Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1374 – 1391
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doi:10.55730/1300-0098.3435

Unipotence in positive characteristic for groups of finite Morley rank

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Received: 29.11.2022 • **Accepted/Published Online:** 11.04.2023 • **Final Version:** 18.07.2023

Abstract: In this article we define a new form of unipotence in groups of finite Morley rank which extends Burdges unipotence to any characteristic. In particular, we show that every connected solvable group of finite Morley rank G has a definable connected subgroup H whose derived subgroup H' is a good unipotent subgroup of finite Morley rank.

Key words: Morley rank, definable,unipotence, indecomposable group

1. Introduction

Groups of finite Morley rank range into the category of groups having a notion of dimension, namely the Morley rank. Morley rank is a generalization of the Zariski dimension and coincides with it over the constructible sets of an algebraically closed field: a linear algebraic group over an algebraically closed field is an example of group of finite Morley rank. Besides, the central question in the study of groups of finite Morley rank lies in the Cherlin-Zilber conjecture which states that simple infinite ones are algebraic. Since the formulation of this conjecture, the search for algebraic analogues in the context of groups of finite Morley rank has occupied a major place among model theorists. This article is a part of it and much of the work is picked from the author's PhD thesis [15].

In groups of finite Morley rank, while a Carter subgroup, a decent torus or even a good torus are some good analogues to the algebraic torus, it is difficult to define an intrinsic notion of the unipotent radical or even of a unipotent group. Indeed, the existing versions of unipotence in groups of finite Morley rank are different among themselves and according to the characteristic. Mainly, it should be noted that in positive characteristic, unipotence is characterized by connected solvable p-groups of bounded exponent: this is p-unipotence. In characteristic zero, it is characterized by $U_{0,r}$-groups [5, 7]: it is 0-unipotence of Burdges.

Unlike the algebraic case where there is only one form of unipotence, p-unipotence and 0-unipotence are indeed disjoint notions. In this paper, we propose a new form of unipotence which takes into account any characteristic; this new one is characterized by U_τ-groups.

Here U_τ-groups are defined following Burdges ideas. In his definition of $U_{0,r}$-groups, Burdges makes use of abelian indecomposable groups A whose quotient $A/J(A)$ by the radical $J(A)$ is torsion-free. In order to preserve nonzero characteristic, torsion-free criterion will be removed. Moreover, Burdges unipotence has a big handicap: it is not preserved by passage to definable subgroups. To remedy this problem, Frécon imposes in [8]

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2010 *AMS Mathematics Subject Classification:* 20F11, 03C45

the condition of homogeneity on $U_{0,r}$-groups and defines a new unipotence characterized by U-groups. We emphasize that this paper is deeply inspired by Frécon's work but we make the choice of not using U-group appellation. Using homogeneity on U_τ-groups, we will rather define two new notions of unipotence, namely unipotence of finite Morley rank and good unipotence of finite Morley rank (Definition 5.7). The following configuration will explain our definitions:

Let $\mathbb{K}_1, ..., \mathbb{K}_r$ be algebraically closed fields of any characteristic. Then $G = GL_{n_1}(\mathbb{K}_1)\times\cdots\times GL_{n_r}(\mathbb{K}_r)$ is a group of finite Morley rank and any group definably embedding in G is said to be definably linear. Intuitively, we would like to say that if U is a unipotent subgroup of G then U decomposes into $U = U_1 \times \cdots \times U_r$ where U_i is a unipotent subgroup of $GL_{n_i}(\mathbb{K}_i)$. We know that not all groups of finite Morley rank are algebraic or even definably linear. Thus, the idea is to define a unipotence form for groups of finite Morley rank which in case of linearity takes into account our intuitive idea of a definably linear unipotent subgroup.

Here are our main results:

Theorem 5.12. Let G be a connected nilpotent group of finite Morley rank. Then $G/Z(G)$ is a unipotent group of finite Morley rank.

Theorem 5.23. Let G be a connected solvable group of finite Morley rank. There exists a pair $(H, \mathbf{T})$ of connected definable subgroups such that

i) $G = H * \mathbf{T}$;

ii) $\mathbf{T}$ is a *generalised* pseudotorus;

iii) H' is a good unipotent subgroup of finite Morley rank.

It is important to note that Burdges unipotence has encountered a certain success since its formulation. Continuing Nesin's study in describing nilpotent groups of finite Morley rank [13], it is used for instance in the structural description of a divisible nilpotent group of finite Morley first by Frécon (Fact 5.2), then in [12] and [1] to prove linearity of nilpotent torsion-free groups of finite Morley rank. Towards a generalization, our unipotence which extends it to positive characteristic, aims to serve for a better structural description of a bounded exponent nilpotent group of finite Morley rank. We think that Proposition 5.4 is the first step in that way.

Now we give the organisation of this paper. Section 2 is devoted to the tools and important results on groups of finite Morley rank we will use. In Section 3, we give a short analysis of indecomposable groups based on a new definition of an indecomposable group (Definition 3.1): in this definition, the group is no longer necessarily abelian as considered by Burdges. In Section 4, we introduce U_τ-groups where τ represents a class of minimal groups modulo a relation of quasiisogeneous (Lemma 4.2). In particular, we will see that depending on the nature of τ, some of U_τ-groups are in fact $U_{0,r}$-groups where $r = \mathrm{rk}(\tau)$. We will see in the last section how U_τ-groups will be useful to the structural description of a connected nilpotent group of finite Morley rank (Corollary 5.5). It is also in the last section that we will define unipotence of finite Morley rank and good unipotence of finite Morley rank (Definition 5.7) and we will present the main results of the paper.

Most of the results on U_τ-groups which generalize those on $U_{0,r}$-groups are proved in [15] following the ideas of the proofs developed in [8]. These results will therefore be presented here without proofs and referenced to their analogues in [8].

2. Prerequisites

First, we invite the reader not familiar with groups of finite Morley rank to visit [4] for a good definition of the topic (especially chapter 4). We follow the usual convention that throughout "definable " means "interpretable with parameters".

Let G be a group of finite Morley rank.

- We recall that a subset $X \subseteq G$ is indecomposable if for all definable subgroups Q of G and finite sequences $g_1, ..., g_n$ of G such that $X \subseteq g_1 Q \sqcup ... \sqcup g_n Q$, there exists i such that $X \subseteq g_i Q$.

  Do not confuse the indecomposable sets that we have just defined here and usually called Zilber's indecomposable set with the indecomposable groups that are mentioned in the introduction and that we will define later.
- For a definable subgroup H, we denote by H° the smallest definable subgroup of finite index in H. We say that H is *connected* if $H = H^\circ$. In particular, a definable subgroup of G is indecomposable if and only if it is connected.
- We say that H/K is a definable quotient if H and K are definable.
- Let S be a subset of a group G. We say that a subgroup A of G is S-minimal if A is infinite, definable, S-invariant and minimal with respect to these conditions.

