Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1350 – 1361
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doi:10.55730/1300-0098.3433

Global existence, asymptotic behavior and blow up of solutions for a Kirchhoff-type equation with nonlinear boundary delay and source terms

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Received: 02.08.2021 • **Accepted/Published Online:** 04.04.2023 • **Final Version:** 18.07.2023

Abstract: The main goal of this work is to study an initial boundary value problem for a Kirchhoff-type equation with nonlinear boundary delay and source terms. This paper is devoted to prove the global existence, decay, and the blow up of solutions. To the best of our knowledge, there are not results on the Kirchhoff type-equation with nonlinear boundary delay and source terms.

Key words: Kirchhoff-type equation, nonlinear boundary conditions, delay term, global existence, decay, blow up

1. Introduction

In this paper, we study the following initial boundary value problem for a Kirchhoff-type equation with nonlinear boundary delay and source terms

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + u_t = 0, & x\in\Omega,\ t>0,\\ u(x,t)=0, & x\in\Gamma_0,\ t>0,\\ M\left(\|\nabla u\|_2^2\right)\dfrac{\partial u}{\partial\nu} + \mu_1|u_t|^{m-2}u_t + \mu_2|u_t(t-\tau)|^{m-2}u_t(t-\tau) = |u|^{p-2}u, & x\in\Gamma_1,\ t>0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega,\\ u(x,t-\tau)=f_0(x,t-\tau), & x\in\Gamma_1,\ t>0, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n\,(n\geq 1)$, $\partial\Omega=\Gamma_0\cup\Gamma_1$, $\text{mes}(\Gamma_0)>0$, $\Gamma_0\cap\Gamma_1=\emptyset$, $\frac{\partial u}{\partial\nu}$ denotes the unit outer normal derivative, $M(s)$ is a positive C^1-function satisfying $M(s)=a+bs^\gamma$, $\gamma>0$, $a>0$, $b\geq 0$, $s\geq 0$, $p, m>2$, μ_1 are positive constants, μ_2 is a real number, $\tau>0$ represents the time delay, and u_0, u_1, f_0 are given functions belonging to suitable spaces.

The Kirchhoff-type equation was introduced by Kirchhoff [14] in order to study nonlinear vibrations of an elastic string. Kirchhoff was the first one to study the oscillations of stretched strings and plates. The existence, decay, and blow-up of solutions in this case have been discussed by many authors. For example, the following Kirchhoff-type equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + g(u_t) = f(u). \tag{1.2}$$

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2010 *AMS Mathematics Subject Classification:* 35L20, 35B40, 35B44

Eq. (1.2) with $M \equiv 1$ is reduced to a nonlinear wave equation, which has been extensively studied, see for instance [8, 10, 15, 16] and the references therein.

When $M \neq 1$, Matsuyama and Ikehata [17] studied (1.2) for $g(u_t) = \delta |u_t|^p u_t$ and $f(u) = \xi |u|^p u$. They proved existence of the global solutions by using Faedo-Galerkin's method and the decay of energy based on the method of Nakao [19]. Ono [23] studied (1.2) with $M(s) = bs$, $g(u_t) = -\Delta u$, and $f(u) = \xi |u|^p u$. They showed that the solutions blow up in finite time with negative initial energy. Later, Wu and Tsai [27] studied (1.2) with different damping terms $\left(u_t,\ \Delta u_t, \text{ and } |u_t|^{m-2}u_t\right)$, they obtained unique local solution and finite time blow-up of solutions, we also refer to other studies [3, 24, 30] and the references therein.

In recent years, there are so many results concerning the wave equation with nonlinear source and boundary damping terms. Vitillaro [26] considered the initial boundary value problem for the following:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ u_\nu = -|u_t|^{m-2}u_t + |u|^{p-2}u, & \text{on } \Gamma_1 \times (0, \infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.3}$$

He proved local existence of the solutions, global existence when $p \leq m$ or the initial data was chosen suitably. Zhang and Hu [31] proved the asymptotic behavior of the solution for problem (1.3) when the initial data are inside a stable set, and the nonexistence of the solution when $p > m$ and the initial data is inside an unstable set. For the wave equation with nonlinear source and boundary damping terms, we also refer to other studies [1, 6, 7] and the references therein.

