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∗-Semiclean rings

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Abstract: A ring R is called semiclean if every element of R can be expressed as sum of a periodic element and a unit. In this paper, we introduce a new class of ring, which is the $*$-version of the semiclean ring, i.e. the $*$-semiclean ring. A $*$-ring is $*$-semiclean if each element is a sum of a $*$-periodic element and a unit. The term $*$-semiclean is a stronger notion than semiclean. In this paper, many properties of $*$-semiclean rings are discussed. It is proved that if $p \in P(R)$ such that pRp and $(1-p)R(1-p)$ are $*$-semiclean rings, then R is also a $*$-semiclean ring. As a result, the matrix ring $M_n(R)$ over a $*$-semiclean ring is $*$-semiclean. A characterization that when the group rings RC_r and RG are $*$-semiclean is done, where R is a finite commutative local ring, C_r is a cyclic group of order r, and G is a locally finite abelian group. We have also found sufficient conditions when the group rings RC_3, RC_4, RQ_8, and RQ_{2n} are $*$-semiclean, where R is a commutative local ring. We have also demonstrated that the group ring $\mathbb{Z}_2D_6$ is a $*$-semiclean ring (which is not a $*$-clean ring).

Key words: Group rings, semiclean rings, $*$-periodic element

1. Introduction

A ring R is called clean if every element of R can be expressed as a sum of an idempotent and a unit. In literature, a lot of work is done on this class of ring; see [14, 19], and [22] for more details on it. A ring R is called $*$-clean if every element of R can be expressed as sum of a projection and a unit. See [1, 3, 6, 8, 12, 16], and [18] for more details on it. So far, much work has been done on the $*$-clean ring, but the $*$-semiclean ring has yet to be discovered. The motivation of the paper is to find out about the $*$ concept in the semiclean ring. In this paper, we are introducing a $*$-semiclean ring. A $*$-semiclean ring is the subclass of a semiclean ring and properly contains the class of a $*$-clean ring. A ring R is a $*$-ring (or ring with involution) if there is an operation $*: R \to R$ such that

$$(a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*, \quad (a^*)^* = a$$

for all $a, b \in R$. An element p of a $*$-ring R is known as a projection if $p^* = p = p^2$, i.e. p is a self-adjoint idempotent. An element a of a $*$-ring R is called $*$-periodic if there exists a positive integer $n > 1$ such that $a^n = p$, where p is a projection. A $*$-ring R is called $*$-semiclean if each element of R is sum of a $*$-periodic element and a unit. Both local and $*$-clean rings are clearly $*$-semiclean, and a $*$-semiclean ring is semiclean. In Section 2, we look at the various basic properties of $*$-periodic elements. In Section 3, we obtain various

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properties of $*$-semiclean rings. Moreover, examples of semiclean rings that are not $*$-semiclean and $*$-semiclean rings that are not $*$-clean are provided. In Section 4, the matrix extension of the $*$-semiclean rings is done. In Section 5, we investigate when a group ring RG is $*$-semiclean. We provide a characterization that when the group ring RC_r and RG are $*$-semiclean, where R is a finite commutative local ring, C_r is a cyclic group of order r, and G is a locally finite abelian group. We obtain several sufficient conditions for the group ring RG to be $*$-semiclean, where R is a commutative local ring and G is one of the groups C_i, $i = 3, 4$ (cyclic group of order 3 and 4), Q_8 (quaternion group of order 8), and Q_{2n} (generalized quaternion group). As a result, numerous examples of $*$-rings that are $*$-semiclean but not $*$-clean have been discovered. Also, we have shown that the group ring $\mathbb{Z}_2 D_6$ is $*$-semiclean but not $*$-clean.