The following result is one of the most important theorems in the study of groups of finite Morley rank and it uses Zilber's indecomposable sets:

Fact 2.1 [4, Theorem 5.26] *Let I be a set and let $(H_i)_{i \in I}$ be a family of indecomposable subsets of a group G of finite Morley rank. Assume that each H_i contains the identity element of G. Then the subgroup H generated by the subsets H_i is definable and connected. Furthermore, there are finitely many indices $i_1, ..., i_m$ such that $H = H_{i_1} ... H_{i_m}$ for $m < 2^{rk(G)+1} - 2$.*

The next result which is a consequence of the previous, will be used several times without being mentioned each time.

Fact 2.2 [4, Corollary 5.29] *Let $H \leqslant G$ be a definable connected subgroup. Let $X \subseteq G$ be any subset. Then the subgroup $[H, X]$ is definable and connected.*

As we will see, minimal infinite subgroups play a major role in this article. Reineke clarifies their nature in the following result.

Fact 2.3 [4, Theorem 6.4] *In a group of finite Morley rank, a minimal infinite definable subgroup A is abelian. Furthermore, either A is divisible or is an elementary abelian p-group for some prime number p.*

In the area of groups of finite Morley rank, the next theorem is one of the most used; it gives the linearization of nonnilpotent solvable groups of finite Morley rank.

Fact 2.4 (Zilber's Field Theorem,[4, Theorem 9.1]) *Let $\mathcal{G} = V \rtimes G$ be a group of finite Morley rank, where V and G are infinite definable abelian subgroups and V is G-minimal. Assume $C_G(V) = 1$. Then the following things happen:*

i. The subring $\mathbb{K}$ *of* $End(V)$ *generated by* G *is a definable algebraically closed field.*

ii. $V \cong \mathbb{K}_+$ *definably,* G *is definably isomorphic to a subgroup* T *of* $\mathbb{K}^*$ *and* G *acts on* V *by multiplication, in other words:*

$$\mathcal{G} \cong \left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} | t \in T, a \in \mathbb{K} \right\} \text{ definably.}$$

iii. In particular, G *acts freely on* V *and there exists* $l \in \mathbb{N}$ *such that*

$$\mathbb{K} = T + \cdots + T(l \text{ times}).$$

Fact 2.5 [4, Theorem 9.19] *Let* G *be a connected group of Morley rank* 2*. Then* G *is solvable. If* G *is not nilpotent then* G *has finite center;* $G/Z(G)$ *is centerless and* $G = G' \rtimes T$ *for some definable subgroup* T *of finite Morley rank containing* $Z(G)$*. If* G *is also centerless, then* $G \cong \mathbb{K} \rtimes \mathbb{K}^*$ *for some algebraically closed field* $\mathbb{K}$*.*

For any group of finite Morley rank G, we let $F(G)$ denote the subgroup generated by all normal nilpotent sugbroups of G. By [4, Theorem 7.3], $F(G)$ is definable and nilpotent.

Fact 2.6 [4, Theorem 9.21] *Let* G *be a connected and solvable group of finite Morley rank. Then* $G/F(G)^\circ$ *(so also* $G/F(G)$*) is a divisible abelian group.*

As we will see further, the following result is important for us.

Fact 2.7 [14, Corollary 3.3] *Let* $\mathbb{K}$ *be a field of finite Morley rank and characteristic zero. Then* $\mathbb{K}_+$ *has no proper definable subgroup.*

An abelian divisible group T of finite Morley rank is a pseudotorus if no definable quotient of T is definably isomorphic to $\mathbb{K}_+$ for a definable field $\mathbb{K}$.

Pseudotori are involved in the structural description of nilpotent groups of finite Morley rank, and they still will play a certain role in this paper. [9] is our best reference for the topic. Below we give some of the results related to the pseudotori that we will need.

Fact 2.8 [9, Theorem 1.7] *The maximal pseudotori in a group* G *of finite Morley rank are conjugate in* G*.*

In any group G of finite Morley rank, we let $T(G)$ denote the subgroup generated by the indecomposable pseudotori of G.

Fact 2.9 [9, Corollary 2.8] *Let* G *be a nilpotent group of finite Morley rank. Then* $T(G)$ *is the unique maximal pseudotorus of* G*.*

Fact 2.10 [9, Corollary 2.9] *In any connected group* G *of finite Morley rank,* $T(F(G))$ *is a pseudotorus central in* G*.*

Fact 2.11 [9, Corollary 2.13] *Let* G *be a group of finite Morley rank,* N *a normal definable subgroup of* G *and* T *a maximal pseudotorus of* G*. Then* TN/N *is a maximal pseudo-torus of* G/N *and every maximal pseudotorus of* G/N *has this form.*

3. Indecomposable groups

Definition 3.1 A connected group of finite Morley rank A is said to be indecomposable if it cannot be generated by its proper definable connected subgroups.

Note that in Burdges's definition, indecomposable groups are imposed to be abelian. Care must be taken not to confuse the indecomposable groups defined here with Zilber's indecomposable sets mentioned in Fact 2.1.

Example 3.2

1. The additive group of an infinite field of finite Morley rank of null characteristic is an example of indecomposable group because it is minimal (Fact 2.7).
2. Let $\mathbb{F}$ be a field of positive characteristic p; another example of an indecomposable group is the following:

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^p \\ 0 & 0 & 1 \end{pmatrix} : (a, b) \in \mathbb{F}^2 \right\}$$

   which is a nilpotent nonabelian group of exponent p^2.

3. Baudisch group which is a nonabelian nilpotent group of bounded exponent and Morley rank 2 is an indecomposable group. Indeed, it is easily shown that, in any connected nonabelian nilpotent group of Morley rank 2, the connected component of the center is the unique proper nontrivial connected subgroup.

Definition 3.3 We say that a proper subgroup A_1 of some group A has a supplement if there is a proper subgroup A_2 such that $A = \langle A_1, A_2 \rangle$.

In an indecomposable group, no definable connected subgroup has a definable supplement. Furthermore, one can see that two definable connected subgroups without a definable supplement generate a definable connected subgroup without a definable supplement.

Definition 3.4 The radical $J(A)$ of a nontrivial group A of finite Morley rank is defined to be the subgroup generated by all definable connected proper subgroups without a definable supplement. We define $J(1) = 1$.

It follows that if A is a nontrivial indecomposable group then $J(A)$ is its unique maximal proper definable connected subgroup.