The time delay occurs in many physical, chemical, biological, thermal, and economical phenomena because this phenomena depend not only on the present state but on the past history of system in a more complicated way. Nicaise and Pignotti [20] studied the following wave equation with a linear boundary term:

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = -\mu_1 u_t(x,t) - \mu_2 u_t(x, t-\tau), & \text{on } \Gamma_1 \times (0, \infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u(x, t-\tau) = f_0(x, t-\tau), & \text{in } \Gamma_1 \times (0, \infty), \end{cases} \tag{1.4}$$

and proved that the energy is exponentially stable, under the condition $\mu_2 < \mu_1$. Then, they extended the result to the time-dependent delay case in the work of Nicaise and Pignotti [21, 22]. Kafini and Messaoudi [12] studied the following nonlinear damping wave equation with delay

$$u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) + \mu_1 u_t + \mu_1 u_t(t-\tau) = b|u|^{p-2}u. \tag{1.5}$$

The authors established the blow-up result in a nonlinear wave equation with delay and negative initial energy and $p \geq m$. For the related equations with time delay, we also refer to other studies [4, 5, 11, 13, 25, 28, 29] and the references therein.

Motivated by previous studies, the main contributions of this paper are as follows: There are not results on the Kirchhoff type-equation with nonlinear boundary delay term. In this paper, we will address the global existence, general decay, and blow-up result for the problem (1.1).

The outline of this paper is as follows: In Section 2, we give some preliminary results. In Section 3, we obtain global existence of the solution of (1.1). Sections 4 and 5 are dedicated to the general decay and blow-up of solutions, respectively.

2. Preliminaries

In this section we give some notation for function spaces and some preliminary lemmas. We denote by $\|u\|_p$ and $\|u\|_{p,\Gamma_1}$ the usual $L^p(\Omega)$ norm and $L^p(\Gamma_1)$ norm, respectively. For Sobolev space $H_0^1(\Omega)$ norm, we use the notation

$$\|u\|_{H_0^1} = \|\nabla u\|_2.$$

To state and prove our results, we need the following assumptions:

(A_1) $p \geq 2\gamma + 2$, if $n = 1, 2$, $2\gamma + 2 \leq p \leq \dfrac{n+2}{n-2}$, if $n \geq 3$.

(A_2) $|\mu_2| < \mu_1$.

Let

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) | u_{|\Gamma_0} = 0 \right\}.$$

According to (A_1), we recall the trace Sobolev embedding inequality $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Omega)$. Let c_p and c_* be the Poincaré's type constants defined as the smallest positive constants such that

$$\|u\|_p \leq c_p \|\nabla u\|_2, \ \forall\, u \in H^1(\Omega), \tag{2.1}$$

and

$$\|u\|_{q,\Gamma_1} \leq c_* \|\nabla u\|_2, \ \forall\, u \in H^1_{\Gamma_0}(\Omega). \tag{2.2}$$

To deal with the time delay term, motivated by Nicaise and Pignotti [20], we introduce a new variable

$$z(x,\rho,t) = u_t(x, t - \tau\rho), \ \ x \in \Gamma_1, \ \ \rho \in (0,1),\, t > 0, \tag{2.3}$$

which gives us

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ \ in\ \Gamma_1 \times (0,1) \times (0,\infty). \tag{2.4}$$

Then, problem (1.1) is equivalent to

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + u_t = 0, & x \in \Omega,\, t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\, t > 0, \\ M\left(\|\nabla u\|_2^2\right)\dfrac{\partial u}{\partial \nu} + \mu_1 |u_t|^{m-2} u_t + \mu_2 |z(1,t)|^{m-2} z(1,t) = |u|^{p-2} u, & x \in \Gamma_1,\, t > 0, \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0, & x \in \Gamma_1,\, \rho \in (0,1),\, t > 0, \\ z(\rho,0) = f_0(-\tau\rho), & x \in \Gamma_1,\, \rho \in (0,1), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{2.5}$$

Let ξ be a positive constant satisfying

$$\tau(m-1)|\mu_2| \leq \xi \leq \tau(m\mu_1 - |\mu_2|). \tag{2.6}$$

We first state a local existence theorem that can be established by Faedo-Galerkin Method, see for instance [2, 9].

Theorem 2.1 *(Local existence). Assume that* $(A_1)-(A_2)$ *hold. Then, for any* $(u_0,u_1,f_0)\in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\cap L^m(\Gamma_1)\times L^2(\Gamma_1\times(0,1))$ *be given. Then, there exists a unique local solution* u *of problem* (1.1) *such that*

$$u\in L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right),\ \ u_t\in L^\infty\left([0,T];L^2(\Omega)\right)\cap L^m([0,T]\times\Gamma_1),$$

for some $T>0$.