In the paper, the ring R represents an associative ring with unity. The terms $J(R)$, $U(R)$, $I(R)$, $N(R)$, $Pri^*(R)$ and $P(R)$ represent the Jacobson radical, the group of all units, the set of all idempotents, the set of all nilpotents, the set of all $*$-periodic elements, and the set of projections of a ring R, respectively. For a group ring RG, the classical (or standard) involution $* : RG \to RG$ is given by $(\sum_{g\in G} \alpha_g g)^* = \sum_{g\in G} \alpha_g g^{-1}$; see [15, Proposition 3.2.11] for more details. Also, for a ring R, the ring homomorphism $\varepsilon : RG \to R$ defined by $\sum_{g\in G} \alpha_g g = \sum_{g\in G} \alpha_g$ is known as the augmentation mapping of RG. Moreover, the terms $\mathbb{Z}_p$, $\mathbb{Z}_{(p)}$, and $\mathbb{Z}$ represent the ring of integers modulo p, the localization of $\mathbb{Z}$ at the prime ideal generated by p, and the ring of integers, respectively.

2. $*$-Periodic elements

Some properties of $*$-periodic elements are given in this section.

Definition 2.1 *Let R be a $*$-ring. An element $x \in R$ is called $*$-periodic if $x^k = x^l$ (where, l and k are positive integers, $l \neq k$) such that $x^{l(k-l)} = p$, where $p \in P(R)$.*

Theorem 2.2 *Let R be a $*$-ring, and $x \in R$. Then the following statements are equivalent:*

1. *There exists $n \in \mathbb{N}$ such that $x^n = p$, where $p \in P(R)$.*
2. *There exists an integer $n \geq 2$ such that $x = f + a$, where $f^n = f$ and $f^{n-1} = p$, with $p \in P(R)$, $a \in N(R)$ and $xf = fx$.*
3. *x is a $*$-periodic element.*

Proof 1. $\Rightarrow$ 2. Since $x^n = p = p^2 = x^{2n}$, which implies $x^n = x^{2n}$ for some $n \in \mathbb{N}$. Rewrite an element x as $x = x^{n+1} + (x - x^{n+1})$ where $(x^{n+1})^{n+1} = x^{n+1}$ (since $(x^{n+1})^{n+1} = (x^n \cdot x)^{n+1} = (px)^{n+1} = px^{n+1} = px = x^n \cdot x = x^{n+1}$) and $(x^{n+1})^n = p$. Also, $(x - x^{n+1})^n = x^n(1 - x^n)^n = p(1 - p)^n = p(1 - p) = 0$, i.e. $x - x^{n+1} \in N(R)$.

2. $\Rightarrow$ 3. It follows from [4, Lemma 4.3, Definition 4.4].

3. $\Rightarrow$ 1. By Definition 2.1, we can say there exist distinct positive integers l and k such that $x^{l(k-l)} = p$, where $p \in P(R)$. Since $l(k - l) \in \mathbb{N}$, therefore, there exists $n = l(k - l) \in \mathbb{N}$ such that $x^n = p$. □

Let R be a $*$-ring. According to [2, Proposition 2.1], [3, Theorem 3.2], and [3, Theorem 3.6], $x \in R$ is a strongly-π-$*$-regular element if and only if there exists an integer $n \geq 1$ such that $x^n = pu = up$, where $p \in P(R)$ and $u \in U(R)$. For more information on strongly-π-$*$-regular, we can see [5].

Theorem 2.3 *Let R be a $*$-ring, and $x \in R$. Then the following statements are equivalent:*

1. *x is $*$-periodic element.*

2. *x is strongly-π-$*$-regular element, with $u = 1 \in U(R)$.*

Proof 1. $\Rightarrow$ 2. From Theorem 2.2, we get $x^n = p = p \cdot 1$, where $p \in P(R)$ and $1 \in U(R)$; therefore, x satisfies the condition of being strongly-π-$*$-regular with $u = 1 \in U(R)$.
2. $\Rightarrow$ 1. As x is a strongly-π-$*$-regular element, there exists an integer $n \geq 1$ such that $x^n = pu$. Since $u = 1$, which implies $x^n = p$, then by Theorem 2.2, x is $*$-periodic element. □

The following concept is based on the above.