Lemma 3.5 *A nontrivial connected group of finite Morley rank A is an indecomposable group if and only if $J(A)$ is maximal among definable and connected subgroups.*

Proof By definition, if A is an indecomposable group, $J(A)$ is proper and contains all the proper connected definable subgroup of A, so it is also maximal. Now suppose that $J(A)$ is proper and maximal. Then A is nontrivial and, since $J(A)$ is maximal, it contains all the proper connected definable subgroups of A, so A is indecomposable. □

Notice that if A is an indecomposable group, $A/J(A)$ is an infinite minimal group.

Lemma 3.6 (Push-forward) *Let G be a group of finite Morley rank and N be a normal definable subgroup of G. If A is an indecomposable subgroup of G, then AN/N is an indecomposable subgroup of G/N and $J(AN/N) = J(A)N/N$.*

Proof We may assume that A is not contained in N. Suppose that AN/N is not an indecomposable group and write $AN/N = \langle B_1, B_2\rangle$ where B_i $(i = 1, 2)$ is a proper definable connected subgroup. Let A_i be a definable connected subgroup of A such that $A_iN/N = B_i$. It follows that $AN/N = \langle A_1, A_2\rangle N/N$ and we get $A = \langle A_1, A_2\rangle(A \cap N)$ which is absurd since A is an indecomposable group not contained in N. We can conclude that AN/N is an indecomposable group.

Now we show that $J(AN/N) = J(A)N/N$. First, note that if $B < A$, then one has $BN/N < AN/N$ otherwise $A = B(A\cap N)$ which leads to the contradiction $A \leqslant N$. So $J(A)N/N$ is contained in $J(AN/N)$ by maximality of the latter. Set $J/N = J(AN/N)$ with J the full preimage in G of $J(AN/N)$. Then we have $J(A) \leqslant (J \cap A)^\circ$. Since $J/N < AN/N$, it follows that $J(A) = (J \cap A)^\circ$ by maximality of $J(A)$; hence,

$$J(A)N/N = (J \cap A)^\circ N/N = J/N$$

□

Lemma 3.7 (Pull-back) *Let G and N be as above. If $\overline{B}$ is an indecomposable subgroup of G/N, then there exists an indecomposable subgroup A of G such that $\overline{B} = AN/N$.*

Proof Let B be the full preimage of $\overline{B}$ in G. We have $\overline{B} = B/N = B^\circ N/N$ and the set of definable connected subgroups A of G such that $\overline{B} = AN/N$ is nonempty. Let A be such a subgroup of minimal Morley rank. Let us show that A is indecomposable.

Let A_1, A_2 be two definable connected subgroups such that $A = \langle A_1, A_2\rangle$. Then either $\overline{B} = A_1N/N$ or $\overline{B} = A_2N/N$. Which means by minimality of $rk(A)$ that $A = A_1$ or $A = A_2$ and the conclusion follows. □

Fact 3.8 [9, Lemma 2.2] *Let G be a group of finite Morley rank and H a definable normal subgroup of G. If $\overline{B}$ is a divisible abelian indecomposable subgroup of G/H, then there is an abelian indecomposable subgroup A of G such that $\overline{B} = AH/H$.*

By Fact 2.3, an abelian indecomposable group A is either divisible or of bounded exponent. Now if the indecomposable group $\overline{B}$ in Lemma 3.7 is abelian and divisible, so is A (Fact 3.8). Unfortunately, the same conclusion is not possible in the bounded exponent case; this is due to the existence of nonabelian indecomposable p-groups (such as given in Example 3.2), whose quotients may be abelian.

Fact 3.9 *[5, Lemma 2.4] Any abelian connected group G of finite Morley rank is generated by its abelian indecomposable subgroups.*

Proposition 3.10 *Any connected group of finite Morley rank is generated by indecomposable subgroups.*

Proof Suppose $G \neq 1$. Let A_1 be an infinite minimal subgroup of G. So A_1 is an indecomposable subgroup, as well as A_1^g for any $g \in G$. Set $V_0 = 1$ and for $i \in \mathbb{N}$, $V_{i+1} = \langle V_i, A_{i+1}^g \mid g \in G\rangle$ where A_{i+1} is chosen to be minimal such that $A_{i+1}V_i/V_i$ is an infinite minimal group. Clearly each A_i is an indecomposable subgroup of G and $(V_i)_i$ is an increasing series of normal definable connected subgroups. By finiteness of Morley rank, $G = V_i$ for some i and we are done. □

4. U_τ-groups

4.1. τ-indecomposables

Definition 4.1 Let G and H be two definable groups in a universe of finite Morley rank $\mathcal{U}$. We say that G and H are quasiisogenous if there are finite normal subgroups I, J of G and H respectively such that G/I and H/J are definably isomorphic.

Lemma 4.2 *Quasi-isogeny defines an equivalence relation on definable groups in a universe of finite Morley rank $\mathcal{U}$.*

Proof Reflexivity and symmetry being obvious, it remains to show transitivity. Let F, G, H be three definable groups in $\mathcal{U}$ such that G is quasi-isogenous to H and F. Let I_1, I_2, J, K be finite normal subgroups such that $\Phi : F/J \to G/I_1$ and $\Psi : G/I_2 \to H/K$ are the associated definable isomorphisms. Now set $\Phi^{-1}(I_2I_1/I_1) = J_0/J$ and $\Psi(I_1I_2/I_2) = K_0/K$. Clearly J_0 and K_0 are finite normal subgroups of F and H respectively and we have

$$F/J_0 \cong G/I_1I_2 \cong H/K_0.$$

Thus, F and H are quasiisogenous and the transitivity is settled. □

Notation 4.3 In what follows, τ will denote a minimal group. We will write $\tau \approx \tau'$ if τ and τ' are quasiisogenous.

Definition 4.4 An indecomposable group A is said to be τ-indecomposable if $A/J(A)$ and τ are quasiisogenous.

Lemma 4.5 *Let A be a divisible abelian indecomposable group. Then either A is a pseudotorus or $A/J(A) \cong \mathbb{K}_+$ definably for some definable field $\mathbb{K}$ of characteristic zero.*

Proof Suppose A is not a pseudotorus. Then there is a definable subgroup B such that $A/B \cong \mathbb{K}_+$ for some definable field $\mathbb{K}$, but A is divisible, so $\mathbb{K}$ has zero characteristic. It follows that A/B is torsion-free and B is divisible so connected; hence, B is contained in $J(A)$. Since $\mathbb{K}_+$ is minimal (Fact 2.7), so is A/B and then $J(A)$ is contained in B. Thus, $B = J(A)$ and we are done. □

Lemma 4.6 *In a universe of finite Morley rank, let A be an indecomposable group and $\mathbb{K}$ a definable field of characteristic zero. Then $A/J(A)$ and $\mathbb{K}_+$ are quasiisogenous if and only if $A/J(A) \cong \mathbb{K}_+$ definably.*

Proof It remains to prove the only if, the if being clear. If $\mathbb{K}_+$ and $A/J(A)$ are quasiisogenous, then the abelian indecomposable quotient group $A/J(A)$ is divisible since $\mathbb{K}$ is zero characteristic. Furthermore, $A/J(A)$ is torsion-free and we have $A/J(A) \cong \mathbb{K}_+$. □

From what procedes together with Fact 2.3, we have:

Proposition 4.7 *A minimal group τ is either:*

- *a pseudotorus or*
- *definably isomorphic to the additive group of a definable field of characteristic zero or*
- *an elementary p-group.*

The first two points are relative to the case τ is divisible and the last point interprets the bounded exponent case. In the following, we will say that a minimal group is 0-indecomposable if it describes the second point and p-indecomposable if it describes the third point.