Now, we define the energy associated with problem (1.1) by

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{2\gamma+2}\|\nabla u\|_2^{2\gamma+2}+\frac{\xi}{m}\int_0^1\|z(\rho,t)\|^m_{m,\Gamma_1}d\rho-\frac{1}{p}\|u\|^p_{p,\Gamma_1}. \tag{2.7}$$

Lemma 2.2 *Let* u *be a solution of problem* (1.1). *Then,*

$$E'(t)\le-\|u_t\|_2^2-m_0\left(\|u_t\|^m_{m,\Gamma_1}+\|z(1,t)\|^m_{m,\Gamma_1}\right)\le 0. \tag{2.8}$$

Proof Multiplying the first equation in (2.5) by u_t and integrating over Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2+\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{2\gamma+2}\|\nabla u\|_2^{2\gamma+2}-\frac{1}{p}\|u\|^p_{p,\Gamma_1}\right]\\&=-\|u_t\|_2^2-\mu_1\|u_t\|^m_{m,\Gamma_1}-\mu_2\int_{\Gamma_1}|z(1,t)|^{m-2}z(1,t)u_tdx.\end{aligned} \tag{2.9}$$

Multiplying the second equation in (2.5) by ξz^{m-1} and integrating over $\Gamma_1\times(0,1)$, we obtain

$$\begin{aligned}\frac{\xi}{m}\frac{d}{dt}\int_{\Gamma_1}\int_0^1|z(\rho,t)|^m d\rho dx&=-\frac{\xi}{m\tau}\int_{\Gamma_1}\int_0^1\frac{\partial}{\partial\rho}|z(\rho,t)|^m d\rho dx\\&=\frac{\xi}{m\tau}\left(\|u_t\|^m_{m,\Gamma_1}-\|z(1,t)\|^m_{m,\Gamma_1}\right).\end{aligned} \tag{2.10}$$

Using Young's inequality, we have

$$-\mu_2\int_{\Gamma_1}|z(1,t)|^{m-2}z(1,t)u_tdx\le\frac{(m-1)|\mu_2|}{m}\|z(1,t)\|^m_{m,\Gamma_1}+\frac{|\mu_2|}{m}\|u_t\|^m_{m,\Gamma_1}. \tag{2.11}$$

Combining (2.9),(2.10), and (2.11), we obtain

$$E'(t)\le-\|u_t\|_2^2-m_0\left(\|u_t\|^m_{m,\Gamma_1}+\|z(1,t)\|^m_{m,\Gamma_1}\right), \tag{2.12}$$

where $m_0=\min\left\{\mu_1-\frac{\xi}{m\tau}-\frac{|\mu_2|}{m},\frac{\xi}{m\tau}-\frac{(m-1)|\mu_2|}{m}\right\}$, which is positive by (2.6) □

Similar as in [18], we can prove the following lemma.

Lemma 2.3 *There exists a positive constant* $C_*>1$ *depending on* Γ_1 *only such that*

$$\|u\|^s_{p,\Gamma_1}\le C_*\left(\|\nabla u\|_2^2+\|u\|^p_{p,\Gamma_1}\right),$$

for any $u\in H^1_{\Gamma_1}(\Omega)$, $2\le s\le p$.

3. Global existence

In this section, we will prove that the solutions established in Theorem 2.1 are global in time. For this purpose, we define the functionals

$$I(t) = I(u(t)) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^{2\gamma+2} - \|u\|_{p,\Gamma_1}^p, \tag{3.1}$$

and

$$J(t) = J(u(t)) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2\gamma+2}\|\nabla u\|_2^{2\gamma+2} + \frac{\xi}{m}\int_0^1 \|z(\rho,t)\|_{m,\Gamma_1}^m d\rho - \frac{1}{p}\|u\|_{p,\Gamma_1}^p. \tag{3.2}$$

Then, it is obvious that

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t). \tag{3.3}$$

In order to show our result, we first establish the following lemma.

Lemma 3.1 *Assume that* $(A_1)-(A_2)$ *hold, and for any* $(u_0,u_1,f_0) \in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\cap L^m(\Gamma_1)\times L^2(\Gamma_1\times(0,1))$, *such that*

$$I(0) > 0 \quad \text{and} \quad \alpha = \frac{c_*^p}{a}\left[\frac{2p}{a(p-2)}E(0)\right]^{\frac{p-2}{2}} < 1, \tag{3.4}$$

then,

$$I(t) > 0, \ \forall t > 0. \tag{3.5}$$

Proof Since $I(0) > 0$, then by continuity of u, there exist a time $T_* < T$ such that

$$I(t) \geq 0, \ \forall t \in [0, T_*]. \tag{3.6}$$

Using (3.1), (3.2), (3.3), and (2.8), we see that

$$\begin{aligned} J(t) &= \frac{1}{p}I(t) + \frac{a(p-2)}{2p}\|\nabla u\|_2^2 + \frac{b(p-2\gamma-2)}{p(2\gamma+2)}\|\nabla u\|_2^{2\gamma+2} + \frac{\xi}{m}\int_0^1 \|z(\rho,t)\|_{m,\Gamma_1}^m d\rho \\ &\geq \frac{a(p-2)}{2p}\|\nabla u\|_2^2 + \frac{b(p-2\gamma-2)}{p(2\gamma+2)}\|\nabla u\|_2^{2\gamma+2}, \end{aligned} \tag{3.7}$$

and

$$\|\nabla u\|_2^2 \leq \frac{2p}{a(p-2)}J(t) \leq \frac{2p}{a(p-2)}E(t) \leq \frac{2p}{a(p-2)}E(0). \tag{3.8}$$