Definition 2.4 *Let R be a $*$-ring. An element $x \in R$ is called $*$-periodic if it satisfies the conditions given in Theorem 2.2 or Theorem 2.3.*

Let R be a $*$-ring. According to [18], an element $x \in R$ is called (strongly) $*$-clean if it can be expressed as $x = p + u$, where $p \in P(R)$ and $u \in U(R)$, with $(pu = up)$.

Lemma 2.5 *Every $*$-periodic element is strongly-$*$-clean.*

Proof Let x be a $*$-periodic element. By Theorem 2.2, an integer $n \geq 1$ exists, and $p \in P(R)$, such that $x^n = p$. Clearly, $1 - p = f$ is a projection. If we prove that $u = x - (1 - p)$ is a unit, then it will complete the proof. Define

$$v = x^{n-1}p - (1 + x + \cdots + x^{n-1})(1 - p).$$

Rewrite the term u as $u = xp - (1 - x)(1 - p)$. Evaluate the term uv, we have

$$\begin{aligned} uv &= (xp - (1 - x)(1 - p))(x^{n-1}p - (1 + x + \cdots + x^{n-1})(1 - p)) \\ &= x^n p + (1 - x)(1 + x + \cdots + x^{n-1})(1 - p) \\ &= p + (1 - x^n)(1 - p) \\ &= 1. \end{aligned}$$

Clearly, $uv = vu$. Therefore, we get $uv = vu = 1$, which implies u is a unit with inverse v. Hence, $x = f + u$, where $f \in P(R)$ and $u \in U(R)$. Clearly, $fu = a + p - ap - 1 = uf$. Hence, element x is strongly $*$-clean. □

3. $*$-Semiclean rings

Let R be a $*$-ring. In 2003, Y. Ye introduced the class of semiclean rings [21]. The notion of $*$-semiclean rings can be perceived as a $*$-versions of the semiclean ring. In this section, the definition and properties of $*$-semiclean rings are given.

Definition 3.1 *A $*$-ring R is $*$-semiclean if every element in it can be written as the sum of a $*$-periodic element and a unit.*

Proposition 3.2 *A $*$-ring R is $*$-semiclean if it is semiclean, and every idempotent is a projection.*

Corollary 3.3 *The group ring $\mathbb{Z}_{(p)}C_3$, where C_3 is a cyclic group of order 3, is $*$-semiclean for every prime p.*

Proof [21, Theorem 3.1] states that the group ring $\mathbb{Z}_{(p)}C_3$ is semiclean, and [21, proposition 3.1] tells us that the only idempotents of the group ring $\mathbb{Z}_{(p)}C_3$ are 0, 1, $\frac{1}{3}+\frac{1}{3}a+\frac{1}{3}a^2$ and $\frac{2}{3}-\frac{1}{3}a-\frac{1}{3}a^2$. Since 0^* is 0, 1^* is 1, $(\frac{1}{3}+\frac{1}{3}a+\frac{1}{3}a^2)^*$ is $\frac{1}{3}+\frac{1}{3}a+\frac{1}{3}a^2$, and $(\frac{2}{3}-\frac{1}{3}a-\frac{1}{3}a^2)^*$ is $\frac{2}{3}-\frac{1}{3}a-\frac{1}{3}a^2$, this implies that every idempotent is a projection. Hence, by Proposition 3.2, $\mathbb{Z}_{(p)}C_3$ is $*$-semiclean for every prime p. □

There exists an example of $*$-ring which is clean but not $*$-clean ring.

Example 3.4 *Let $R=\mathbb{Z}_2\times\mathbb{Z}_2$ be a commutative ring. Now, define a map $*: R\to R$ such that $(a,b)^*=(b,a)$. Then R is a clean ring, but it is not $*$-clean ring as idempotents do not coincide with projection.*

Similarly, there exists a $*$-ring that is semiclean but not $*$-semiclean; in fact, we obtain the following relations between the classes of rings:

$$\begin{array}{ccccccc} *\text{-periodic} & \Rightarrow & \text{strongly-}\pi\text{-}*\text{-regular} & \Rightarrow & *\text{-clean} & \Rightarrow & *\text{-semiclean} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{periodic} & \Rightarrow & \text{strongly-}\pi\text{-regular} & \Rightarrow & \text{clean} & \Rightarrow & \text{semiclean} \end{array}$$

The examples given below show that the above relations are irreversible.

