Atomic systems in Krein spaces

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Abstract: In the present article, we establish a definition of atomic systems in the Krein spaces, specifically, we establish the fundamental tools of the theory of atomic systems in the formalism of the Krein spaces and give a complete characterization of them. We also show that the atomic systems do not depend on the decomposition of the Krein space.

Key words: Krein space, Fundamental decomposition, atomics system, projectors

1. Introduction

Spaces of the indefinite metric are spaces with an indefinite inner product, the most studied of such spaces are the Krein spaces, which were formally defined by L. Pontryagin [13], a complete development of this theory is given in [2] by Azizov, I. Iokhvidov, and in [3] by J. Bognar. The concept of frames in Hilbert spaces was introduced by Duffin and Schaeffer in 1952, when studying some deep problems of the nonharmonic Fourier series see [8], the frames have also been studied for the Banach spaces see [4]. Today the frame theory is a fundamental research area in mathematics, computer science, and engineering with many interesting applications in a variety of different fields, the frames have proven to be a powerful tool in signal processing and wavelet analysis see [5, 6]. On the other hand, the frame theory for spaces of the indefinite metric was introduced by [9, 12] and developed in the articles [1, 10]. Because of the close relationship between atomic systems and K-frames that are a generalization of the usual frames, it is interesting to develop the theory of atomic systems in the Krein spaces, which we introduce and characterize in this paper.

In the first section, the definition of atomic systems for bounded linear operators on Hilbert spaces is presented. In the second section, we introduce the Krein spaces and the most important properties for the development of this article. In the third section, we present the definition of the Bessel sequence in Krein spaces. In the last section, the most important for the development of the present investigation, the definition of atomic systems for Krein spaces is introduced. In the Example 4.4 we give an example of an atomic system for the Krein space $(\mathbb{R}^2, [\cdot, \cdot])$, with $[(a,b), (c,d)] = ac - bd$. Finally we characterize the atomic systems in the Krein spaces see, Theorems 4.5 and 4.7.

Definition 1.1 *[11] Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T \in B(H)$ a bounder linear operator on H. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset H$ is called an atomic system for T, if the following conditions are satisfied:*

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(i) There exists a constant $C > 0$ such that for all $x \in H$ there exists $a_x = \{a_n\}_{n \in \mathbb{N}} \in \ell_2(\mathbb{N})$ such that

$$\|a_x\|_{\ell_2(\mathbb{N})} \leq C\|x\| \ \text{ and } \ Tx = \sum_{n=1}^{\infty} a_n x_n.$$

(ii) The series $\sum_{n=1}^{\infty} c_n x_n$ converges for all $c = \{c_n\}_{n \in \mathbb{N}} \in \ell_2(\mathbb{N})$.

Theorem 1.2 *[7] Let* $(H_1, \langle \cdot, \cdot \rangle_1), (H_2, \langle \cdot, \cdot \rangle_2)$ *and* $(H, \langle \cdot, \cdot \rangle)$ *are Hilbert spaces and* $T_1 \in B(H_1, H), T_2 \in B(H_2, H)$ *bounded operators. The following statements are equivalent:*

(i) $R(T_1) \subset R(T_2)$;

(ii) $T_1 T_1^* \leq \lambda^2 T_2 T_2^*$ *for some* $\lambda \geq 0$ *and*

(iii) *There exists a bounded operator* $X \in L(H_1, H_2)$ *such that* $T_1 = T_2 X$.

2. Krein spaces

Definition 2.1 *[3] A space* K *with an indefinite inner product* $[\cdot, \cdot]$, *that admits a fundamental decomposition of the form* $K = K^+ [\dot{+}] K^-$ *such that* $(K^+, [\cdot, \cdot])$ *and* $(K^-, -[\cdot, \cdot])$ *are Hilbert spaces is called a Krein space.*

Definition 2.2 *[3] Let* $(K, [\cdot, \cdot])$ *be a Krein space with fundamental decomposition* $K = K^+ [\dot{+}] K^-$, *then exist unique operators*

$$P^+ : (K, [\cdot, \cdot]) \longrightarrow (K^+, [\cdot, \cdot]), \ \ P^- : (K, [\cdot, \cdot]) \longrightarrow (K^-, -[\cdot, \cdot]),$$

defined as follows: $P^+(k) = k^+$ *and* $P^-(k) = k^-$ *for all* $k \in K$, *where* $k^+ \in K^+$, $k^- \in K^-$ *and* $k = k^+ + k^-$. *The operators* P^+ *and* P^- *are known as fundamental projectors. The operator* $J : (K, [\cdot, \cdot]) \longrightarrow (K, [\cdot, \cdot])$ *defined by* $J := P^+ - P^-$, *i.e. for all* $k \in K$

$$Jk = P^+ k - P^- k = k^+ - k^-,$$

is called the fundamental symmetry of Krein space K *associated with the fundamental decomposition. From now on we will write* $(K, [\cdot, \cdot], J)$ *to denote the Krein space* $(K, [\cdot, \cdot])$ *with fundamental symmetry* J *associated to the fundamental decomposition* $K = K^+ [\dot{+}] K^-$.

Definition 2.3 *[2, 3] Let* $(K = K^+ [\dot{+}] K^-, [\cdot, \cdot])$ *be a Krein space and* J *the fundamental symmetry associated with the given decomposition. The function* $[\cdot, \cdot]_J : K \times K \longrightarrow \mathbb{C}$ *is defined by*

$$[x, y]_J = [Jx, y] \ \ x, y \in K.$$

is a usual positive definite inner product and is called J-*inner product.*

Definition 2.4 *[2] The fundamental symmetry* J *associated with the Krein space* $(K = K^+ [\dot{+}] K^-, [\cdot, \cdot])$ *induces a norm in* K *defined by:*

$$\|x\|_J := \sqrt{[x, x]_J} \ \text{ for all } \ x \in K,$$

this norm is called the J-norm of K. Explicitly,

$$\|x\|_J = ([x^+, x^+] - [x^-, x^-])^{1/2} \text{ for all } x \in K.$$

Unless otherwise stated, we assume that the topology of Krein spaces is directly related with the J-norm of K.