Exploiting (2.2), (3.4), and (3.7), we get □

$$\|u\|_{p,\Gamma_1}^p \leq c_*^p\|\nabla u\|_2^p \leq \frac{c_*^p}{a}\left[\frac{2p}{a(p-2)}E(0)\right]^{\frac{p-2}{2}} a\|\nabla u\|_2^2 = \alpha a\|\nabla u\|_2^2 < a\|\nabla u\|_2^2, \ \forall t \in [0,T_*]. \tag{3.9}$$

Therefore, we have

$$I(t) > 0, \ \forall t \in [0, T_*]. \tag{3.10}$$

By repeating the procedure, T_* is extended to T. The proof is completed.

Theorem 3.2 *Assume that the conditions of Lemma 3.1 hold, then the solution of problem (1.1) is global and bounded.*

Proof It suffices to show that

$$\|u_t\|_2^2 + \|\nabla u\|_2^2,$$

is bounded independently of t. By using (2.8), (3.3), and (3.7), we have

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{a(p-2)}{2p}\|\nabla u\|_2^2, \tag{3.11}$$

which means,

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(0), \tag{3.12}$$

where C is a positive constant. □

4. General decay

In this section, we state and prove the decay result of solution to problem (1.1). For this goal, we set

$$F(t) := E(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon}{2}\|u\|_2^2, \tag{4.1}$$

where ε is a positive constant to be specified later.

Lemma 4.1 *Let u be a solution of problem (1.1). Then, there exist two positive constants α_1 and α_2 depending on ε such that*

$$\alpha_1 E(t) \leq F(t) \leq \alpha_2 E(t). \tag{4.2}$$

Theorem 4.2 *Let $(u_0, u_1, f_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \cap L^m(\Gamma_1) \times L^2(\Gamma_1 \times (0,1))$. Assume that $(A_1)-(A_2)$ hold. Then, there exist two positive constant K and k such that*

$$E(t) \leq Ke^{-kt}, \quad t \geq 0.$$

Proof Taking a derivative of (4.1) with respect to t, using (2.5) and (2.8), we obtain

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon \int_\Omega u u_{tt} dx + \varepsilon \int_\Omega u u_t dx \\ &\leq -m_0\|u_t\|_{m,\Gamma_1}^m - m_0\|z(1,t)\|_{m,\Gamma_1}^m - (1-\varepsilon)\|u_t\|_2^2 - a\varepsilon\|\nabla u\|_2^2 - b\varepsilon\|\nabla u\|_2^{2\gamma+2} \\ &\quad + \varepsilon\|u\|_{p,\Gamma_1}^p - \varepsilon\mu_1 \int_{\Gamma_1} |u_t|^{m-2}u_t u d\Gamma - \varepsilon\mu_2 \int_{\Gamma_1} |z(1,t)|^{m-2}z(1,t) u d\Gamma. \end{aligned} \tag{4.3}$$

By using Young's inequality for $\eta > 0$, we get

$$\begin{aligned} \mu_1 \int_{\Gamma_1} |u_t|^{m-2}u_t u d\Gamma &\leq \mu_1^m \eta \|u\|_{m,\Gamma_1}^m + c(\eta)\|u_t\|_{m,\Gamma_1}^m \leq \mu_1^m \eta c_*^m \|\nabla u\|_2^m + c(\eta)\|u_t\|_{m,\Gamma_1}^m \\ &\leq \eta c_1 \|\nabla u\|_2^2 + c(\eta)\|u_t\|_{m,\Gamma_1}^m, \end{aligned} \tag{4.4}$$

and

$$\mu_2 \int_\Omega |z(1,t)|^{m-2}z(1,t) u dx \leq \eta c_2 \|\nabla u\|_2^2 + c(\eta)\|z(1,t)\|_{m,\Gamma_1}^m, \tag{4.5}$$

where c_1 and c_2 are positive constants which depend only on m and $E(0)$. Combining (4.4)-(4.5) with (4.3), we obtain

$$\begin{aligned} F'(t) &\leq -(m_0 - \varepsilon c(\eta))\|u_t\|_{m,\Gamma_1}^m - (m_0 - \varepsilon c(\eta))\|z(1,t)\|_{m,\Gamma_1}^m - (1-\varepsilon)\|u_t\|_2^2 \\ &\quad -\varepsilon(a - \eta(c_1 + c_2))\|\nabla u\|_2^2 - \varepsilon b\|\nabla u\|_2^{2\gamma+2} + \varepsilon\|u\|_{p,\Gamma_1}^p. \end{aligned} \tag{4.6}$$