Example 3.5 1. *Let $R=\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix}\}$ (where $0,1\in\mathbb{Z}_2$) be a commutative ring under the usual addition and multiplication. Clearly, the ring R is semiclean. Now, define a map $*: R\to R$ such that $\begin{bmatrix}x&y\\z&w\end{bmatrix}^*=\begin{bmatrix}x+y&y\\x+y+z+w&y+w\end{bmatrix}$. The only way of representing the element $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ as sum of the periodic and the unit is $\begin{bmatrix}0&1\\0&1\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}$, but $\begin{bmatrix}0&1\\0&1\end{bmatrix}\notin Pri^*(R)$. Hence, it is not $*$-semiclean.*

2. *By Corollary 3.3, the group ring $\mathbb{Z}_{(7)}C_3$, where C_3 is a cyclic group of order 3, generated by a, is $*$-semiclean. However, the element $2+3a$ of $\mathbb{Z}_{(7)}C_3$ is not clean. Thus, the group ring $\mathbb{Z}_{(7)}C_3$ is not $*$-clean.*

3. *The ring F_3C_8 is finite; therefore, it is clean, but by [16, Example 3.12], it is not $*$-clean.*

4. *Let $R=\mathbb{Z}_5\bigoplus\mathbb{Z}_5$ be a ring. Define an involution map $*: R\to R$ such that $(a,b)^*=(b,a)$. The ring R is strongly-π-regular, but it is not strongly-π-$*$-regular as idempotents do not coincide with projections.*

5. *The ring $R=F_{7^2}C_8$ is finite, so it is periodic, but by [16, Example 3.10], it is not $*$-clean, and thus according to Lemma 2.5, it is not $*$-periodic.*

Theorem 3.6 *Let R be a $*$-ring, with $2\in U(R)$. Then R is semiclean, and every unit is self-adjoint, i.e. $v^*=v$ for all $v\in U(R)$ if and only if R is $*$-semiclean and $*=1_R$.*

Proof $\Rightarrow$ Let $a\in R$. Then, by Definition 3.1, we have $a=f+v$, where $f^{2n}=f^n$ and $v\in U(R)$. Observe that $(1-2f^n)^2=1$. Because every unit of R is self-adjoint, $2f^{n*}=2f^n$. As a result, $2(f^{n*}-f^n)=0$.

Because $2 \in U(R)$, $f^{n*} = f^n$, implying that an element $a \in R$ is $*$-semiclean. Because $f \in R$ is periodic, and every periodic is clean, so $f = f^{'} + v^{'}$, where $f^{'} \in I(R)$ and $v^{'} \in U(R)$. Observe that $(1-2f^{'})^2 = 1$. Because every unit of R is self-adjoint, $2f^{'*} = 2f^{'}$. As a result, $2(f^{'*} - f^{'}) = 0$. Because $2 \in U(R)$, $f^{'*} = f^{'}$, implying that $f^* = f$. Hence, $a^* = a$, so $* = 1_R$.

$\Leftarrow$ Obvious. □

If an element x is self-adjoint square root of 1, it fulfills the conditions $x^2 = 1$ and $x^* = x$.

Every element of a $*$-clean ring in which 2 is invertible is shown to have a sum of no more than 2 units by Jian Cui and Zhou Wang [3]. We extended this finding to $*$-semiclean rings using Theorem 3.7 and demonstrated that each element of a $*$-semiclean ring can be expressed as the sum of three units.