Lemma 4.8 *Let τ be a minimal divisible group. Then any τ-indecomposable group A is abelian. Furthermore, if τ is a pseudotorus, so is A.*

Proof Let A be τ-indecomposable. By Proposition 4.7 above, $A/J(A)$ is either a pseudotorus or definably isomorphic to the additive group $\mathbb{K}_+$ for a definable field $\mathbb{K}$ of characteristic zero.

Suppose that $A/J(A) \cong \mathbb{K}_+$ definably. Let $x \in A \setminus J(A)$ and set $D = d(x)$ the definable closure of x. By Fact 2.7, we have $A = DJ(A)$, and by the definition of $J(A)$, we obtain $A = D$ and it is done.

Suppose now that $A/J(A)$ is a pseudotorus. By Fact 2.11, $A/J(A) = TJ(A)/J(A)$ where T is a maximal pseudotorus of A. Thus, we get $A = TJ(A)$ and once again, we have $A = T$ and the proof is finished. □

This means that nonabelian indecomposable groups are necessarily related to the positive characteristic.

4.2. U_τ-groups

Before giving the definition of U_τ-groups, we first recall that of $U_{0,r}$-groups. Let $r \in \mathbb{N}^*$. The reduced rank of an abelian indecomposable group A is defined to be $\overline{r}(A) = \text{rk}(A/J(A))$. A group of finite Morley rank G is said to be a $U_{0,r}$-group if $G = U_{0,r}(G)$ where

$$U_{0,r}(G) = \langle A \mid A \text{ abelian indecomposable, } A/J(A) \text{ is torsion-free, } \overline{r}(A) = r\rangle$$

Without saying more, just note that $U_{0,r}$-groups have been introduced by Burdges in his PhD thesis [6] and then in [5] and [7].

Definition 4.9 Let G be a group of finite Morley rank and τ a minimal group. We say that G is a U_τ-group if $G = U_\tau(G)$ where

$$U_\tau(G) = \langle A \leqslant G \mid A \text{ is } \tau\text{-indecomposable}\rangle$$

.

Remark 4.10

1. For a fixed minimal group τ, $U_\tau(G)$ is a connected definable and characteristic subgroup of G.
2. In the case τ is a 0-indecomposable group, say $\tau = \mathbb{K}_+$ for a definable field $\mathbb{K}$ of characteristic zero, then a definable subgroup A is τ-indecomposable if and only if $A/J(A) \cong \mathbb{K}_+$ definably (Lemma 4.6); so instead of $U_\tau(G)$, we will write $U_\mathbb{K}(G)$ and set

$$U_\mathbb{K}(G) = \langle A \leqslant G \mid A/J(A) \cong \mathbb{K}_+ \text{ definably}\rangle;$$

this last notation is due to Frécon [10].

The notion of U_τ-groups thus defined is derived from the Burdges notion of $U_{0,r}$-groups.

From now on, rather than using the unipotence $U_{0,r}$ operator, we will use its analog $U_{\mathbb{K}}$ introduced by Frécon which has the advantage of bringing out a definable field $\mathbb{K}$ of characteristic zero, necessary when it comes to attack the mattters of linearity. However, the use of $U_{\mathbb{K}}$ only concerns the null characteristic, we reserve U_τ for the general context.

Fact 4.11 (Compare with [7, Fact 2.1]) *In a universe of finite Morley rank, let $f : G \to H$ be a definable homomorphism between two definable groups and τ an infinite minimal group. Then the following hold.*

1. (Push-forward) *$f(U_\tau(G)) \leqslant U_\tau(H)$ is a U_τ-group.*
2. (Pull-back) *If $U_\tau(H) \leqslant f(G)$ then $f(U_\tau(G)) = U_\tau(H)$.*

In particular, an extension of a U_τ-group by a U_τ-group is a U_τ-group.

This naturally follows from the Push-forward and Pull-back on indecomposable groups. Also, if G is a group of finite Morley rank and N a definable normal subgroup, the Pull-back can be reformulated as follows:

$$U_\tau(G/N) = U_\tau(G)N/N,$$

that is U_τ is preseved through definable quotients. As a closed subgroup of a unipotent affine algebraic group is still unipotent, one would like U_τ to be preserved by passage to definable subgroup. Unfortunately, it is not the case in general. In order to remedy this, we are going to define a new category of groups in the next subsection.

4.3. Homogeneous U_τ-groups

Definition 4.12 Let τ be a minimal group. We say that a U_τ-group G is homogeneous or G is a homogeneous U_τ-group if every definable connected subgroup of G is a U_τ-group.

It is clear that a definable subgroup of some homogeneous U_τ-goup is again a homogeneous U_τ-group. Also, in [8], it is established that homogeneity is preserved through definable quotient and by extension of an homogeneous $U_{0,r}$-group by another one. Following the proofs in [8], this still holds for U_τ-groups.

Fact 4.13 (Compare with [8, Lemma 3.3]) *A definable quotient of a homogeneous U_τ-group is a homogeneous U_τ-group.*

Fact 4.14 (Compare with [8, Lemma 3.4]) *An extension of a homogeneous U_τ-group by a homogeneous U_τ-group is a homogeneous U_τ-group.*

Unipotent algebraic groups are known to be nilpotent groups. Homogeneity provides this property for $U_{0,r}$-groups of finite Morley rank which are not bad. Recall that a bad group is a nonsolvable connected group of finite Morley rank in which every proper definable connected subgroup is nilpotent.

Fact 4.15 [8, Proposition 3.8] *A homogeneous $U_{0,r}$-group is either nilpotent or contains a bad group.*

Here we extend this result to U_τ-groups:

Proposition 4.16 *A homogeneous U_τ-group is either nilpotent or contains a bad group. In particular, a solvable homogeneous U_τ-group is nilpotent.*

Proof Let G be a nonnilpotent homogeneous U_τ-group. We have just to prove that G contains a bad group. Let H be a minimal nonnilpotent connected definable subgroup of G.