First, we choose η so small satisfying

$$a - \eta(c_1 + c_2) > 0.$$

For any fixed η, we choose ε so small that (4.2) remains valid and

$$m_0 - \varepsilon c(\eta) > 0, \;\; 1 - \varepsilon > 0.$$

Consequently, inequality (4.6) becomes

$$F'(t) \leq -c_3 E(t), \;\; \forall t > 0. \tag{4.7}$$

Using (4.2), we obtain

$$F'(t) \leq -c_3 E(t) \leq \frac{c_3}{\alpha_2} F(t), \;\; \forall t > 0. \tag{4.8}$$

A simple integration of (4.8), leads to

$$F(t) \leq c_4 e^{-kt}, \;\; \forall t > 0. \tag{4.9}$$

Again (4.2), gives

$$E(t) \leq K e^{-kt}, \;\; \forall t > 0. \tag{4.10}$$

□

5. Blow-up

In this section, we state and prove the finite time blow-up of solutions to problem (1.1) with $E(0) < 0$.

Theorem 5.1 *Let $(A_1) - (A_2)$ and $E(0) < 0$ holds. Then, the solution of problem (1.1) blows up in finite time T^* and*

$$T^* \leq \frac{1-\sigma}{\omega\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

Proof Set

$$H(t) = -E(t), \tag{5.1}$$

then (2.8) gives

$$H'(t) = -E'(t) \geq m_0 \left(\|u_t\|_{m,\Gamma_1}^m + \|z(1,t)\|_{m,\Gamma_1}^m\right) \geq 0, \tag{5.2}$$

and $H(t)$ is an increasing function. From (2.7) and (5.1), we see that

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_{p,\Gamma_1}^p. \tag{5.3}$$

Next, we define

$$\Psi(t) = H(t)^{1-\sigma} + \varepsilon \int_\Omega u_t u dx + \frac{\varepsilon}{2}\|u\|_2^2, \tag{5.4}$$

where ε is a positive constants to be specified later and

$$0 < \sigma \leq \frac{p-m}{p(m-1)}. \tag{5.5}$$

Taking a derivative of $\Psi(t)$ and using (2.5), we have

$$\begin{aligned}\Psi'(t) &= (1-\sigma)H'(t)H(t)^{-\sigma} + \varepsilon\|u_t\|_2^2 + \varepsilon\int_\Omega uu_{tt}dx + \varepsilon\int_\Omega uu_t dx \\ &= (1-\sigma)H'(t)H(t)^{-\sigma} + \varepsilon\|u_t\|_2^2 - \varepsilon a\|\nabla u\|_2^2 - \varepsilon b\|\nabla u\|_2^{2\gamma+2} + \varepsilon\|u\|_{p,\Gamma_1}^p \\ &\quad -\varepsilon\mu_1\int_{\Gamma_1}|u_t|^{m-2}u_t u d\Gamma - \varepsilon\mu_2\int_{\Gamma_1}|z(1,t)|^{m-2}z(1,t)ud\Gamma.\end{aligned} \tag{5.6}$$

Applying Young's inequality for $\eta > 0$, we have

$$\begin{aligned}&\mu_1\int_{\Gamma_1}|u_t|^{m-2}u_t u d\Gamma \leq \frac{\mu_1^m\eta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{m}\eta^{-\frac{m}{m-1}}\|u_t\|_{m,\Gamma_1}^m \\ &\leq \frac{\mu_1^m\eta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}H'(t).\end{aligned} \tag{5.7}$$

Similarly,

$$\mu_2\int_\Omega|z(1,t)|^{m-2}z(1,t)ud\Gamma \leq \frac{|\mu_2|^m\eta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}H'(t). \tag{5.8}$$

A substitution of (5.7)-(5.8) into (5.6), we have

$$\begin{aligned}\Psi'(t) &\geq \left\{(1-\sigma)H(t)^{-\sigma} - \varepsilon\frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}\right\}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon a\|\nabla u\|_2^2 - \varepsilon b\|\nabla u\|_2^{2\gamma+2} \\ &\quad +\varepsilon\|u\|_{p,\Gamma_1}^p - \frac{(\mu_1^m+|\mu_2|^m)\eta^m}{m}\|u\|_{m,\Gamma_1}^m.\end{aligned} \tag{5.9}$$