Theorem 3.7 *Let R be a $*$-semiclean ring with $2 \in U(R)$. Then every element of R is the sum of a self-adjoint square root of 1 and two units.*

Proof Let $a \in R$. Then $\frac{a+1}{2} = f + v$, where $f \in Pri^*(R)$ and $v \in U(R)$. Because $f \in Pri^*(R)$, $f^n = f^{2n}$, and $f^n = p = p^*$. According to Lemma 2.5, $f = f^{'} + v^{'}$, where $f^{'} = (1-p) \in P(R)$ and $v^{'} \in U(R)$. Thus, $a = (2-2p) - 1 + 2v^{'} + 2v = (1-2p) + 2v^{'} + 2v$, where $(1-2p)^* = 1-2p$ and $(1-2p)^2 = 1$, with $2v^{'}$, $2v \in U(R)$. □

An ideal I of a $*$-ring R is called $*$-invariant if $I^* \subseteq I$. Lemma 3.8 extends an involution $*$ of R to the factor ring R/I, which is still denoted by $*$.

Lemma 3.8 *Let R be $*$-semiclean and I be $*$-invariant ideal, then the ring R/I is $*$-semiclean. In particular, the ring $R/J(R)$ is $*$-semiclean.*

Proof By [21, Proposition 2.1], the homomorphic image of semiclean is semiclean. Also, the homomorphic image of projection is projection. Thus, the result holds. Since an ideal $J(R)$ is $*$-invariant, therefore, $R/J(R)$ is $*$-semiclean. □

Every polynomial ring over a commutative ring is not $*$-semiclean, as shown in Example 3.9.

Example 3.9 *Let R be a commutative ring. Then the polynomial ring $R[x]$ is not $*$-semiclean.*

Proof By [21, Example 3.2], the polynomial ring $R[x]$ is never semiclean. Hence, for any involution $*$, the ring $R[x]$ is not $*$-semiclean. □

Let R be a $*$-ring and $R[[x]]$ be a power series ring. Then, on $R[[x]]$, an induced involution $*$ is defined as $(\sum_{i=0}^{\infty} \alpha_i x^i)^* = \sum_{i=0}^{\infty} \alpha_i^* x^i$. In 2003, Yuanqing Ye [21] proved that the ring $R[[x]]$ is semiclean if and only if R is semiclean. This result has been extended to $*$-semiclean by Proposition 3.10.

Proposition 3.10 *The ring $R[[x]]$ is $*$-semiclean if and only if R is $*$-semiclean.*

Proof $\Rightarrow$ Let $R[[x]]$ be $*$-semiclean. Because $R \cong R[[x]]/(x)$ and (x) is a $*$- invariant ideal of $R[[x]]$, R is $*$-semiclean according to Lemma 3.8.

$\Leftarrow$ Let R be $*$-semiclean and $g(x) = \sum_{i=0}^{\infty} \alpha_i x^i \in R[[x]]$. If $\alpha_0 = f + v$, where $f \in Pri^*(R)$ and $v \in U(R)$, then $g(x) = f + (v + \sum_{i=1}^{\infty} \alpha_i x^i)$, where $f \in Pri^*(R) \subseteq Pri^*(R[[x]])$ and $v + \sum_{i=1}^{\infty} \alpha_i x^i \in U(R[[x]])$. As a result, $g(x) \in R[[x]]$ is $*$-semiclean. □

Every $*$-clean ring is a $*$-semiclean ring, but the converse is not true. By Theorem 3.11, we demonstrate that, under certain conditions, the converse will also hold.

Theorem 3.11 *Let R be a torsion free ring, and $z \in R$ such that $z = b+v$, where $b \in Pri^*(R)$ and $v \in U(R)$. If $v = \pm 1$, then z is $*$-clean.*

Proof Case I: Let $v = 1$
Rewrite an element $z \in R$ as $z = b+1$, $b^k = b^l$ (where, l and k are positive integers such that $l > k$), and $b^{k(l-k)} = p = p^* \in P(R)$.
We have $(z-1)^k = (z-1)^l$ because $b^k = b^l$, which implies that $(1-z)^{2k} = (1-z)^{2l}$ and $(1-z)^{2k(2l-2k)} = p$. As a result, $1-z$ is $*$-periodic, and thus, according to Lemma 2.5, an element $1-z$ is $*$-clean, i.e. $1-z = f+u$, where $f = (1-p) \in P(R)$, and $u \in U(R)$. To put it simply, $z = p+u^{'}$, where $p \in P(R)$ and $u^{'} = -u \in U(R)$.