Remark 2.5 *We define*

$$\|x^+\|_+ = \sqrt{[x^+, x^+]}, \ x^+ \in K^+ \quad and \quad \|x^-\|_- = \sqrt{-[x^-, x^-]}, \ x^- \in K^-.$$

Proposition 2.6 *Let $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$ be a Krein space, then the following is true:*

$$\|x\|_J^2 = \|x^+\|_+^2 + \|x^-\|_-^2 \text{ for all } x = x^+ + x^- \in K.$$

Proof Let $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$ be a Krein space with the fundamental symmetry J. Let us consider $x \in K$ such that $x = x^+ + x^-$, with $x^+ \in K^+$ and $x^- \in K^-$. Then,

$$\|x\|_J^2 = [x, x]_J = [x^+ + x^-, x^+ + x^-]_J = [J(x^+ + x^-), x^+ + x^-] = [x^+ - x^-, x^+ + x^-]$$

$$= [x^+, x^+] - [x^-, x^-] = \left(([x^+, x^+])^{1/2}\right)^2 + \left((-[x^-, x^-])^{1/2}\right)^2 = \|x^+\|_+^2 + \|x^-\|_-^2.$$

$\square$

Definition 2.7 *[2] Let $(K_1 = K_1^+[\dot{+}]K_1^-, [\cdot, \cdot]_1, J_1)$ and $(K_2 = K_2^+[\dot{+}]K_2^-, [\cdot, \cdot]_2, J_2)$ are Krein spaces. Let $W : K_1 \longrightarrow K_2$, be a bounded linear operator, there is a bounded linear operator $W^{[*]} : K_2 \longrightarrow K_1$ such that*

$$[Wk_1, k_2]_2 = [k_1, W^{[*]}k_2]_1, \quad for \ all \ \ k_1 \in K_1 and \ k_2 \in K_2.$$

There is also a bounded linear operator $W^{*_J} : K_2 \longrightarrow K_1$ such that

$$[Wk_1, k_2]_{J_2} = [k_1, W^{*_J}k_2]_{J_1}, \ for \ all \ \ k_1 \in K_1 \ and \ k_2 \in K_2.$$

Proposition 2.8 *[2, 3] Let $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$ be a Krein space and J the fundamental symmetry associated with the given decomposition. The symmetry has the following properties*

1. *J is invertible with $J^{-1} = J$ and $J^2 = I$.*

2. *J is a self-adjoint operator in $(K, [\cdot, \cdot])$ and $(K, [\cdot, \cdot]_J)$. That is $J^{[*]} = J$ and $J^{*_J} = J$, additionally $[Jx, y] = [x, Jy]$ and $[Jx, y]_J = [x, Jy]_J$ for all $x, y \in K$.*

3. *J is a unitary operator in $(K, [\cdot, \cdot])$ and $(K, [\cdot, \cdot]_J)$. That is, $[Jx, Jy] = [x, y]$ and $[Jx, Jy]_J = [x, y]_J$ for all $x, y \in K$.*

Theorem 2.9 *[2] Let $(K, [\cdot, \cdot])$ be a Krein space and let*

$$K = K_1^+[\dot{+}]K_1^-, \ K = K_2^+[\dot{+}]K_2^-,$$

two fundamental decompositions of K. If J_1 and J_2 are the respective fundamental symmetries, then $\|.\|_{J_1}$ and $\|.\|_{J_2}$ are equivalents norms.

Proposition 2.10 *[15] Let $(K, [\cdot, \cdot], J)$ be a Krein space and $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis for the Hilbert space $(K, [\cdot, \cdot]_J)$ then*

$$x = \sum_{n=1}^{\infty} [x, e_n]_J e_n = \sum_{n=1}^{\infty} [x, Je_n]e_n \quad \text{for all} \;\; x \in K.$$

2.1. Example

We consider the vector space $\mathbb{C}^2$ over $\mathbb{C}$, with the usual sum and product and the function $[\cdot, \cdot] : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by

$$[(x_1, y_1), (x_2, y_2)] = x_1\overline{x_2} - y_1\overline{y_2}. \tag{2.1}$$

Well, it turns out that the space with inner product $(\mathbb{C}^2, [\cdot, \cdot])$ is a Krein space with fundamental decomposition $\mathbb{C}^2 = K^+[\dotplus]K^-$, where $K^+ = \{(x, 0) : \;\; x \in \mathbb{C}\}$ and $K^- = \{(0, y) : \;\; y \in \mathbb{C}\}$. Then, the fundamental symmetry

$$J((x, y)) = P^+(x, y) - P^-(x, y) = (x, -y),$$

determines the $J-$norm $\|\cdot\|_J$ such that

$$\|(x, y)\|_J = [J(x, y), (x, y)]^{1/2} = (x \cdot \overline{x} - (-y) \cdot \overline{y})^{1/2} = \sqrt{|x|^2 + |y|^2}.$$

The following example allows us to see the Hilbert space $\ell_2(\mathbb{N})$ as a nontrivial Krein space.

Example 2.11 *[9] Now, $\ell_2(\mathbb{N})$ can also be seen as Krein space with an inner product whose inner J-product coincides with the usual one. For instance, we may define the following mapping,*

$$[\cdot, \cdot]_{\ell_2} : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \to \mathbb{C}, [\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}}]_{\ell_2} := \sum_{n\in\mathbb{N}} (-1)^n \alpha_n \overline{\beta_n},$$

for all $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$. Thus, if $\{e_n\}_{n\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell_2(\mathbb{N})$, then $\ell_2(\mathbb{N})$ accepts the following fundamental decomposition:

$$\ell_2(\mathbb{N}) = \ell_2^+(\mathbb{N})[\dotplus]\ell_2^-(\mathbb{N}),$$

where $\ell_2^+(\mathbb{N}) = \overline{span}\{e_{2n} : n \in \mathbb{N}\}$ and $\ell_2^-(\mathbb{N}) = \overline{span}\{e_{2n+1} : n \in \mathbb{N}\}$ with associated fundamental symmetry

$$J_{\ell_2} : (\ell_2(\mathbb{N}), [\cdot, \cdot]_{\ell_2}) \to (\ell_2(\mathbb{N}), [\cdot, \cdot]_{\ell_2}),$$

given by $J_{\ell_2}(\{\alpha_n\}_{n\in\mathbb{N}}) = \{(-1)^n \alpha_n\}_{n\in\mathbb{N}}$ for all $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$. Therefore, $[\cdot, \cdot]_{J_{\ell_2}} = \langle\cdot, \cdot\rangle_{\ell_2}$.

Remark 2.12 *From now on, whenever we view $\ell_2(\mathbb{N})$ as Krein space, we will understand that it is endowed with a fundamental symmetry J_{ℓ_2} such that $[\cdot, \cdot]_{J_{\ell_2}} = \langle\cdot, \cdot\rangle_{\ell_2}$. An example of it is the one developed above and a more trivial one is the symmetry given by the identity operator on $\ell_2(\mathbb{N})$. Thus we will write $\mathfrak{k}_2(\mathbb{N})$ instead of $\ell_2(\mathbb{N})$ when viewed as a Krein space with such properties and denote the fundamental symmetry by $\mathfrak{J}_{\mathfrak{k}_2}$ to avoid confusion.*

Theorem 2.13 *[2] Let $(K = K^+[\dotplus]K^-, [\cdot, \cdot], J)$ be a Krein space and $T \in L(K)$ a bounded linear operator. Then $T^{[*]} = JT^{*J}J$.*

Proposition 2.14 *[10]* *Let* $(K, [\cdot, \cdot])$ *be a Krein space with fundamental symmetry* J *associated with the decomposition* $K = K^+[\dot{+}]K^-$. *If* $T \in B(K)$ *then* K^+ *is* T, $T^{[*]}$-*invariant if and only if* $TJ = JT$.