Using (2.7) and (5.1), for a constant $\mu > 0$, we see that

$$\begin{aligned}\Psi'(t) &\geq \left\{(1-\sigma)H(t)^{-\sigma} - \varepsilon\frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}\right\}H'(t) + \varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2 + \varepsilon a\left(\frac{\mu}{2}-1\right)\|\nabla u\|_2^2 \\ &\quad +\varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2} + \varepsilon\left(1-\frac{\mu}{p}\right)\|u\|_{p,\Gamma_1}^p - \frac{(\mu_1^m+|\mu_2|^m)\eta^m}{m}\|u\|_{m,\Gamma_1}^m \\ &\quad +\frac{\mu\xi}{m}\int_0^1\|z(\rho,t)\|_{m,\Gamma_1}^m d\rho + \mu\varepsilon H(t).\end{aligned} \tag{5.10}$$

Therefore, by taking $\eta = (kH(t)^{-\sigma})^{-\frac{m-1}{m}}$ where $k > 0$ to be specified later, we see that

$$\begin{aligned}\Psi'(t) &\geq \left\{(1-\sigma) - \varepsilon k\frac{(m-1)}{mm_0}\right\}H(t)^{-\sigma}H'(t) + \varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2 + \varepsilon a\left(\frac{\mu}{2}-1\right)\|\nabla u\|_2^2 \\ &\quad +\varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2} + \varepsilon\left(1-\frac{\mu}{p}\right)\|u\|_{p,\Gamma_1}^p + \frac{\mu\xi}{m}\int_0^1\|z(\rho,t)\|_{m,\Gamma_1}^m d\rho \\ &\quad -\frac{(\mu_1^m+|\mu_2|^m)}{m}k^{1-m}H(t)^{\sigma(m-1)}\|u\|_{m,\Gamma_1}^m + \mu\varepsilon H(t).\end{aligned} \tag{5.11}$$

Exploiting (5.3), we have

$$H(t)^{\sigma(m-1)}\|u\|_{m,\Gamma_1}^m \le C_p^m H(t)^{\sigma(m-1)}\|u\|_{p,\Gamma_1}^m \le \frac{C_p^m}{p^\sigma}\|u\|_{p,\Gamma_1}^{\sigma p(m-1)+m}. \tag{5.12}$$

Combining (5.11) and (5.12), we get

$$\begin{aligned}\Psi'(t) \quad &\ge \left\{(1-\sigma)-\varepsilon k\frac{(m-1)}{mm_0}\right\}H(t)^{-\sigma}H'(t)+\varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2+\varepsilon a\left(\frac{\mu}{2}-1\right)\|\nabla u\|_2^2\\ &+\varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2}+\varepsilon\left(1-\frac{\mu}{p}\right)\|u\|_{p,\Gamma_1}^p+\frac{\mu\xi}{m}\int_0^1\|z(\rho,t)\|_{m,\Gamma_1}^m d\rho\\ &-\varepsilon\frac{(\mu_1^m+|\mu_2|^m)}{m}\frac{C_p^m k^{1-m}}{p^\sigma}\|u\|_{p,\Gamma_1}^{\sigma p(m-1)+m}+\mu\varepsilon H(t).\end{aligned} \tag{5.13}$$

Applying Lemma 2.3 for $s=\sigma p(m-1)+m<p$, we get

$$\|u\|_{p,\Gamma_1}^{\sigma p(m-1)+m} \le C_*\left(\|\nabla u\|_2^2+\|u\|_{p,\Gamma_1}^p\right). \tag{5.14}$$

Combining (5.14) with (5.13), we obtain

$$\begin{aligned}\Psi'(t) \quad &\ge \left\{(1-\sigma)-\varepsilon\frac{(m-1)k}{mm_0}\right\}H(t)^{-\sigma}H'(t)+\varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2\\ &+\varepsilon\left(a\left(\frac{\mu}{2}-1\right)-c_\sigma k^{1-m}\right)\|\nabla u\|_2^2+\varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2}\\ &+\varepsilon\left(\left(1-\frac{\mu}{p}\right)-c_\sigma k^{1-m}\right)\|u\|_{p,\Gamma_1}^p+\frac{\mu\xi}{m}\int_0^1\|z(\rho,t)\|_{m,\Gamma_1}^m d\rho+\mu\varepsilon H(t),\end{aligned} \tag{5.15}$$

where $c_\sigma=\dfrac{C_*(\mu_1^m+|\mu_2|^m)}{m}\dfrac{C_p^m}{p^\sigma}$.