We assume toward a contradiction that H is solvable. By Fact 2.6, the quotient group $H/F(H)^\circ$ is abelian and divisible. Therefore, by the minimality of H, the quotient group $H/F(H)^\circ$ is a minimal infinite definable group. Now $H/F(H)^\circ$ is either a pseudotorus or it is definably isomorphic to the additive group $\mathbb{K}_+$ of a definable field $\mathbb{K}$ of characteristic zero. In the last case, we have $\tau \cong \mathbb{K}_+$ and H is a homogeneous $U_{0,r}$-group for $r = \mathrm{rk}(\mathbb{K}_+)$, contradicting Fact 4.15. Consequently, $H/F(H)^\circ$ is a pseudotorus, so τ is a pseudotorus. This proves that $F(H)^\circ$ is a pseudotorus by Fact 2.9, so $F(H)^\circ$ is central in H by Fact 2.10, contradicting that H is nonnilpotent.

We have proved that H is not solvable, so it is a bad group as desired. □

Unipotent elements in algebraic groups over a field of characteristic zero are elements of infinite order. This property is also provided by some homogeneous U_τ-groups where $\tau = \mathbb{K}_+$, for a definable field $\mathbb{K}$ of characteristic zero.

Fact 4.17 [8, Corollary 3.9] *A homogeneous $U_{0,r}$-group not containing a bad group is torsion-free.*

Now we say a few words about the intersection of homogeneous U_τ-groups.

Fact 4.18 (Compare with [8, Corollary 3.12]) *Let τ, τ' be two minimal groups, H a homogeneous U_τ-group and K a homogeneous $U_{\tau'}$-group. If $\tau \not\approx \tau'$ then the intersection $H \cap K$ is finite. Furthermore, if τ and τ' are 0-indecomposable and H (or K) does not contain a bad group, then the intersection is trivial.*

5. Unipotence and structure theorems

Let $\mathbb{K}$ and $\mathbb{L}$ be two fields of finite Morley rank and characteristic zero. We know that $\mathbb{K}$ and $\mathbb{L}$ are definably isomorphic if and only if $\mathbb{K}_+$ and $\mathbb{L}_+$ are (see [10, Corollary 2.8]). Now let H be a homogeneous $U_{\mathbb{K}}$-group and K a homogeneous $U_{\mathbb{L}}$-group. By Fact 4.18 above, one has $H \cap K = 1$ if $\mathbb{K}$ and $\mathbb{L}$ are not definably isomorphic. This will be used in the following without necessarily being mentioned.

5.1. Nilpotent groups and unipotence

We start with a recall of a structural result of Nesin on nilpotent groups of finite Morley rank.

Fact 5.1 [13] *Let G be a nilpotent group of finite Morley rank. Then G is a central product $G = D * C$ where*

i) D is definable, connected, characteristic, and divisible,

ii) C is definable and of bounded exponent.

Further, by Burdges (and Frécon), we have the description of the divisible part D above.

Fact 5.2 [7, Theorem 3.5], Frécon [10, Fact 3.10] *Let G be a divisible nilpotent group of finite Morley rank, and let T be its maximal pseudotorus. Then G interprets some fields $\mathbb{K}_1, ..., \mathbb{K}_n$ of characteristic zero such that*

$$G = T * U_{\mathbb{K}_1}(G) * \cdots * U_{\mathbb{K}_n}(G).$$

More recently, Myasnikov and Sohrabi in [12] gave a linear description of a torsion-free nilpotent groups of finite Morley rank.

Fact 5.3 [12, Theorem 1.5] *A torsion-free nilpotent group G has finite Morley rank in the language of groups if and only if G is a finite direct product*

$$G = G_1 \times \cdots \times G_n \times G_0$$

where each G_i, $i \neq 0$, is a unipotent algebraic group over $\mathbb{K}_i$, $\mathbb{K}_i$ is a characteristic zero algebraically closed field, and G_0 is a torsion-free divisible abelian group.

Note that the subgroups G_i in the decomposition above are not necessarily definable while in the previous all $U_{\mathbb{K}_i}(G)$ are.

At this stage, one would legitimately like to know what occurs on the bounded part C of Fact 5.1. Unfortunately, it is unexpected to obtain the configuration described in Fact 5.3. Indeed, while a zero characteristic field can be interpreted by a nonabelian nilpotent-per-finite group, Baudisch in [3] kills any hope for a general method in interpreting a field of positive characteristic from any connected nilpotent group of positive exponent p. However, in the following, we give a more general decomposition of a connected nilpotent group of finite Morley rank.

Proposition 5.4 *Let G be a connected nilpotent group of finite Morley rank. Then*

$$G = U_{\tau_1}(G) * \cdots * U_{\tau_n}(G)$$

where the minimal groups $\tau_1, ..., \tau_n$ are such that

$$[U_{\tau_i}(G)\ ,\ U_{\tau_j}(G)] = 1 \quad \textit{when } i \neq j$$

Proof By Proposition 3.10, it is clear that $G = U_{\tau_1}(G) \cdots U_{\tau_n}(G)$ for some minimal groups $\tau_1, ..., \tau_n$. We may assume $\tau_i \not\approx \tau_j$ when $i \neq j$. Now we show that $[U_{\tau_i}(G), U_{\tau_j}(G)] = 1$. We proceed by induction on $\mathrm{rk}(G)$. We may assume that G is not abelian and let A be a G-minimal central subgroup; such one exists since the center of a nilpotent group of finite Morley rank is infinite. In particular, A is an indecomposable subgroup of G. Now by induction hypothesis, one has $[U_{\tau_i}(G/A), U_{\tau_j}(G/A)] = 1$, and using Fact 4.11, this yields $[U_{\tau_i}(G), U_{\tau_j}(G)] \leqslant A$. Let $u \in U_{\tau_i}(G)$ not centralizing $U_{\tau_j}(G)$. Consider the definable map $U_{\tau_j}(G) \to A : x \mapsto [u, x]$, which is a morphism since $A \leqslant Z(G)$, and thanks to Fact 4.11, its image $[u, U_{\tau_j}(G)]$ is a U_{τ_j}-subgroup of A. By minimality of A, one has $A = [u, U_{\tau_j}(G)]$ and $A \approx \tau_j$. Let $v \in U_{\tau_j}(G)$ not centralizing $U_{\tau_i}(G)$. Similarly, we show that $A = [v, U_{\tau_i}(G)]$ and $A \approx \tau_i$. This leads to the contradiction $\tau_i \approx \tau_j$. □

Corollary 5.5 *Let G be a connected nilpotent group of finite Morley rank. Then*

$$G = T * U_{\mathbb{K}_1}(G) * \cdots U_{\mathbb{K}_r}(G) * U_{\tau_1}(G) * \cdots U_{\tau_s}(G)$$

where $\mathbb{K}_1, ..., \mathbb{K}_r$ are definable fields of zero characteristic, $\tau_1, ..., \tau_s$ are p_i-indecomposable groups for some primes $p_1, ..., p_s$, and T is the maximal pseudotorus of G.