Lemma 2.15 *[9]* *Let* $(K, [\cdot, \cdot])$ *be a Krein space with associated fundamental symmetry* J *and* P *an orthogonal projection that commutes with* J, *then the spaces* PK *and* $(I - P)K$ *are Krein spaces with fundamental symmetries* PJ *and* $(I - P)J$, *respectively.*

3. Bessel sequences in Krein spaces

In this section, we show some properties of Bessel sequences necessary for the introduction of atomic systems for Krein spaces.

Definition 3.1 *[9]* *Let* $(K, [\cdot, \cdot], J)$ *be a Krein space, a sequence* $\mathbb{X} = \{x_n\}_{n \in \mathbb{N}} \subset K$ *is called a* Bessel sequence *for the Krein space* $(K, [\cdot, \cdot])$, *if there exists a constant* $0 < B < \infty$ *such that*

$$\sum_{n \in \mathbb{N}} |[x, x_n]|^2 \leq B\|x\|_J^2 \quad \text{for all } x \in K. \tag{3.1}$$

Remark 3.2 *The sequence* $\mathbb{X} = \{x_n\}_{n \in \mathbb{N}} \subset K$ *is a Bessel sequence for the Hilbert space* $(K, [\cdot, \cdot]_J)$, *if there exists a constant* B *with* $0 < B < \infty$ *such that*

$$\sum_{n \in \mathbb{N}} |[x, x_n]_J|^2 \leq B\|x\|_J^2 \quad \text{for all } x \in K. \tag{3.2}$$

The following results obtained in this investigation allow us to introduce atomic systems in spaces of indefinite metric and to study their main properties.

Theorem 3.3 *Let* $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$ *be a Krein space and* $\{x_n\}_{n \in \mathbb{N}} \subset K$. *The following statements are equivalent.*

  i) $\{x_n\}_{n \in \mathbb{N}}$ *is a Bessel sequence for the Krein space* $(K, [\cdot, \cdot])$.

 (ii) $\{Jx_n\}_{n \in \mathbb{N}}$ *is a Bessel sequence for the Hilbert space* $(K, [\cdot, \cdot]_J)$.

 (iii) $\{x_n\}_{n \in \mathbb{N}}$ *is a Bessel sequence for the Hilbert space* $(K, [\cdot, \cdot]_J)$.

 (iv) $\{Jx_n\}_{n \in \mathbb{N}}$ *is a Bessel sequence for the Krein space* $(K, [\cdot, \cdot])$.

Proof

$i) \to ii)$ Suppose that $\{x_n\}_{n \in \mathbb{N}} \subset K$ is a Bessel sequence for the Krein space $(K, [\cdot, \cdot])$, then there exists $0 < B < \infty$ such that

$$\sum_{n \in \mathbb{N}} |[x, x_n]|^2 \leq B\|x\|_J^2 \quad \text{for all } x \in K.$$

Therefore, $\sum_{n \in \mathbb{N}} |[x, Jx_n]_J|^2 = \sum_{n \in \mathbb{N}} |[Jx, Jx_n]|^2 = \sum_{n \in \mathbb{N}} |[x, x_n]|^2 \leq B\|x\|_J^2.$

Thus we obtain $\sum_{n \in \mathbb{N}} |[x, Jx_n]_J|^2 \leq B\|x\|_J^2$. From the above it follows that $\{Jx_n\}_{n \in \mathbb{N}} \subset K$ is a Bessel sequence for the Hilbert space $(K, [\cdot, \cdot]_J)$.

$ii) \to iii)$ Suppose that $\{Jx_n\}_{n\in\mathbb{N}} \subset K$ is a Bessel sequence for the Hilbert space $(K, [\cdot, \cdot]_J)$, then there exists a constant $0 < B < \infty$ such that

$$\sum_{n\in\mathbb{N}} |[x, Jx_n]_J|^2 \leq B\|x\|_J^2 \quad \text{for all } x \in K.$$

Therefore, $\displaystyle\sum_{n\in\mathbb{N}} |[x, x_n]_J|^2 = \sum_{n\in\mathbb{N}} |[Jx, Jx_n]_J|^2 \leq B\|Jx\|_J^2 = B[Jx, Jx]_J = B[x, x]_J \leq B\|x\|_J^2.$

Thus we obtain $\sum_{n\in\mathbb{N}} |[x, x_n]_J|^2 \leq B\|x\|_J^2$, whence it follows that $\{x_n\}_{n\in\mathbb{N}} \subset K$ is a Bessel sequence for the Hilbert space $(K, [\cdot, \cdot]_J)$.

$iii) \to iv)$ Suppose that $\{x_n\}_{n\in\mathbb{N}} \subset K$ is a Bessel sequence for the Hilbert space $(K, [\cdot, \cdot]_J)$, then there exists a constant $0 < B < \infty$ such that

$$\sum_{n\in\mathbb{N}} |[x, x_n]_J|^2 \leq B\|x\|_J^2 \quad \text{for all } x \in K.$$

Therefore, $\displaystyle\sum_{n\in\mathbb{N}} |[x, Jx_n]|^2 = \sum_{n\in\mathbb{N}} |[Jx, x_n]|^2 = \sum_{n\in\mathbb{N}} |[x, x_n]_J|^2 \leq B\|x\|_J^2.$

Thus we obtain $\sum_{n\in\mathbb{N}} |[x, Jx_n]|^2 \leq B\|x\|_J^2$, therefore $\{Jx_n\}_{n\in\mathbb{N}} \subset K$ is a Bessel sequence for the Krein space $(K, [\cdot, \cdot])$.

$iv) \to i)$ Suppose that $\{Jx_n\}_{n\in\mathbb{N}} \subset K$ is a Bessel sequence for the Krein space $(K, [\cdot, \cdot])$, then exists a constant $0 < B < \infty$ such that

$$\sum_{n\in\mathbb{N}} |[x, Jx_n]|^2 \leq B\|x\|_J^2 \quad \text{for all } x \in K.$$

Therefore, $\displaystyle\sum_{n\in\mathbb{N}} |[x, x_n]|^2 = \sum_{n\in\mathbb{N}} |[Jx, Jx_n]_J|^2 \leq B\|Jx\|_J^2 = B[Jx, Jx]_J = B[x, x]_J \leq B\|x\|_J^2.$

Thus we obtain $\sum_{n\in\mathbb{N}} |[x, x_n]|^2 \leq B\|x\|_J^2$, therefore $\{x_n\}_{n\in\mathbb{N}} \subset K$ is a Bessel sequence for the Krein space $(K, [\cdot, \cdot])$. □

The following result shows that orthogonal projectors preserve the Bessel sequence in spaces of indefinite metrics.