At this point, we choose $2\gamma+2<\mu<p$ such that

$$\frac{\mu}{2}-1>0,\quad \frac{\mu}{2\gamma+2}-1>0,\quad 1-\frac{\mu}{p}>0.$$

When μ is fixed, we choose k large enough such that

$$a\left(\frac{\mu}{2}-1\right)-c_\sigma k^{1-m}>0,\quad \left(1-\frac{\mu}{p}\right)-c_\sigma k^{1-m}>0.$$

Once k and μ are fixed, we select $\varepsilon>0$ small enough so that

$$(1-\sigma)-\varepsilon k\frac{(m-1)}{mm_0}>0,\quad \Psi(0)=H(0)^{1-\sigma}+\varepsilon\int_\Omega u_1u_0dx+\frac{\varepsilon}{2}\|u_0\|_2^2>0.$$

Then inequality (5.15) becomes

$$\Psi'(t)\ge K\left(\|u_t\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2\gamma+2}+\|u\|_{p,\Gamma_1}^p+H(t)\right), \tag{5.16}$$

where K is a positive constant.

On the other hand, we will estimate $\Psi^{\frac{1}{1-\sigma}}(t)$. Applying Hölder and Youngs inequalities, we have

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \le C\|u\|_p^{\frac{1}{1-\sigma}}\|u_t\|_2^{\frac{1}{1-\sigma}} \le C\left(\|u\|_p^{\frac{\mu}{1-\sigma}} + \|u_t\|_2^{\frac{\theta}{1-\sigma}}\right), \tag{5.17}$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. Take $\theta = 2(1-\sigma)$ which gives $\frac{\mu}{1-\sigma} = \frac{2}{1-2\sigma}$. Then, (5.17) becomes

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \le C\left(\|u\|_p^{\frac{2}{1-2\sigma}} + \|u_t\|_2^2\right), \tag{5.18}$$

It follows from (3.12) and (5.3), we have

$$\|u\|_p^{\frac{2}{1-2\sigma}} \le c_p^{\frac{2}{1-2\sigma}}\|\nabla u\|_2^{\frac{2}{1-2\sigma}} \le c_p^{\frac{2}{1-2\sigma}}(CE(0))^{\frac{1}{1-2\sigma}} \le c_p^{\frac{2}{1-2\sigma}}(CE(0))^{\frac{2}{1-2\sigma}}\frac{H(t)}{H(0)}. \tag{5.19}$$

Similar to (5.19), we have

$$\|u\|_2^{\frac{2}{1-\sigma}} \le c_2^{\frac{2}{1-\sigma}}(CE(0))^{\frac{1}{1-\sigma}} \le c_2^{\frac{2}{1-\sigma}}(CE(0))^{\frac{1}{1-\sigma}}\frac{H(t)}{H(0)} \le c_2^{\frac{2}{1-\sigma}}(CE(0))^{\frac{1}{1-\sigma}}\frac{\|u\|_{p,\Gamma_1}^p}{pH(0)}. \tag{5.20}$$

Combining (5.19)-(5.20) and (5.4), we get

$$\Psi^{\frac{1}{1-\sigma}}(t) \le \widetilde{K}\left(\|u_t\|_2^2 + \|u\|_{p,\Gamma_1}^p + H(t)\right), \tag{5.21}$$

where $\widetilde{K}$ is a positive constant.
It follows from (5.16) and (5.21), we find that

$$\Psi'(t) \ge \omega\Psi^{\frac{1}{1-\sigma}}(t), \quad \forall t > 0, \tag{5.22}$$

where κ is a positive constant.
A simple integration of (5.22) over $(0,t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\omega\sigma t}{1-\sigma}}.$$

Consequently, the solution of problem (1.1) blows up in finite time T^*. □

Acknowledgment

This work was supported by the Directorate-General for Scientific Research and Technological Development, Algeria (DGRSDT).

References

[1] Aassila M, Cavalcanti MM, Domingos Cavalcanti VN. Existence and uniform decay of the wave equation with nonlinear boundary damping and boundary memory source term. Calculus of Variations and Partial Differential Equations 2002; 15: 155-183.

[2] Boumaza N, Gheraibia B. On the existence of a local solution for an integro-differential equation with an integral boundary condition. Boletín de la Sociedad Matemática Mexicana 2020; 26: 521-534.

[3] Boumaza N, Gheraibia B. General decay and blowup of solutions for a degenerate viscoelastic equation of Kirchhoff type with source term. Journal of Mathematical Analysis and Applications 2020; 489 (2): 124185.

[4] Boumaza N, Saker M, Gheraibia B. Asymptotic behavior for a viscoelastic Kirchhoff-Type equation with delay and source terms. Acta Applicandae Mathematicae 2021; 171 (1): 18.

[5] Boumaza N, Gheraibia B. Global existence, nonexistence, and decay of solutions for a wave equation of p-Laplacian type with weak and p-Laplacian damping, nonlinear boundary delay and source terms. Asymptotic Analysis. 2021; 1-16.

[6] Cavalcanti MM, Domingos Cavalcanti VN, Martinez P. Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term. Journal of Differential Equations 2004; 203 (1): 119-158.