Proof Consider the decomposition of G given in the previous proposition and fix a minimal τ. Then by Proposition 4.7, we have three cases.

If τ is 0-indecomposable, then $U_\tau(G) = U_{\mathbb{K}}(G)$ where $\mathbb{K}$ is a definable field such that $\tau \cong \mathbb{K}_+$ definably. If τ is a pseudotorus, then by Lemma 4.8 and Fact 2.9, we have $U_\tau(G) \leqslant T$. In the last case, τ is p-indecomposable for some prime p; hence, the decomposition is proved. □

Remark 5.6 In Proposition 5.4:

- Each $U_{\tau_i}(G)$ is not necessarily homogeneous;
- $U_{\tau_i}(G) \cap U_{\tau_j}(G)$ is not necessarily finite.

Definition 5.7 Let G be a connected nilpotent group of finite Morley rank. We say that G is unipotent of finite Morley rank if G is the central product of homogeneous U_{τ_i}-groups for minimal definable groups $\tau_1, ..., \tau_r$, that is

$$G = G_1 * \cdots * G_r$$

where G_i is a homogeneous U_{τ_i}-group and $G_i \cap G_j$ is finite whenever $i \neq j$. Furthermore, if for all $i = 1, ..., r$, there exists a definable field $\mathbb{K}_i$ such that τ_i embeds in $(\mathbb{K}_i)_+$ definably, we say that G is a good unipotent group of finite Morley rank.

Remark 5.8

1. If G is a unipotent group of finite Morley rank, we have

$$G = (B_1 * \cdots * B_r) \times (U_1 \times \cdots \times U_s)$$

   where U_i is a homogeneous $U_{\mathbb{K}_i}$-group with $\mathbb{K}_i$ a definable field of characteristic zero and B_i is a homogeneous U_{τ_i}-group with τ_i a p_i-indecomposable group. The good unipotence suggests that each p_i-indecomposable group τ_i can be considered a minimal additive subgroup of a definable field of positive characteristic. This will be useful for studying the linearity of groups of finite Morley rank.

2. The choice of central product instead of direct product in Definition 5.7 is due to the fact that the intersection of subgroups B_i is not necessarily trivial. However, in the case we have direct product, we will say that the (good) unipotent group of finite Morley rank has a good form.

If $\mathbb{K}$ is a field of finite Morley rank and of characteristic zero, then by minimality (Fact 2.7) $\mathbb{K}_+$ is a homogeneous U_τ-group with $\tau = \mathbb{K}_+$; so $\mathbb{K}_+$ is unipotent of finite Morley rank as expected. Proposition 5.10 below extends this to the positive characteristic.

Lemma 5.9 *Let $\mathbb{K}$ be an algebraically closed field and E a finite subgroup of $\mathbb{K}_+^n$ for any positive integer n. Then $\mathbb{K}_+^n/E$ and $\mathbb{K}_+^n$ are definably isomorphic.*

Proof We can assume $\mathbb{K}$ of positive characteristic p. Then being of p exponent, $\mathbb{K}_+^n/E$ is isomorphic as an algebraic group to some $\mathbb{K}_+^d$ with $d \geq 1$ (see [11, Corollary 20.4]). Dimension consideration yields $d = n$ and the isomorphism is definable in the pure field $\mathbb{K}$. □

Proposition 5.10 *Let $\mathbb{K}$ be a field of finite Morley rank and τ a minimal subgroup of $\mathbb{K}_+$. Then $\mathbb{K}_+$ is a homogeneous U_τ-group.*

Proof For convenience, we set $A = \tau$. We have $\mathbb{K}_+ = \langle xA \mid x \in \mathbb{K}^* \rangle$. For all $x \in \mathbb{K}^*$, xA is a minimal subgroup definably isomorphic to A. So $\mathbb{K}_+$ is a U_τ-group. Moreover, by Zilber Indecomposable Theorem (Fact 2.1), there are $x_1, ..., x_n \in \mathbb{K}^*$ such that $\mathbb{K}_+ = x_1 A + \cdots + x_n A$ with n minimal, and by Fact 4.14 $\mathbb{K}_+$ is therefore a homogeneous U_τ-group since each $x_i A$ is. □

The quotient of a connected nilpotent linear algebraic group by its center is unipotent. In the following, we prove this for groups of finite Morley rank.

Lemma 5.11 *Let G be a nilpotent U_τ-group, then $G/Z(G)^\circ$ is U_τ-homogeneous.*

Proof First, note that $G/Z(G)^\circ$ is U_τ-group (Fact 4.11). By Proposition 5.4, we can write

$$G = U_{\tau_1}(G) * \cdots * U_{\tau_n}(G)$$

so $[G, U_{\tau_i}(G)] = [U_\tau(G), U_{\tau_i}(G)] = 1$ for all $\tau_i \not\approx \tau$. It follows that $U_{\tau_i}(G) \leqslant Z(G)^\circ$ for all $\tau_i \not\approx \tau$. Thus, for every connected subgroup $H \leqslant G$ containing $Z(G)^\circ$, we still have $U_{\tau_i}(H) \leqslant Z(G)^\circ$ for all $\tau_i \not\approx \tau$. Then $H/Z(G)^\circ$ is U_τ-group by Proposition 5.4 and we can conclude. □

Theorem 5.12 *Let G be a connected nilpotent group of finite Morley rank. Then $G/Z(G)$ is a unipotent group of finite Morley rank of* good form*:*

$$G/Z(G) = B_1 \times \cdots \times B_r \times U_1 \times \cdots \times U_s$$

where B_i is a homogeneous U_τ-group with τ_i a p_i-indecomposable group and U_i is a homogeneous $U_{\mathbb{K}_i}$-group with $\mathbb{K}_i$ a definable field of zero characteristic.

Proof By proposition 5.4, we can write $G = U_{\tau_1}(G) * \cdots * U_{\tau_n}(G)$ and

$$\frac{G}{Z(G)} = U_{\tau_1}\left(\frac{G}{Z(G)}\right) * \cdots * U_{\tau_n}\left(\frac{G}{Z(G)}\right) \tag{5.1}$$

where τ_i are minimal groups. It follows that $Z(U_{\tau_i}(G)) = Z(G) \cap U_{\tau_i}(G)$, and

$$U_{\tau_i}\left(\frac{G}{Z(G)}\right) \cong \frac{U_{\tau_i}(G)}{Z(U_{\tau_i}(G))}.$$

Now remember that any definable quotient of a homogeneous U_τ-group is homogeneous, so by Lemma 5.11 above, $U_{\tau_i}\left(\frac{G}{Z(G)}\right)$ is homogeneous for all i and $G/Z(G)$ is unipotent of finite Morley rank.