Case II: Let $v = -1$
Then an element $z \in R$ is rewritten as $z = b-1$.

1. Let $b = b^n$ (where, n is a positive integer such that $n > 1$).
   Then $z = b^{n-1} + (-1+b-b^{n-1})$. Because $b \in Pri^*(R)$ and $b = b^n$, an element $b^{n-1} \in P(R)$. An element $-1+b-b^{n-1}$ is a unit in R, with the inverse $(2^{n-1} - 1 + 2^{n-3}b + 2^{n-4}b^2 + \cdots + b^{n-2} + (1 - 2^{n-2})b^{n-1})(1-2^{n-1})^{-1} \in R$. Hence, $z = b-1$ is $*$-clean.

2. Let $b^k = b^l$ (where, l and k are positive integers such that $l > k$).
   Then $z = b^{k(l-k)} + (-1+b-b^{k(l-k)})$. Because $b \in Pri^*(R)$ and $b^k = b^l$, an element $b^{k(l-k)} \in P(R)$. An element $-1+b-b^{k(l-k)}$ is a unit in R. Hence, $z = b-1$ is $*$-clean.

□

4. Matrix extension of $*$-semiclean rings

If R is a $*$-ring, then $M_n(R)$ the ring of $n \times n$ matrices over R inherits the natural involution from R: if $A = (a_{ij})$, then A^* is the transpose of (a_{ij}^*). In 2010, Lia Vaš [18] proved that if both pRp and $(1-p)R(1-p)$ are $*$-clean rings (here p is a projection), then R is $*$-clean. As a result, the $M_n(R)$ (ring of $n \times n$ matrices over R) is $*$-clean. This result has been extended to $*$-semiclean rings in this section.

Lemma 4.1 *If pRp and $(1-p)R(1-p)$ are both $*$-semiclean, where $p \in P(R)$, then R is also $*$-semiclean.*

Proof For each $p \in R$, write $1-p = \overline{p}$. Apply the Pierce decomposition of the ring R:

$$R = \begin{bmatrix} pRp & pR\overline{p} \\ \overline{p}Rp & \overline{p}R\overline{p} \end{bmatrix}.$$

Let $M = \begin{bmatrix} m & n \\ o & q \end{bmatrix} \in R$. Thus, $m = a+u$, where $a \in Pri^*(pRp)$ such that $a^{k_1} = a^{l_1}$ (where, l_1 and k_1 are possitive integers such that $l_1 > k_1$) and u is a unit in pRp with inverse u_1. Then, $q - nu_1o \in \overline{p}R\overline{p}$. So $q - ou_1n = b+v$, where $b \in Pri^*(\overline{p}R\overline{p})$ such that $b^{k_2} = b^{l_2}$ (where, l_2 and k_2 are possitive integers such that

$l_2 > k_2$) and v is a unit in $\overline{p}R\overline{p}$ with inverse v_1. Thus,

$$M = \begin{bmatrix} a+u & n \\ o & b+v+nu_1o \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} u & n \\ o & v+ou_1n \end{bmatrix}.$$

To show: $\begin{bmatrix} u & n \\ o & v+ou_1n \end{bmatrix}$ is unit in R.

Compute, $\begin{bmatrix} p & 0 \\ -ou_1 & \overline{p} \end{bmatrix} \begin{bmatrix} u & n \\ o & v+ou_1n \end{bmatrix} \begin{bmatrix} p & -u_1n \\ 0 & \overline{p} \end{bmatrix} = \begin{bmatrix} u & n \\ 0 & v \end{bmatrix} \begin{bmatrix} p & -u_1n \\ 0 & \overline{p} \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$. Since the matrices $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$, $\begin{bmatrix} p & 0 \\ -ou_1 & \overline{p} \end{bmatrix}$, and $\begin{bmatrix} p & -u_1n \\ 0 & \overline{p} \end{bmatrix}$ are units in $\begin{bmatrix} pRp & pR\overline{p} \\ \overline{p}Rp & \overline{p}R\overline{p} \end{bmatrix}$, therefore, $\begin{bmatrix} u & n \\ o & v+ou_1n \end{bmatrix}$ is unit in R.

