(\zeta)-\overline{\zeta}f(\zeta)\right)}{\zeta-q\overline{\zeta}}\frac{d\zeta}{\zeta}.\tag{3.24}$$

Thus, we prove the following result:

Theorem 3.1 *The Neumann problem (3.1)-(3.2) is solvable if and only if the conditions (3.21) and (3.22) are satisfied. Moreover if γ and f satisfy the condition (3.23), then the solution is uniquely expressed by (3.16).*

The following result can be obtained if the function f is taken as zero in the above theorem:

Corollary 3.2 *The Neumann problem for the homogeneous Beltrami equation in the ring domain R,*

$$w_{\overline{z}}+qw_z=0,\;\; z\in R,\tag{3.25}$$

$$\lambda|z|\partial_{\nu_z}w=\gamma\;\text{ on }\partial R,\;\; w(z_0)=c,\;\;\lambda=\begin{cases}1, & |z|=1,\\-1, & |z|=r,\end{cases}\tag{3.26}$$

where $\gamma\in C(\partial R;\mathbb{C})$, $c\in\mathbb{C}$, $z_0\in R$, and $q\in\mathbb{C}$, $|q|<1$ is solvable if and only if the conditions

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\zeta}{\zeta-q\overline{\zeta}}\frac{d\zeta}{1-\overline{z}\zeta}\\&\quad+\frac{q}{2\pi i}\int\limits_{\partial R}w(\zeta)\frac{d\overline{\zeta}}{\left(1-\overline{z}\zeta\right)^2}+\frac{2\overline{z}q}{\pi}\int\limits_{R}w(\zeta)\frac{d\xi d\eta}{\left(1-\overline{z}\zeta\right)^3}=0,\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\zeta}{\zeta-q\overline{\zeta}}\frac{d\zeta}{r^2-\overline{z}\zeta}\\&\quad+\frac{q}{2\pi i}\int\limits_{\partial R}w(\zeta)\frac{d\overline{\zeta}}{\left(r^2-\overline{z}\zeta\right)^2}+\frac{2\overline{z}q}{\pi}\int\limits_{R}w(\zeta)\frac{d\xi d\eta}{\left(r^2-\overline{z}\zeta\right)^3}=0,\end{aligned}\tag{3.28}$$

are satisfied.

Moreover if γ satisfies the condition

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta-q\overline{\zeta}}=0, \tag{3.29}$$

then the solution is uniquely expressed by

$$\begin{aligned} w(z)=&c-\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\gamma(\zeta)}{\zeta-q\overline{\zeta}}\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)d\zeta \\ &+\frac{q}{\pi}\int\limits_{R}\sum_{m=0}^{\infty}(-1)^m(q\Pi)^m[\varphi](\zeta)\frac{z-z_0}{(\zeta-z_0)(\zeta-z)}d\xi d\eta, \end{aligned} \tag{3.30}$$

where φ and Π are defined by

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\gamma(\zeta)}{\zeta-q\overline{\zeta}}\frac{d\zeta}{\zeta-z}, \tag{3.31}$$

and

$$\Pi[w_z](z)=\frac{1}{\pi}\int\limits_{R}w_\zeta(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2}. \tag{3.32}$$

4. Neumann problem for the Bitsadze/Laplace operator in the ring domain

In this section, we discuss a Neumann problem for second order operators of the following form

$$w_{\overline{z}\,\overline{z}}+qw_{z\overline{z}}=f \tag{4.1}$$

for $f\in L_p(R;\mathbb{C}), p>2,$ and $q\in\mathbb{C}, |q|<1$.