Theorem 3.4 *Let $(K, [\cdot, \cdot], J)$ be a Krein space with associated fundamental symmetry J and P an orthogonal projection that commutes with J. If $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(K, [\cdot, \cdot])$ then $\{Px_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for PK.*

Proof As $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(K, [\cdot, \cdot], J)$, there exists a constant $0 < B < \infty$ such that $\sum_{n\in\mathbb{N}} |[x, x_n]|^2 \leq B\|x\|_J^2$ for all $x \in K$.

We consider $z \in PK$, there is $x \in K$ such that $z = Px$.

$$\sum_{n\in\mathbb{N}} |[z, Px_n]|^2 = \sum_{n\in\mathbb{N}} |[Px, Px_n]|^2 = \sum_{n\in\mathbb{N}} |[PPx, x_n]|^2 = \sum_{n\in\mathbb{N}} |[Px, x_n]|^2 = \sum_{n\in\mathbb{N}} |[z, x_n]|^2$$

$$\leq B\|z\|_J^2 = B\|Px\|_J^2 = B[Px, Px]_J = B[x, PPx]_J = B[Jx, PPx] = B[PJx, Px]$$

$$= B[PPJx, Px] = B[PJPx, Px] = B[PJz, z] = B[z, z]_{PJ} = B\|z\|_{PJ}^2 \text{ for all } z \in PK.$$

$\square$

Theorem 3.5 *Let $(K_1, [\cdot, \cdot]_1, J_1)$ and $(K_2, [\cdot, \cdot]_2, J_2)$ are two Krein space. If $\mathbb{X} = \{x_n\}_{n \in \mathbb{N}} \subset K_1$ and $\mathbb{Y} = \{y_n\}_{n \in \mathbb{N}} \subset K_2$ Bessel sequences for $(K_1, [\cdot, \cdot]_1, J_1)$ and $(K_2, [\cdot, \cdot]_2, J_2)$, respectively. Then,*

$$\mathbb{X} + \mathbb{Y} := \{x_n + y_n\}_{n \in \mathbb{N}} \subset K_1[\dot{+}]K_2,$$

is a Bessel sequence for $\left(K_1[\dot{+}]K_2, [\cdot, \cdot] = [\cdot, \cdot]_1 + [\cdot, \cdot]_2, J = J_1[\dot{+}]J_2\right).$

Proof Let $\mathbb{X} = \{x_n\}_{n \in \mathbb{N}} \subset K_1$ and $\mathbb{Y} = \{y_n\}_{n \in \mathbb{N}} \subset K_2$ are Bessel sequences for $(K_1, [\cdot, \cdot]_1, J_1)$ and $(K_2, [\cdot, \cdot]_2, J_2)$ respectively, then there exist constants $0 < A, B < \infty$, such that

$$\sum_{n \in \mathbb{N}} |[x, x_n]_1|^2 \leq A\|x\|_{J_1}^2 \quad \text{for all } x \in K_1. \tag{3.3}$$

$$\sum_{n \in \mathbb{N}} |[x, y_n]_2|^2 \leq B\|x\|_{J_2}^2 \quad \text{for all } x \in K_2. \tag{3.4}$$

Let $x \in K_1[\dot{+}]K_2$, then $x = x_1 + x_2$, with $x_1 \in K_1$, $x_2 \in K_2$, and we have that:

$$\sum_{n \in \mathbb{N}} |[x, x_n + y_n]|^2 = \sum_{n \in \mathbb{N}} |[x_1, x_n]_1 + [x_2, y_n]_2|^2 \leq \sum_{n \in \mathbb{N}} (|[x_1, x_n]_1| + |[x_2, y_n]_2|)^2$$

$$\leq \sum_{n \in \mathbb{N}} \left(|[x_1, x_n]_1|^2 + 2|[x_1, x_n]_1||[x_2, y_n]_2| + |[x_2, y_n]_2|^2\right)$$

$$= \sum_{n \in \mathbb{N}} |[x_1, x_n]_1|^2 + \sum_{n \in \mathbb{N}} 2|[x_1, x_n]_1||[x_2, y_n]_2| + \sum_{n \in \mathbb{N}} |[x_2, y_n]|_2^2.$$

Using Hölder's inequality and Equations (3.3) and (3.4) we obtain the following

$$\sum_{n \in \mathbb{N}} |[x, x_n + y_n]|^2 \leq \sum_{n \in \mathbb{N}} |[x_1, x_n]_1|^2 + 2\left(\sum_{n \in \mathbb{N}} |([x_1, x_n]_1)|^2\right)^{1/2}\left(\sum_{n \in \mathbb{N}} |([x_2, y_n]_2)|^2\right)^{1/2} + \sum_{n \in \mathbb{N}} |[x_2, y_n]|_2^2$$

$$\leq A\|x_1\|_{J_1}^2 + 2\left(\sqrt{A}\|x_1\|_{J_1}\right)\left(\sqrt{B}\|x_2\|_{J_2}\right) + B\|x_2\|_{J_2}^2 = \left(\sqrt{A}\|x_1\|_{J_1} + \sqrt{B}\|x_2\|_{J_2}\right)^2$$

$$\leq M^2\left(\|x_1\|_{J_1} + \|x_2\|_{J_2}\right)^2, \quad \text{with } M = \max\{\sqrt{A}, \sqrt{B}\}.$$

Then there exists $0 < M^2 < \infty$, such that

$$\sum_{n \in \mathbb{N}} |[x, x_n + y_n]|^2 \leq M^2\|x\|_{J_1[\dot{+}]J_2}^2.$$

Which means that the sequence $\mathbb{X} + \mathbb{Y} := \{x_n + y_n\}_{n \in \mathbb{N}}$, is a Bessel sequence for the Krein space $K_1[\dot{+}]K_2$. $\square$

4. Atomic systems in Krein spaces

In this section, we present the most important results of the present investigation. We introduce the definition of atomic systems for bounded operators in Krein spaces, showing also that these systems are independent of

the fundamental decomposition of the Krein space, see Theorem 4.7. We construct an example of an atomic system in Krein space $(\mathbb{R}^2, [\cdot, \cdot])$. Finally, we characterize atomic systems for bounded operators in the Krein spaces, see Theorem 4.8 and we show how to transfer atomic systems from a Krein space to the associated Hilbert space, see Theorem 4.5.

Definition 4.1 *Let* $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$ *be a Krein space and* $T \in \mathcal{B}(K)$. *We say that* $\{x_n\}_{n \in \mathbb{N}} \subset K$ *is an atomic system for* T *in* $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$ *if:*

(i) $\{x_n\}_{n \in \mathbb{N}}$ *is a Bessel sequence for* $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$.