Now we show that we have a direct product in (5.1). Let $x \in G$ such that

$$\overline{x} \in U_{\tau_i}\left(\frac{G}{Z(G)}\right) \cap \langle U_{\tau_j}\left(\frac{G}{Z(G)}\right) \mid j \neq i \rangle.$$

Writing $x = uz_1$ where $u \in U_{\tau_i}(G)$ and $z_1 \in Z(G)$, we get

$$[x, U_{\tau_j}(G)] = 1 \quad \text{for all } j \neq i$$

We can also write $x = vz_2$ where $v \in \langle U_{\tau_j}(G) \mid j \neq i \rangle$, $z_2 \in Z(G)$ and get $[x, U_{\tau_i}(G)] = 1$. Finally, we deduce from the above decomposition of G that $x \in Z(G)$ and $\overline{x} = 1$. Therefore, we can write

$$\frac{G}{Z(G)} = U_{\tau_1}\left(\frac{G}{Z(G)}\right) \times \cdots \times U_{\tau_n}\left(\frac{G}{Z(G)}\right).$$

Finally, we define U_i to be subgroups $U_{\tau_i}\left(\frac{G}{Z(G)}\right)$ where $\tau_i = (\mathbb{K}_i)_+$ for a definable field $\mathbb{K}_i$ of characteristic zero, and $B_i = U_{\tau_j}\left(\frac{G}{Z(G)}\right)$ where τ_j is some p_j-indecomposable group. □

5.2. U-groups and unipotence

Definition 5.13 Let G be a group of finite Morley rank. We let $U(G)$ denote the subgroup of G generated by its normal homogeneous U_τ-groups, τ varying among minimal indecomposable groups. If $G = U(G)$, we say that G is a U-group.

U-groups have been introduced by Frécon [8] in order to define a sort of unipotence which provides a unipotent radical for groups of finite Morley rank. We emphasize that the definition of U-group set here is not quite different from [8, Definition 5.1].

The following result can be compared with [8, Proposition 5.7]. However, even if it is inspired by the proof given in [8], we think that our proof is much more conceptual, and the use of new definitions justifies our giving it here.

Proposition 5.14 *Let G be a group of finite Morley rank. Then G is a U-group if and only if $U_\tau(G)$ is homogeneous for all τ. In that case, there are minimal groups $\tau_1, ..., \tau_n$ such that*

$$G = U_\tau(G) * \cdots * U_{\tau_n}(G).$$

Proof It is clear that if $U_\tau(G)$ is homogeneous for every τ, then G is a U-group since it is generated by all (Proposition 3.10). Suppose now that G is a U-group. There are distinct minimal groups $\tau_1, ..., \tau_n$ such that $G = V_1 * \cdots * V_n$ where V_i is the largest normal homogeneous U_{τ_i}-subgroup of G; such one exists by Fact 4.14. We show that $V_i = U_{\tau_i}(G)$. It suffices to prove that $U_{\tau_i}(G) \leqslant V_i$, the inverse inclusion being trivial.

For $i \in \{1, ..., n\}$ fixed let A be a τ_i-indecomposable subgroup of G. Set $W_0 = V_i$ and for every integer $j \in \{1, ..., n\}$, $W_j = \langle V_k \mid k \in \{1, ..., j\} \cup \{i\} \rangle$. In particular, $W_i = W_{i-1}$.

Let s be minimal such that $A \leqslant W_s$. Suppose that $s \neq 0$. Since $W_i = W_{i-1}$, we have $s \neq i$ and $W_s/W_{s-1} = V_s W_{s-1}/W_{s-1}$ is a nontrivial homogeneous U_{τ_s}-group. If AW_{s-1}/W_{s-1} is not trivial, it is τ_i-indecomposable and also a τ_s-indecomposable subgroup of W_s/W_{s-1}. This is absurd since τ_i is not quasiisogenous to τ_s. So $A \leqslant W_{s-1}$ which contradicts the minimality of s. We deduce that $s = 0$ that is $A \leqslant V_i$ hence $U_{\tau_i}(g) \leqslant V_i$ as desired. The rest of the proof is clear. □

This means that a nilpotent U-group is a unipotent group of finite Morley rank and conversely. Therefore, all the results established for U-groups are valuable for unipotent groups of finite Morley rank. In particular :

Fact 5.15 (Compare with [8, Theorem 6.8]) *Let G be connected solvable group of finite Morley rank. Then G' is unipotent of finite Morley rank.*

To close this part, we give this result which will be useful later.

Fact 5.16 (Compare with [8, Theorem 4.11]) *Let G be a group of finite Morley rank, and H a normal nilpotent U_τ-group. Then $[G,H]$ is an homogeneous U_τ-group.*

5.3. Solvable groups and unipotence

Definition 5.17 Let G be a group of finite Morley rank. We define $C(G)$ to be the intersection of all centralizers $C_G(A)$ with A varying over the class of definable connected groups of automorphisms of G.

By the descending chain condition for groups of finite Morley rank (see [4, Theorem 5.2]) $C(G)$ is definable and it is a central subgroup.

Fact 5.18 [8, Lemma 6.19] *Let G be a solvable connected group of finite Morley rank. Then* $F(G) = F(G)^\circ C(G)$

Theorem 5.12 can be retrieved by the following result.

Proposition 5.19 *Let G be a solvable connected group of finite Morley rank. Then $F(G)/Z(G)$ is a unipotent group of finite Morley rank of* good form.

Proof By Proposition 5.4, we have the decomposition

$$F(G) = F(G)^\circ C(G) = U_{\tau_1}(F(G)^\circ) * \cdots * U_{\tau_n}(F(G)^\circ) * C(G)$$

for minimal groups $\tau_1, ..., \tau_n$ such that $\tau_i \not\approx \tau_j$ for all $i \neq j$. Let A be a definable connected groups of automorphisms of G and set $U_{\tau_i} = U_{\tau_i}(F(G)^\circ)$. We have

$$[A, U_{\tau_i} \cap U_{\tau_j}] \leqslant [A, U_{\tau_i}] \cap [A, U_{\tau_j}].$$

However, by Fact 5.16, $[A, U_{\tau_i}]$ is τ_i-homogeneous and $[A, U_{\tau_j}]$ is τ_j-homogeneous. It follows that $[A, U_{\tau_i}] \cap [A, U_{\tau_j}]$ is finite by Fact 4.18. Since $[A, U_{\tau_i} \cap U_{\tau_j}]$ is connected, it is trivial so $U_{\tau_i} \cap U_{\tau_j} \leqslant C_G(A)$. Thus, we can deduce that $U_{\tau_i} \cap U_{\tau_j} \leqslant C(G) \leqslant Z(G)$; hence, $U_i := U_{\tau_i}Z(G)/Z(G)$ is a homogeneous U_{τ_i}-group and $F(G)/Z(G) = U_1 \times \cdots \times U_n$ is unipotent of finite Morley rank. □

Definition 5.20 A group of finite Morley rank $\mathbf{T}$ is said to be a generalized pseudotorus if it is nilpotent, connected with no definable quotient definably embedding in the additive group of a definable field $\mathbb{K}$.