Indeed, Operator (4.1) is nothing but a combining of operators

$$w_{\overline{z}} \text{ and } w_{\overline{z}}+cw_z.$$

We are thus led to the following Neumann problem

$$\begin{aligned} &w_{\overline{z}\,\overline{z}}+qw_{z\overline{z}}=f(z),\ z\in R, \\ &\lambda|z|\partial_{\nu_z}w=\gamma_0 \text{ on } \partial R, \lambda|z|\partial_{\nu_z}w_{\overline{z}}=\gamma_1 \text{ on } \partial R,\ \lambda=\begin{cases}1, & |z|=1,\\ -1, & |z|=r,\end{cases} \\ &w(z_0)=c_0, w_{\overline{z}}(z_0)=c_1, z_0\in R. \end{aligned} \tag{4.2}$$

Problem (4.2) is resolved into

$$w_{\overline{z}}=g,\ z\in R, \lambda|z|\partial_{\nu_z}w=\gamma_0 \text{ on } \partial R, w(z_0)=c_0, \tag{4.3}$$

$$g_{\overline{z}}+qg_z=f(z),\ z\in R, \lambda|z|\partial_{\nu_z}g=\gamma_1 \text{ on } \partial R, g(z_0)=c_1 \tag{4.4}$$

with λ defined as above.

According to Theorem 3.1, solution of the Neumann Problem (4.4) can be given by

$$\begin{aligned} g(z) =& c - \frac{1}{2\pi i} \int_{\partial R} \frac{\gamma_1(\zeta)}{\zeta - q\overline{\zeta}} \log\left(\frac{|\zeta|^2 - z\overline{\zeta}}{|\zeta|^2 - z_0\overline{\zeta}} \right) d\zeta \\ &+ \frac{q}{\pi} \int_R \sum_{m=0}^{\infty} (-1)^m (q\Pi)^m [\psi_1](\zeta) \frac{z - z_0}{(\zeta - z_0)(\zeta - z)} d\xi d\eta, \end{aligned} \tag{4.5}$$

where ψ_1 and Π are defined by

$$\psi_1(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{\gamma_1(\zeta) - \overline{\zeta} f(\zeta)}{\zeta - q\overline{\zeta}} \frac{d\zeta}{\zeta - z}, \tag{4.6}$$

and

$$\Pi[k](z) = \frac{1}{\pi} \int_R k(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2} \tag{4.7}$$

with condition

$$\frac{1}{2\pi i} \int_{\partial R} \frac{\gamma_1(\zeta)}{\zeta - q\overline{\zeta}} d\zeta = \frac{1}{2\pi i} \int_{\partial R} \frac{\overline{\zeta} f(\zeta)}{\zeta - q\overline{\zeta}} d\zeta \tag{4.8}$$

if and only if the following conditions are satisfied:

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\partial R} \frac{\left(\zeta\gamma_1(\zeta) - \overline{\zeta} f(\zeta)\right)}{\zeta - q\overline{\zeta}} \frac{d\zeta}{1 - \overline{z}\zeta} + \frac{1}{\pi} \int_R f(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} \\ &+ \frac{q}{2\pi i} \int_{\partial R} g(\zeta) \frac{d\overline{\zeta}}{(1 - \overline{z}\zeta)^2} + \frac{2\overline{z}q}{\pi} \int_R g(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^3} = 0, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\partial R} \frac{\left(\zeta\gamma_1(\zeta) - \overline{\zeta} f(\zeta)\right)}{\zeta - q\overline{\zeta}} \frac{d\zeta}{r^2 - \overline{z}\zeta} + \frac{r^2}{\pi} \int_R f(\zeta) \frac{d\xi d\eta}{(r^2 - \overline{z}\zeta)^2} \\ &+ \frac{q}{2\pi i} \int_{\partial R} g(\zeta) \frac{d\overline{\zeta}}{(r^2 - \overline{z}\zeta)^2} + \frac{2\overline{z}q}{\pi} \int_R g(\zeta) \frac{d\xi d\eta}{(r^2 - \overline{z}\zeta)^3} = 0. \end{aligned} \tag{4.10}$$

By Theorem 2.5, the Neumann problem (4.3) is uniquely solvable if and only if

$$\frac{1}{2\pi i} \int_{\partial R} \left[\gamma_0(\zeta) - \overline{\zeta} g(\zeta)\right] \frac{d\zeta}{1 - \overline{z}\zeta} + \frac{1}{\pi} \int_R g(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} = 0, \tag{4.11}$$

$$\frac{1}{2\pi i} \int_{\partial R} \left[\gamma_0(\zeta) - \overline{\zeta} g(\zeta)\right] \frac{d\zeta}{r^2 - \overline{z}\zeta} + \frac{r^2}{\pi} \int_R g(\zeta) \frac{d\xi d\eta}{(r^2 - \overline{z}\zeta)^2} = 0. \tag{4.12}$$