(ii) *There exists a Bessel sequence* $\{y_n\}_{n \in \mathbb{N}}$ *in the Krein space* $(K = K^+[\dot{+}]K^-, [\cdot, \cdot], J)$ *such that*

$$Tx = \sum_{n=1}^{\infty} [x, y_n] x_n \quad \text{for all } x \in K.$$

Remark 4.2 *In the theory of atomic systems in Hilbert spaces, we can observe in [14] how orthonormal basis can produce atomic systems for bounded operators. In the case of Krein spaces using the Proposition 2.10 and following [11] as in the case of Hilbert spaces, easily proven that orthonormal bases produce atomic systems for bounded operators in Krein spaces.*

Remark 4.3 *With the following simple examples, we give a finite sequence of elements in the Krein space* $(\mathbb{R}^2, [\cdot, \cdot])$, *where* $[(a, b), (c, d)] = ac - bd$, *which is not a basis, however, it is an atomic system for the operator* T *defined by* $T(m, r) = (m + r, 0)$. *The calculations are quite simple since we are dealing with a finite sequence and a continuous operator in finite-dimensional spaces. For the convenience of the reader and for clarity of the example we proceed to complete them.*

Example 4.4 *We consider the Krein space* $(\mathbb{R}^2, +, \cdot)$ *with indefinite inner product:*

$[\cdot, \cdot] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, *given by* $[(a, b), (c, d)] = ac - bd$.

Set $K^+ = \{(x, 0) : x \in \mathbb{R}\}$ *and* $K^- = \{(0, y) : y \in \mathbb{R}\}$, *so* $\mathbb{R}^2 = K^+[\dot{+}]K^-$.

$$J : \mathbb{R}^2 \to \mathbb{R}^2$$

$$(a, b) \longmapsto J(a, b) = (a, -b).$$

Then, $[(a, b), (c, d)]_J = [J(a, b), (c, d)] = [(a, -b), (c, d)] = ac + bd$.

We consider $\{x_n\}_{n=1}^4 = \{(1, 0), (1, 0), (0, 1), (0, 1)\} \subset \mathbb{R}^2$ *and* $T(m, r) = (m + r, 0)$. *Let us prove that* $\{x_n\}_{n=1}^4$ *is an atomic system for* T *in* $(\mathbb{R}^2, [\cdot, \cdot], J)$.

Statement 1.

$\{x_n\}_{n=1}^4$ *is a Bessel sequence in* $(\mathbb{R}^2, [\cdot, \cdot], J)$. *In fact, let* $x = (m, r)$, *then,*

$$\sum_{n=1}^4 |[x, x_n]|^2 = 2|[(m, r), (1, 0)]|^2 + 2|[(m, r), (0, 1)]|^2 = 2m^2 + 2r^2 \le 2\|x\|_J^2.$$

Statement 2.

T is a linear and bounded operator.

In fact, $T(\alpha(m,r)) = T(\alpha m, \alpha r) = (\alpha m + \alpha r, 0) = \alpha T(m,r)$ *and* $T((m,r) + (u,v)) = T((m+r, u+v)) = T(m,r) + T(u,v)$. *Therefore, T is a linear operator. Moreover,*

$$\|T(m,r)\|_J^2 = \|(m+r,0)\|_J^2 = [(m+r,0),(m+r,0)]_J = [J(m+r,0),(m+r,0)] = [(m+r,0),(m+r,0)]$$

$$= (m+r)^2 \leq 2(m^2 + r^2) = 2\|(m,r)\|_J^2.$$

Therefore, T is a bounded operator.

Statement 3.

There is a Bessel sequence $\{y_n\}_{n\in\mathbb{N}}$ *such that* $Tx = \sum\limits_{n=1}^{4}[x,y_n]x_n$. *In fact, consider the sequence* $\{y_n\}_{n=1}^{4} = \{(1,0),(0,-1),(0,0),(0,0)\}$. *Let* $y = (m,r)$, *then,*

$$\sum_{n=1}^{4}|[y,y_n]|^2 = |[(m,r),(1,0)]|^2 + |[(m,r),(0,-1)]|^2 = m^2 + r^2 \leq \|(m,r)\|_J^2 = \|y\|_J^2.$$

On the other hand, let $x = (m,r)$, *then*

$$\sum_{n=1}^{4}[x,y_n]x_n = \sum_{n=1}^{4}[(m,r),y_n]x_n = [(m,r),(1,0)](1,0) + [(m,r),(0,-1)](1,0) + [(m,r),(0,0)](0,1)$$

$$+ [(m,r),(0,0)](0,1) = (m,0) + (r,0) + (0,0) + (0,0) = (m+r,0) = T(m,r) = Tx.$$

Based on the above, we can conclude that $\{x_n\}_{n=1}^{4}$ *is a atomic system for T in Krein space* $(\mathbb{R}^2, [\cdot,\cdot], J)$.

With the following results, we see that the atomic systems for a Krein space and the Hilbert space associated with the $J-$symmetry are related.

Theorem 4.5 *Let* $(K = K^+[\dot{+}]K^-, [\cdot,\cdot], J)$ *be a Krein space,* $T \in B(K)$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset K$, *then the following statements are equivalent:*

(i) $\{x_n\}_{n\in\mathbb{N}}$ *is an atomic system for T in Krein space* $(K, [\cdot,\cdot])$.

(ii) $\{x_n\}_{n\in\mathbb{N}}$ *is an atomic system for T in Hilbert space* $(K, [\cdot,\cdot]_J)$.

(iii) $\{Jx_n\}_{n\in\mathbb{N}}$ *is an atomic system for JT in Krein space* $(K, [\cdot,\cdot])$.

Proof

$i) \rightarrow ii)$ Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for T in the Krein space $(K, [\cdot,\cdot])$, then there exist a Bessel sequence $\{y_n\}_{n\in\mathbb{N}} \subset K$ in $(K, [\cdot,\cdot])$ such that $Tx = \sum\limits_{n=1}^{\infty}[x,y_n]x_n$ for all $x \in K$. By Theorem 3.3 we have that $\{x_n\}_{n\in\mathbb{N}}$ and $\{Jy_n\}_{n\in\mathbb{N}}$ are Bessel sequences in the Hilbert space $(K, [\cdot,\cdot]_J)$, furthermore,

$$Tx = TIx = T(J(Jx)) = \sum_{n\in\mathbb{N}}[JJx,y_n]x_n = \sum_{n\in\mathbb{N}}[Jx,y_n]_J x_n = \sum_{n\in\mathbb{N}}[x,Jy_n]_J x_n.$$

Therefore $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for T in the Hilbert space $(K, [\cdot,\cdot]_J)$.