To show: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is $*$-periodic, i.e. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^l$ and $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{k(l-k)} \in P(R)$ (where, l and k are the possitive integer such that $l > k$).

Without loss of generality, let $k_2 \geqslant k_1$.

$a^{k_1} = a^{l_1} = a^{(l_1-k_1)+k_1} = a^{s(l_1-k_1)+k_1}$,
$b^{k_2} = b^{l_2} = b^{(l_2-k_2)+k_2} = b^{s(l_2-k_2)+k_2}$, and
$a^{k_2} = a^{k_1+(k_2-k_1)} = a^{s(l_1-k_1)+k_2}$.
Let $k = k_2$ and $l = (l_1-k_1)(l_2-k_2)+k_2$. Then, $a^k = a^l$ and $b^k = b^l$.

Thus, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} a^l & 0 \\ 0 & b^l \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^l$. Hence, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is periodic.

As $a \in Pri^*(pRp)$ and $a^k = a^l$. Thus, $a^{k(l-k)} = p_1$, where $p_1 \in P(pRp)$.
Similarly, $b \in Pri^*(\overline{p}R\overline{p})$ and $b^k = b^l$. Thus, $b^{k(l-k)} = 1 - p_2$, where $p_2 \in P(\overline{p}R\overline{p})$.

Compute, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{k(l-k)} = \begin{bmatrix} a^{k(l-k)} & 0 \\ 0 & b^{k(l-k)} \end{bmatrix} = \begin{bmatrix} p_1 & 0 \\ 0 & 1-p_2 \end{bmatrix} \in P(R)$.

This proves that matrix M is $*$-semiclean. Therefore, R is $*$-semiclean. □

By Lemma 4.1, and an inductive argument, the next result holds.

Theorem 4.2 *If $p_1, p_2, \cdots, p_n$ are orthogonal projections with $1 = p_1+p_2+\cdots+p_n$, and p_iRp_i is $*$-semiclean for each i, then R is $*$-semiclean.*

The following two conclusions follow directly from Theorem 4.2.

Corollary 4.3 *If R is $*$-semiclean, then so is $M_n(R)$.*

Corollary 4.4 *If $N = N_1 \bigoplus N_2 \bigoplus \cdots \bigoplus N_n$ are modules and $End(N_i)$ is $*$-semiclean for each i, then $End(N)$ is $*$-semiclean.*

5. $*$-Semiclean group rings

In this section, we obtain several results pertaining to commutative and noncommutative $*$-semiclean group rings. Throughout this section, we are considering standard involution on the group ring RG.

Theorem 5.1 *If RG is a $*$-semiclean ring, then so is $((R/J(R))G$.*

Proof Define a map $\Psi : RG \to (R/J(R))G$ as $\Psi(\sum_{g\in G}\alpha_g g) = \sum_{g\in G}\Psi(\alpha_g)g$, $\Psi(\alpha_g) = \alpha_g + J(R)$. Note that Ψ is an onto map. The map Ψ preserves an involution $*$ as $\Psi(\sum_{g\in G}\alpha_g g)^* = (\Psi(\sum_{g\in G}\alpha_g g))^*$. Let $\overline{x} \in (R/J(R))G$. Since Ψ is an onto map, there exists an element $x \in RG$, which is defined as $x = f + u$, where $f \in Pri^*(RG)$ and $u \in U(RG)$. So, $\overline{x} = \Psi(f) + \Psi(u)$, where $\Psi(f) \in Pri^*((R/J(R))G)$ and $\Psi(u) \in U((R/J(R))G)$. Hence, $((R/J(R))G$ is a $*$-semiclean ring. □

5.1. Abelian group rings

In 2015 [6], Gao, Chen, and Li found out that when the group rings RC_3, RC_4, RS_3, and RQ_8 are $*$-clean, where R is a commutative local ring. In this section, we have extended this result to $*$-semiclean rings. As a consequence, many examples of group rings that are $*$-semiclean but not $*$-clean have been obtained. In Theorem 5.7 and 5.8, a characterization that when the group rings RC_r and RG are $*$-semiclean is obtained (respectively). Here, R is a finite commutative local ring, C_r is a cyclic group of order r, and G is a locally finite abelian group.