Moreover if γ_0 and g satisfy the condition

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_0(\zeta)-\overline{\zeta}g(\zeta)\right]\frac{d\zeta}{\zeta}=0 \tag{4.13}$$

then the solution is uniquely expressed by

$$\begin{aligned} w(z)=&c_0-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left[\gamma_0(\zeta)-\overline{\zeta}g(\zeta)\right]\log\left(\frac{1-z\overline{\zeta}}{1-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\\ &+\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma_0(\zeta)-\overline{\zeta}g(\zeta)\right]\log\left(\frac{r^2-z\overline{\zeta}}{r^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\\ &-\frac{1}{\pi}\int\limits_{R}g(\zeta)\frac{z-z_0}{(\zeta-z_0)(\zeta-z)}d\xi d\eta, \end{aligned} \tag{4.14}$$

where $g(z)$ is defined by (4.5).

Theorem 4.1 *The Neumann problem* (4.2) *is uniquely solvable if and only if the functions* f, γ_0, γ_1 *satisfy conditions* (4.9)-(4.10) *and* (4.11)-(4.12)*. Moreover,* (4.8) *and* (4.13) *are satisfied, then the solution is uniquely given by* (4.14)*.*

Acknowledgement

The author wishes to express his thank to the referees for carefully reading his manuscript and giving such constructive comments which substantially helped improve the quality of the paper. The author also would like to express his gratitude to Prof. Dr. Kerim Koca who taught him the theory of complex boundary value problems.

References

[1] Abdymanapov S, Begehr H, Harutyunyan G, Tungatarov A. Four boundary value problems for the Cauchy-Riemann equation in a quarter plane. In: Begehr H, Nicolosi F, editors. More Progress in Analysis - ISAAC 2005 5th International Conference Proceedings; 25–30 July 2005; Catania, Italy. Singapore: World Scientific, 2009, pp. 1137-1147.

[2] Akel M, Aldawsari M. Neumann boundary value problems in fan-shaped domains. Turkish Journal of Mathematics 2018; 42 (4): 1571-1589. https://doi.org/10.3906/mat-1604-83

[3] Akel M, Mondal SR. Dirichlet problems in lens and lune. Bulletin of the Malaysian Mathematical Sciences Society 2018; 41 (2): 1029-1043. https://doi.org/10.1007/s40840-017-0478-8

[4] Aksoy Ü, Çelebi AO. Neumann problem for generalized n-Poisson equation. Journal of Mathematical Analysis and Applications 2009; 357 (2): 438-446. https://doi.org/10.1016/j.jmaa.2009.04.032

[5] Begehr H, Vaitekhovich T. Modified harmonic Robin function. Complex Variables and Elliptic Equations 2013; 58 (4): 483-496. https://doi.org/10.1080/17476933.2011.625092

[6] Begehr H. Complex Analytic Methods for Partial Differential Equations. Singapore: World Scientific, 1994.

[7] Begehr H. Boundary value problems in complex analysis I. Boletín de la Asociación Matemática Venezolana 2005; 12 (1): 65-85.

[8] Begehr H, Harutjunjan G. Neumann problem for the Beltrami operator and for second order operators with Poisson/Bitsadze operator as main part. Complex Variables and Elliptic Equations 2009; 54 (12): 1129-1150. https://doi.org/10.1080/17476930903276084

[9] Begehr H, Obolashvili E. Some boundary value problems for a Beltrami equation. Complex Variables and Elliptic Equations 1994; 26 (1-2): 113-122. https://doi.org/10.1080/17476939408814769

[10] Castro L, Duduchava R, Speck FO, et al. Mixed impedance boundary value problems for the Laplace-Beltrami equation. Journal of Integral Equations and Applications 2020; 32 (3): 275-292. https://doi.org/10.1216/jie.2020.32.275

[11] Castro L, Duduchava R, Speck FO. Diffraction from polygonal-conical screens, an operator approach. In: Amelia Bastos M, Lebre A, Samko S, Spitkovsky IM, editors. Operator Theory, Operator Algebras and Applications 242. Basel, Switzerland: Springer, 2014, pp. 113-137.