$ii) \to iii)$ Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for T in the Hilbert space $(K, [\cdot,\cdot]_J)$, then there exists a

Bessel sequence $\{y_n\}_{n\in\mathbb{N}} \subset K$ in $(K, [\cdot,\cdot]_J)$ such that $Tx = \sum_{n=1}^{\infty} [x, y_n]_J x_n$ for all $x \in K$.

$$JTx = J\left(\sum_{n\in\mathbb{N}}[x, y_n]_J x_n\right) = \sum_{n\in\mathbb{N}}[x, y_n]_J Jx_n = \sum_{n\in\mathbb{N}}[Jx, y_n]Jx_n = \sum_{n\in\mathbb{N}}[x, Jy_n]Jx_n.$$

By the Theorem 3.3 we have that $\{Jx_n\}_{n\in\mathbb{N}}$ and $\{Jy_n\}_{n\in\mathbb{N}}$ are Bessel sequences in the Krein space $(K, [\cdot,\cdot])$. Therefore $\{Jx_n\}_{n\in\mathbb{N}}$ is an atomic system for JT in the Krein space $(K, [\cdot,\cdot])$.

$iii) \to i)$ Suppose that $\{Jx_n\}_{n\in\mathbb{N}}$ is an atomic system for JT in the Krein space $(K, [\cdot,\cdot])$, then there exists a

Bessel sequence $\{y_n\}_{n\in\mathbb{N}} \subset K$ in $(K, [\cdot,\cdot])$ such that $JTx = \sum_{n=1}^{\infty}[x, y_n]Jx_n$ for all $x \in K$. As $\{Jx_n\}_{n\in\mathbb{N}}$

is a Bessel sequence in the Krein space $(K, [\cdot,\cdot])$, by Theorem 3.3 we have that $\{JJx_n\}_{n\in\mathbb{N}} = \{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence in the Krein space $(K, [\cdot,\cdot])$, then

$$Tx = J(JTx) = J\left(\sum_{n\in\mathbb{N}}[x, y_n]Jx_n\right) = \sum_{n\in\mathbb{N}}[x, y_n]JJx_n = \sum_{n\in\mathbb{N}}[x, y_n]x_n.$$

Therefore $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for T in the Krein space $(K, [\cdot,\cdot])$. □

Remark 4.6 *The following result shows that atomic systems do not depend on the decomposition of the Krein space.*

Theorem 4.7 *Let $(K, [\cdot,\cdot])$ be a Krein space with fundamental decompositions $K = K_1^+[\dot{+}]K_1^-$, $K = K_2^+[\dot{+}]K_2^-$ and fundamental symmetries J_1, J_2, respectively, and $T : K \to K$ a bounded operator. If $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for T with respect to J_1, then $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for T with respect to J_2.*

Proof Let $\{x_n\}_{n\in\mathbb{N}} \subset K$ be an atomic system for T in $(K = K_1^+[\dot{+}]K_1^-, [\cdot,\cdot], J_1)$, then we have

(i) $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(K = K_1^+[\dot{+}]K_1^-, [\cdot,\cdot], J_1)$.

(ii) There exists a Bessel sequence $\{y_n\}_{n\in\mathbb{N}}$ for $(K = K_1^+[\dot{+}]K_1^-, [\cdot,\cdot], J_1)$ such that

$$Tx = \sum_{n=1}^{\infty}[x, y_n]x_n.$$

Since the norms $\|.\|_{J_1}$ and $\|.\|_{J_2}$ are equivalent, there exist constants $A, B > 0$ such that

$$A\|x\|_{J_1} \le \|x\|_{J_2} \le B\|x\|_{J_1} \text{ for all } x \in K. \tag{4.1}$$

As $\{x_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence for* $(K = K_1^+[\dot{+}]K_1^-, [\cdot,\cdot], J_1)$, there exists a constant $0 < D < \infty$ such that

$$\sum_{n\in\mathbb{N}}|[x, x_n]|^2 \le D\|x\|_{J_1}^2 \le \frac{D}{A}\|x\|_{J_2}^2, \tag{4.2}$$

therefore $\{x_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence for* $(K = K_2^+[\dot{+}]K_2^-, [\cdot,\cdot], J_2)$. Similarly, $\{y_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(K = K_2^+[\dot{+}]K_2^-, [\cdot,\cdot], J_2)$.

Thus, we have $\{x_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence for* $(K = K_2^+[\dot{+}]K_2^-, [\cdot,\cdot], J_2)$ and $\{y_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(K = K_2^+[\dot{+}]K_2^-, [\cdot,\cdot], J_2)$ such that

$$Tx = \sum_{n=1}^{\infty}[x, y_n]x_n.$$

Consequently $\{x_n\}_{n\in\mathbb{N}}$ is an *atomic system for* T in $(K = K_2^+[\dot{+}]K_2^-, [\cdot,\cdot], J_2)$. $\qquad\square$

Theorem 4.8 *Let* $(K = K^+[\dot{+}]K^-, [\cdot,\cdot])$ *be a Krein space and* J *the fundamental symmetry induced by the given fundamental decomposition,* $T \in \mathcal{B}(K)$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset K$*, then the following conditions are equivalent:*

(i) $\{x_n\}_{n\in\mathbb{N}}$ *is an atomic system for* T *in* $(K = K^+[\dot{+}]K^-, [\cdot,\cdot])$*.*

(ii) (a). There is a constant $C > 0$ *such that for all* $x \in K$ *there exists* $a_x = \{a_n\}_{n\in\mathbb{N}} \in \mathfrak{k}_2(\mathbb{N})$ *such that*

$$\|a_x\|_{\mathfrak{I}_{\mathfrak{k}_2}} \leq C\|x\|_J \text{ and } Tx = \sum_{n=1}^{\infty} a_n x_n.$$

(b). The series $\sum_n z_n x_n$ *converges for all* $\{z_n\}_{n\in\mathbb{N}} \in \mathfrak{k}_2(\mathbb{N})$*.*

Proof

$i) \to ii)$ If $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for T in $(K = K^+[\dot{+}]K^-, [\cdot,\cdot])$, then there exists a Bessel sequence $\{y_n\}_{n\in\mathbb{N}}$ in $(K = K^+[\dot{+}]K^-, [\cdot,\cdot])$ such that

$$Tx = \sum_{n=1}^{\infty}[x, y_n]x_n \text{ for all } x \in K.$$

We consider $a_x = \{[x, y_n]\}_{n\in\mathbb{N}}$ and $a_n(x) = [x, y_n]$. As $\{y_n\}_{n\in\mathbb{N}}$ is a Bessel sequence in $(K = K^+[\dot{+}]K^-, [\cdot,\cdot])$, there is $0 < C < \infty$ such that

$$\|a_x\|_{\mathfrak{I}_{\mathfrak{k}_2}}^2 = \|a_x\|_{\ell_2}^2 = \sum_{n=1}^{\infty}|[x, y_n]|^2 \leq C\|x\|_J^2 \text{ and } Tx = \sum_{n=1}^{\infty} a_n x_n.$$

On the other hand, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of elements in $\mathfrak{k}_2(\mathbb{N})$. Let us prove that $\sum_{n\in\mathbb{N}} z_n x_n$ converges.