Remark 5.21 According to Proposition 5.4, a generalized pseudotorus $\mathbf{T}$ admits the following decomposition

$$\mathbf{T} = T * U_{\tau_1}(\mathbf{T}) * \cdots * U_{\tau_s}(\mathbf{T})$$

where $\tau_1, ..., \tau_s$ are p_i-indecomposable groups for some primes $p_1, ..., p_s$, and none of them is embedding in the additive group of a definable field; T is the (unique) maximal pseudotorus of $\mathbf{T}$. In particular, if $\mathbf{T}$ is divisible, then $\mathbf{T} = T$ is a pseudo-torus.

As we can see, generalized pseudotori serve to isolate the unipotent pieces given by nonlinear indecomposable groups and are therefore not allowed by good unipotence.

For any group G, we let G^∞ denote the smallest subgroup by inclusion of the descending central series $(G^n)_n$. If G has finite Morley rank then G^∞ is definable.

Proposition 5.22 *Let G be a connected solvable group of finite Morley rank. Then G^∞ is a good unipotent group of finite Morley rank.*

Proof We may assume that G is not nilpotent. By Fact 5.15 and Proposition 5.14, we have

$$G^\infty = U_{\tau_1}(G^\infty) * \cdots * U_{\tau_n}(G^\infty)$$

with $U_{\tau_i}(G^\infty) \cap U_{\tau_j}(G^\infty)$ finite whenever $i \neq j$, and each $U_{\tau_i}(G^\infty)$ is homogeneous. It remains to prove that each minimal τ_i is an additive subgroup of some definable field $\mathbb{K}_i$.

Set $U_i = U_{\tau_i}(G^\infty)$. For every i, let $A_i \leqslant U_i$ be a G-normal definable connected subgroup such that $\overline{U_i} = U_i/A_i$ is G-minimal. If $[G, U_i] \leqslant A_i$, then

$$G^\infty = [G, G^\infty] \leqslant U_1 * \cdots * U_{i-1} * A_i * U_{i+1} * \cdots * U_n < G^\infty;$$

a contradiction. So $[G, \overline{U_i}]$ is not trivial and by Fact 2.4, there exists a definable field $\mathbb{K}_i$ such that $\overline{U_i} \cong (\mathbb{K}_i)_+$. This implies that τ_i embeds in $(\mathbb{K}_i)_+$ definably. □

Theorem 5.23 *Let G be a connected solvable group of finite Morley rank. There exists a pair $(H, \mathbf{T})$ of connected definable subgroups such that*

i) $G = H * \mathbf{T}$*;*

ii) $\mathbf{T}$ *is a generalised pseudotorus;*

iii) H' *is a good unipotent group of finite Morley rank.*

Proof We proceed by induction on Morley rank. By Proposition 5.4 and Fact 5.15, we may assume that G is not nilpotent. Then G^∞ is nontrivial and by Proposition 5.22 there is a definable field $\mathbb{K}$ and a minimal indecomposable group τ embedding in $\mathbb{K}_+$ such that $U_\tau(G^\infty)$ is nontrivial. Let A be a G-minimal subgroup of $U_\tau(G^\infty)$. By induction, we have

$$G/A = H/A * \mathbf{T}_1/A$$

where H/A and $\mathbf{T}_1/A$ have the desired form.

Suppose $G = H$. Then $G'A/A$ is a good unipotent group of finite Morley rank. By Fact 5.15, $G' = U_{\tau_1}(G') * \cdots * U_{\tau_n}(G')$, then for every integer i such that $U_{\tau_i}(G')A/A = U_{\tau_i}(G'A/A)$ is nontrivial, there exists a definable field $\mathbb{K}_i$ with τ_i embedding in $(\mathbb{K}_i)_+$. In fact, such a definable field exists for all i because A is a homogeneous U_τ-group with τ embedding in $\mathbb{K}_+$. Therefore, G' is a good unipotent group of finite Morley rank.

Now suppose that $G \neq H$. In particular, $G \neq \mathbf{T}_1$ since G is not nilpotent, and by induction, we have $\mathbf{T}_1 = \mathbf{T} * A$ where $\mathbf{T}$ is a generalised pseudotorus. Also $H' = U_{\tau_1}(H') * \cdots * U_{\tau_n}(H')$ with τ_i embedding in

$(\mathbb{K}_i)_+$ for a definable field $\mathbb{K}_i$. Since $G = H\mathbf{T}$, it remains to show that $[\mathbf{T}, H]$ is trivial, but $[\mathbf{T}, H] \leqslant A$, so for every minimal group $\tau' \not\approx \tau$, $[U_{\tau'}(\mathbf{T}), H]$ is both τ-homogeneous and τ'-homogeneous (Fact 5.16), whence is trivial. Thus, because of the decompostion of $\mathbf{T}$ (Remark 5.21), it suffices to show that $[T, H]$ is trivial where T is the maximal pseudotorus of $\mathbf{T}$. Let $h \in H \setminus \{1\}$. Then

$$\begin{array}{cccc} ad_h & : & T & \rightarrow & A \\ & & t & \mapsto & [h,t] \end{array}$$

is a definable morphism and $T/C_T(h) \cong [h, T]$ is a U_τ-group with τ embedding in $\mathbb{K}_+$ definably, and since no nontrivial definable section of T embeds in $\mathbb{K}_+$, $[T, h]$ is necessarily trivial and the theorem is proved. □

An example of a solvable group of finite Morley rank such that H is as in the above theorem is given in Fact 2.4 and also Fact 2.5. Of course the interest of H lies in the fact that its derived subgroup is a good unipotent group of finite Morley and this, we hope, will be useful when it will come to attack matters of linearity.

Acknowledgments

I would like to thank Olivier Frécon for his valuable contribution and careful guidance. I also thank Adrien Deloro for his relevant remarks and encouragement. Finally, I would like to thank the referee for his careful reading and the comments he made for the improvement of the exposition.

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