[12] Gençtürk İ. Robin boundary value problem depending on parameters in a ring domain. Fundamental Journal of Mathematics and Applications 2020; 3 (2): 161-167. https://doi.org/10.33401/fujma.795538

[13] Gençtürk İ. The Dirichlet-Neumann boundary value problem for the inhomogeneous Bitsadze equation in a ring domain. Thai Journal of Mathematics 2021; 19 (1): 145-155.

[14] Gençtürk İ, Koca K. Neumann boundary value problem for Bitsadze equation in a ring domain. The Journal of Analysis 2020; 28: 799-815. https://doi.org/10.1007/s41478-019-00202-3

[15] Gençtürk, İ, Linares, YR. A Robin boundary value problem for the Cauchy-Riemann operator in a ring domain. Georgian Mathematical Journal 2022; 29(2): 205-209. https://doi.org/10.1515/gmj-2021-2131

[16] Guo T, Shen E, Lu Z, Wang Y, Dong L. Implicit heat flux correction-based immersed boundary-finite volume method for thermal flows with Neumann boundary conditions. Journal of Computational Physics 2019; 386: 64-83. https://doi.org/10.1016/j.jcp.2019.02.016

[17] Harutyunyan G. Boundary value problems for the Beltrami operator. Complex Variables and Elliptic Equations 2007; 52 (6): 475-484. https://doi.org/10.1080/17476930701200922

[18] Karaca B. Dirichlet problem for higher-order equations in a ring domain. Complex Variables and Elliptic Equations 2022; 1-10. https://doi.org/10.1080/17476933.2022.2102613

[19] Kolomenskiy D, Nguyen R, Schneider K. Analysis and discretization of the volume penalized Laplace operator with Neumann boundary conditions. Applied Numerical Mathematics 2015; 95: 238-249. https://doi.org/10.1016/j.apnum.2014.02.003

[20] Linares YR, Vanegas CJ. A Robin boundary value problem in the upper half plane for the Bitsadze equation. Journal of Mathematical Analysis and Applications 2014; 419 (1): 200-217. https://doi.org/10.1016/j.jmaa.2014.04.056

[21] Medková D, Krutitskii P. Neumann and Robin problems in a cracked domain with jump conditions on cracks. Journal of Mathematical Analysis and Applications 2005; 301 (1): 99-114. https://doi.org/10.1016/j.jmaa.2004.06.062

[22] Pidcock M, Ciulli S, Ispas S. Boundary Modelling in Electrical Impedance Tomography. In: Engl, H.W., Louis, A.K., Rundell, W. (eds) Inverse Problems in Medical Imaging and Nondestructive Testing. Vienna, Austria: Springer, 1997, pp. 188-195. https://doi.org/10.1007/978-3-7091-6521-8_13

[23] Sakurai T, Yoshimatsu K, Okamoto N, Schneider K. Volume penalization for inhomogeneous Neumann boundary conditions modeling scalar flux in complicated geometry. Journal of Computational Physics 2019; 390 (2019): 452-469. https://doi.org/10.1016/j.jcp.2019.04.008

[24] Vaitekhovich TS. Boundary value problems for complex partial differential equations in a ring domain. PhD, Freie Universität Berlin, Berlin, Germany, 2008.

[25] Vaitekhovich TS. Boundary value problems to second order complex partial differential equations in a ring domain. Sauliai Mathematical Seminar 2007; 2 (10): 117-146.

[26] Vaitekhovich TS. Boundary value problems to first-order complex partial differential equations in a ring domain. Integral Transforms and Special Functions 2008; 19 (3): 211-233. https://doi.org/10.1080/10652460701736569

[27] Vekua IN. Generalized analytic functions. Oxford, UK: Pergamon Press, 1962.

[28] Wang Y. Tri-harmonic boundary value problems in a sector. Complex Variables and Elliptic Equations 2014; 59 (5): 732-749. https://doi.org/10.1080/17476933.2012.759566

[29] Zhevandrov P, Merzon A. On the Neumann problem for the Helmholtz equation in a plane angle. Mathematical Methods in the Applied Sciences 2000; 23 (16): 1401-1446.