We consider $N, M \in \mathbb{N}$ with $N > M > 0$, then

$$
\left\| \sum_{n=1}^{N} z_n x_n - \sum_{n=1}^{M} z_n x_n \right\|_J = \left\| \sum_{n=M+1}^{N} z_n x_n \right\|_J = \sup_{g \in K, \ \|g\|_J=1} \left\{ \left| \left[\sum_{n=M+1}^{N} z_n x_n, g \right]_J \right| \right\}
$$

$$
\leq \sup_{g \in K, \ \|g\|_J=1} \left\{ \sum_{n=M+1}^{N} |z_n [x_n, g]_J| \right\} = \left\{ \sum_{n=M+1}^{N} |z_n|^2 \right\}^{1/2} \sup_{g \in K, \ \|g\|_J=1} \left\{ \sum_{n=M+1}^{N} |[x_n, g]_J|^2 \right\}^{1/2}
$$

$$
\leq \left\{ \sum_{n=M+1}^{N} |z_n|^2 \right\}^{1/2} \sup_{g \in K, \ \|g\|_J=1} \left\{ \sum_{n=1}^{\infty} |[x_n, Jg]|^2 \right\}^{1/2}
$$

$$
\leq \left\{ \sum_{n=M+1}^{N} |z_n|^2 \right\}^{1/2} \sup_{g \in K, \ \|g\|_J=1} \left\{ B\|Jg\|_J^2 \right\}^{1/2} \leq \left\{ \sum_{n=M+1}^{N} |z_n|^2 \right\}^{1/2} \sup_{g \in K, \ \|g\|_J=1} \left\{ B\|g\|_J^2 \right\}^{1/2}
$$

$$
\leq \sqrt{B} \left\{ \sum_{n=M+1}^{N} |z_n|^2 \right\}^{1/2} .
$$

In the previous calculations, we have made use of the following arguments: $(K, [\cdot, \cdot]_J)$ is a Hilbert space, the Hölder's inequality and the fact that $\{x_n\}_{n \in \mathbb{N}}$ is a Bessel sequence in $(K, [\cdot, \cdot])$ and $\{x_n\}_{n \in \mathbb{N}}$ is a Bessel sequence in $(K, [\cdot, \cdot]_J)$, see Theorem 3.3. Then there exists $0 < B < \infty$ such that $\sum_{n=1}^{\infty} |[x_n, g]_J|^2 \leq B\|g\|_J^2$. Now, as we take $\{z_n\}_{n \in \mathbb{N}}$ in $\mathfrak{k}_2(\mathbb{N})$, we have that $\{z_n\}_{n \in \mathbb{N}}$ is a square summable sequence in $\ell_2(\mathbb{N})$, therefore it is convergent and additionally Cauchy. That is, $\left\{ \sum_{n=1}^{N} |z_n|^2 \right\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$.

The above shows that $\sum_{n=1}^{\infty} z_n x_n$ is a Cauchy sequence in the Hilbert space $(K, [\cdot, \cdot]_J)$ and therefore converges.

$ii) \to i)$ Suppose that:

(a) There exists a constant $C > 0$ such that for all $x \in K$ there exists $a_x = \{a_n\}_{n \in \mathbb{N}} \in \mathfrak{k}_2(\mathbb{N})$ such that

$$
\|a_x\|_{\mathfrak{J}_{\mathfrak{k}_2}} \leq C\|x\|_J \text{ and } Tx = \sum_{n=1}^{\infty} a_n x_n.
$$

(b) The series $\sum_n z_n x_n$ converges for all $\{z_n\}_{n \in \mathbb{N}} \in \mathfrak{k}_2(\mathbb{N})$.

We remember that the convergence in the Krein space is related to the J−norm, as the convergence in the space $\mathfrak{k}_2(\mathbb{N})$ is related to the inner product of the space $\ell_2(\mathbb{N})$.

Let us consider the sequence of operators $W_k : \ell_2(\mathbb{N}) \to (K, \|\cdot\|_J)$ defined by $W_k(\{z_n\}_{n=1}^{\infty}) = \sum_{n=1}^{k} z_n x_n$, it can be easily proved that $W_k \longrightarrow W$ converges pointwise when $k \longrightarrow \infty$, where $W : \ell_2(\mathbb{N}) \longrightarrow (K, \|\cdot\|_J)$ is defined by $W(\{z_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} z_n x_n$. Since W_k be a sequence of bounded operators, which converges pointwise to W, we can assert that the operator $W : \ell_2(\mathbb{N}) \to (K, \|\cdot\|_J)$ is a bounded operator.

We now proceed to calculate the adjoint operator of W. Since $W : \ell_2(\mathbb{N}) \longrightarrow (K, \|\cdot\|_J)$ is bounded, we already know that W^{*J} is a bounded operator where $W^{*J} : (K, \|\cdot\|_J) \longrightarrow \ell_2(\mathbb{N})$. Therefore, the kth coordinate

function is bounded from $(K, \| \cdot \|_J)$ to $\mathbb{C}$; by Riesz representation theorem, W^{*J} has the following form

$$W^{*J}k = \{[k, h_n]_J\}_{n=1}^\infty,$$

for some $\{h_n\}_{n=1}^\infty$ in $(K, \| \cdot \|_J)$.

Let $k \in (K, \| \cdot \|_J)$ and $\{z_n\}_{n \in \mathbb{N}} \in \mathfrak{k}_2(\mathbb{N})$, then

$$[k, W\{z_n\}_{n \in \mathbb{N}}]_J = \left[k, \sum_{n=1}^\infty z_n x_n\right] = \sum_{n=1}^\infty \overline{z_n}[k, x_n],$$

$$\left[W^{*J}k, \{z_n\}_{n \in \mathbb{N}}\right]_{\mathfrak{J}_{\mathfrak{k}_2}} = [\{[k, h_n]_J\}_{n=1}^\infty, \{z_n\}_{n \in \mathbb{N}}]_{\mathfrak{J}_{\mathfrak{k}_2}} = \sum_{n=1}^\infty [k, h_n]\overline{z_n},$$

$$\sum_{n=1}^\infty \overline{z_n}[k, x_n] = \sum_{n=1}^\infty \overline{z_n}[k, h_n] \text{ for all } k \in (K, \| \cdot \|_J) \text{ and for all } \{z_n\}_{n \in \mathbb{N}} \in \mathfrak{k}_2(\mathbb{N}),$$

it follows from the above that $h_n = x_n$ and $W^{*J}k = \{[k, x_n]_J\}_{n=1}^\infty$.