[7] Cavalcanti MM, Domingos Cavalcanti VN, Lasiecka I. Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction. Journal of Differential Equations 2007; 236 (2): 407-459.

[8] Chen H, Liu G. Global existence, uniform decay and exponential growth for a class of semilinear wave equation with strong damping. Acta Mathematica Scientia 2013; 33 (1): 41-58.

[9] Ferhat M, Hakem Ali. Global existence and energy decay result for a weak viscoelastic wave equations with a dynamic boundary and nonlinear delay term. Computers and Mathematics with Applications 2016; 71 (3): 779-804.

[10] Georgiev V, Todorova D. Existence of solutions of the wave equations with nonlinear damping and source terms. Journal of Differential Equations 1994; 109 (2): 295-308.

[11] Gheraibia B, Boumaza N. General decay result of solutions for viscoelastic wave equation with Balakrishnan-Taylor damping and a delay term. Zeitschrift für angewandte Mathematik und Physik 2020; 71 (6): 198.

[12] Kafini M, Messaoudi SA. A blow-up result in a nonlinear wave equation with delay. Mediterranean Journal of Mathematics 2016; 13(1): 237-247.

[13] Kamache H, Boumaza N, Gheraibia B. General decay and blow up of solutions for the Kirchhoff plate equation with dynamic boundary conditions, delay and source terms. Zeitschrift für angewandte Mathematik und Physik 2022; 73 (2): 76.

[14] Kirchhoff G. Vorlesungen über Mechanik. Teubner, Leipzig, 1883.

[15] Levine HA. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. SIAM Journal on Mathematical Analysis 1974; 5 (1): 138-146.

[16] Levine HA, Park SR, Serrin J. Global existence and global nonexistence of solutions of the Cauchy problems for a nonlinearly damped wave equation. Journal of Mathematical Analysis and Applications 1998; 228 (1): 181-205.

[17] Matsuyama T, Ikehata R, On global solutions and energy decay for the wave equation of Kirchhoff type with nonlinear damping terms. Journal of Mathematical Analysis and Applications 1996; 204 (3): 729-753.

[18] Messaoudi SA. Blow up and global existence in a nonlinear viscoelastic wave equation. Mathematische Nachrichten 2003; 260 (1) 58-66.

[19] Nakao M. Existence of global smooth solutions to the initial-boundary value problem for the quasi-linear wave equation with a degenerate dissipative term. Journal of Differential Equations 1992; 98 (2): 299-327.

[20] Nicaise S, Pignotti C. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM Journal on Control and Optimization 2006; 45 (5): 1561-1585.

[21] Nicaise S, Valein J, Fridman E. Stabilization of the heat and the wave equations with boundary time-varying delays. Discrete and Continuous Dynamical Systems-s 2009; 2 (3): 559-581.

[22] Nicaise S, Pignotti C, Valein J. Exponential stability of the wave equation with boundary time-varying delay. Discrete and Continuous Dynamical Systems-s 2011; 4 (3): 693-722.

[23] Ono K. Blowing up and global existence of solutions for some degenerate nonlinear wave equations with some dissipation. Nonlinear Analysis: Theory, Methods and Applications 1997; 30 (7): 4449-4457.

[24] Ono K. Global existence, decay and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings. Journal of Differential Equations 1997; 137 (2): 273-301.

[25] Pişkin E, Yüksekkaya H. Blow up of solution for a viscoelastic wave equation with m-Laplacian and delay terms. Tbilisi Mathematical Journal 2021; SI (8): 21-32.

[26] Vitillaro E. Global existence for the wave equation with nonlinear boundary damping and source terms. Journal of Differential Equations 2002; 186 (1): 259-298.

[27] Wu ST, Tsai LY. Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with some dissipation. Nonlinear Analysis: Theory, Methods and Applications 2006; 65 (2) 243-264.

[28] Yüksekkaya H , Pişkin E, Boulaaras SM, Cherif BB, Zubair SA. Existence, nonexistence, and stability of solutions for a delayed plate equation with the logarithmic source. Advances in Mathematical Physics 2021; 2021: 1-11.

[29] Yüksekkaya H , Pişkin E, Boulaaras SM, Cherif BB. Existence, decay and blow-up of solutions for a higher-order Kirchhoff-type equation with delay term. Journal of Function Spaces 2021; 2021: 1-11.

[30] Zeng R, Mu CL, Zhou SM. A blow-up result for Kirchhoff-type equations with high energy. Mathematical Methods in the Applied Sciences 2011; 34 (4): 479-486.

[31] Zhang H, Hu Q. Asymptotic behavior and nonexistence of wave equation with nonlinear boundary condition. Communications on Pure and Applied Analysis 2005; 4 (4): 861-869.