Since $\|W\| = \|W^{*J}\|$ and $\|W^{*J}k\|^2 \leq \|W\|^2\|k\|^2$ for all $k \in (K, \| \cdot \|_J)$, it follows that

$$\sum_{n=1}^\infty |[k, x_n]|^2 = \|W^{*J}k\|^2 \leq \|W\|^2\|k\|^2 \text{ for all } k \in (K, \| \cdot \|_J),$$

that is, $\{x_n\}_{n \in \mathbb{N}}$ is a Bessel sequence.

By hypothesis we have that $T : (K, \| \cdot \|_J) \to (K, \| \cdot \|_J)$ is a bounded operator. Using the Theorem 1.2, there exists a bounded linear operator $D : (K, \| \cdot \|_J) \to \ell_2(\mathbb{N})$ such that $T = WD$.

Let $A_n : (K, \| \cdot \|_J) \to \mathbb{C}$ be the linear functional defined by $A_n x = A_n(x) := (Dx)_n$.

(The n-th term of the sequence , $A^x := Dx = \{(Dx)_n\}_{n \in \mathbb{N}} = \{A_n(x)\}_{n \in \mathbb{N}}$).

Then we have that:

$$|A_n(x)| \leq \left(\sum_{n=1}^\infty |A_n(x)|^2\right)^{1/2} = \|A^x\|_{\mathfrak{J}_{\mathfrak{k}_2}} = \|A^x\|_{\ell_2} = \|Dx\|_{\ell_2} \leq \|D\|\|x\|_J.$$

Since D is a bounded operator, $|A_n(x)| \leq \|D\|\|x\|_J$ means that $A_n(x) := (Dx)_n$ is a bounded linear functional. By the Riesz representation theorem, there is $y_n \in (K, \| \cdot \|_J)$ such that

$$A_n(x) = [x, y_n]_J.$$

Then we have that

$$Tx = WDx = W\left(\{A_n(x)\}_{n=1}^\infty\right) = \sum_{n=1}^\infty (A_n(x))x_n = \sum_{n=1}^\infty [x, y_n]_J x_n = \sum_{n=1}^\infty [x, Jy_n]x_n.$$

Furthermore $\sum_{n=1}^\infty |[x, y_n]_J|^2 = \sum_{n=1}^\infty |A_n(x)|^2 \leq \|D\|^2\|x\|_J^2$, which means that $\{y_n\}_{n \in \mathbb{N}}$ is a Bessel sequence in $(K, \| \cdot \|_J)$. By Theorem 3.3, we have that $\{Jy_n\}_{n \in \mathbb{N}}$ is a Bessel sequence in Krein space $(K, \| \cdot \|_J)$. This implies that $\{x_n\}_{n \in \mathbb{N}}$ is an atomic system for T in $(K, [\cdot, \cdot])$.

$\square$

Theorem 4.9 *Let $(K, [\cdot, \cdot])$ be a Krein space with associated fundamental symmetry J and P an orthogonal projection that commutes with J. If $\{x_n\}_{n \in \mathbb{N}}$ is an atomic system for T in $(K, [\cdot, \cdot])$ then $\{Px_n\}_{n \in \mathbb{N}}$ is an atomic system for PT in PK and $\{(I - P)x_n\}_{n \in \mathbb{N}}$ is an atomic system for $(I - P)T$ in $(I - P)K$.*

Proof Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is an atomic system for T in $(K, [\cdot, \cdot])$, then there exists a Bessel sequence $\{y_n\}_{n \in \mathbb{N}}$ in $(K, [\cdot, \cdot])$ such that

$$Tx = \sum_{n=1}^{\infty} [x, y_n] x_n. \tag{4.3}$$

By Theorem 3.4 $\{Px_n\}_{n \in \mathbb{N}}$ and $\{Py_n\}_{n \in \mathbb{N}}$ are Bessel sequences in the Krein space $(PK, [\cdot, \cdot])$. Consider $z \in PK$, then there exists $x \in K$ such that $z = Px$.

$$PTz = PTPx = PTPPx = P \sum_{n=1}^{\infty} [PPx, y_n] x_n = \sum_{n=1}^{\infty} [Px, Py_n] Px_n = \sum_{n=1}^{\infty} [z, Py_n] Px_n.$$

Thus $\{Px_n\}_{n \in \mathbb{N}}$ is an atomic system for PT in the Krein spaces $(PK, [\cdot, \cdot])$. Similarly, it is proved that $\{(I - P)x_n\}_{n \in \mathbb{N}}$ is an atomic system for $(I - P)K$. $\qquad \square$

Theorem 4.10 *Let $(K, [\cdot, \cdot], J)$ be a Krein space, $\mathbb{X} = \{x_n\}_{n \in \mathbb{N}}$ an atomic system for $T \in B(K)$ in $(K, [\cdot, \cdot])$ and $L \in B(K)$ a unitary operator that commutes with T. Then $L(\mathbb{X})$ is an atomic system for T in $(K, [\cdot, \cdot])$.*

Proof If $\{x_n\}_{n \in \mathbb{N}}$ is an atomic system for $(K = K^+ [\dot{+}] K^-, [\cdot, \cdot], J)$, there is a Bessel sequence $\{y_n\}_{n \in \mathbb{N}}$ in the Krein space $(K = K^+ [\dot{+}] K^-, [\cdot, \cdot], J)$ such that $Tx = \sum_{n=1}^{\infty} [x, y_n] x_n$ for all $x \in K$. Therefore

$$Tx = TIx = TLL^{-1}x = LTL^{-1}x = L \sum_{n=1}^{\infty} [L^{-1}x, y_n] x_n = \sum_{n=1}^{\infty} [L^{[*]}x, y_n] L x_n = \sum_{n=1}^{\infty} [x, Ly_n] L x_n \text{ for all } x \in K.$$

Furthermore there exists $0 < B < \infty$ such that

$$\sum_{n \in \mathbb{N}} |[x, x_n]|^2 \leq B \|x\|_J^2 \quad \text{for all } x \in K.$$

Therefore $\sum_{n \in \mathbb{N}} |[x, Lx_n]|^2 = \sum_{n \in \mathbb{N}} \left| [L^{[*]}x, x_n] \right|^2 \leq B \|L^{[*]}x\|_J^2 \leq B \|L^{[*]}\|^2 \|x\|_J^2$ for all $x \in K$. This shows that $L(\mathbb{X})$ is a Bessel sequence. Therefore $L(\mathbb{X})$ is an atomic system for T in $(K, [\cdot, \cdot])$. $\qquad \square$

5. Conclusion

Atomic systems associated with a bounded operator in a Krein space are independent of the fundamental decomposition, see Theorem 4.7. Orthogonal projectors commuting with the fundamental symmetry generate new atomic systems, see Theorem 4.9. Atomic systems for bounded operators that commute with unitary operators produce new atomic systems for the initial operator in the Krein space, see Theorem 4.10.

Conflicts of interest
The authors declare no conflict of interest.

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