Proposition 5.2 ([13]) *If R is local, G is a locally finite p-group, and $p \in J(R)$, then the group ring RG is local.*

We now investigate when RC_3 is $*$-semiclean.

In 2015 [6], Gao, Chen, and Li investigated the group rings RC_3 and $\mathbb{Z}_pC_3$ and proved that if $(-3)^{\frac{p-1}{2}} \equiv 1(mod\ p)$, then the group ring $\mathbb{Z}_pC_3$ is not $*$-clean; however, Theorem 5.3(3) demonstrates that it is $*$-semiclean. Furthermore, in Theorem 5.3(2), we relaxed the requirement that RC_3 be clean, allowing us to broaden the class of rings (rings that are $*$-semiclean but not $*$-clean are obtained). One such example is $\mathbb{Z}_{(7)}C_3$, which is explained below.

Theorem 5.3 *Let R be a commutative local ring and $G = C_3 = \langle x \rangle$ be a cyclic group of order 3.*

1. *If $3 \notin U(R)$, then RC_3 is $*$-semiclean.*
2. *If $3 \in U(R)$ and the equation $z^2 + z + 1 = 0$ has no solutions in R, then the ring RC_3 is $*$-semiclean.*
3. *If $2 \in U(R)$, then RC_3 is $*$-semiclean if RC_3 is clean and $U(RC_3)$ is a torsion group.*

Proof

1. Since $3 \in J(R)$, by Proposition 5.2, RC_3 is local. Hence, RC_3 is a $*$-semiclean.
2. According to [10, Theorem 2.7], the ring RC_3 is a semiclean ring. By [6, Theorem 2.4], if the equation $z^2 + z + 1 = 0$ has no solution in R, then every idempotent of the ring RC_3 is a projection. Hence, by Proposition 3.2, the ring RC_3 is a $*$-semiclean ring.
3. If RC_3 is clean and $2 \in U(RC_3)$, then by [20, Proposition 2.5], RC_3 is a 2-*good* ring. If an element $a \in RC_3$, then there exist $u_1, u_2 \in U(RC_3)$ such that $a = u_1 + u_2$, according to the definition of a 2-*good* ring. Because $U(RC_3)$ is a torsion group, there exists $m \in \mathbb{N}$ such that $u_1^m = 1 = 1^*$, implying that

$u_1 \in Pri^*(RC_3)$ and $u_2 \in U(RC_3)$. Thus, element a is $*$-semiclean. Since a is an arbitary element of RC_3, therefore, every element of RC_3 is $*$-semiclean. Hence, RC_3 is a $*$-semiclean ring.

□

The examples given below are the direct consequences of Theorem 5.3.

Example 5.4 *1. By Theorem 5.3(1), the ring $\mathbb{Z}_3C_3$ is $*$-semiclean.*

2. *The ring $\mathbb{Z}_{(7)}C_3$ is $*$-semiclean because the equation $z^2+z+1=0$ has no solution in $\mathbb{Z}_{(7)}$, but it is not $*$-clean because, according to [14], $\mathbb{Z}_{(p)}C_3$ is clean if and only if $p \not\cong 1(mod 3)$.*

3. *By [22, Corollary 19], we can say that $\mathbb{Z}_pC_3$, where $p>2$ is prime, is clean. Also, as $2 \in U(\mathbb{Z}_pC_3)$, by Theorem 5.3(3a), we conclude $\mathbb{Z}_pC_3$ is $*$-semiclean, but by [6, Example 2.7], for $p>3$, if $(-3)^{\frac{p-1}{2}} \equiv 1(mod\ p)$, it is not $*$-clean.*

We now investigate when RC_4 is $*$-semiclean.

In 2015 [6], Gao, Chen, and Li investigated the group rings RC_4 and $\mathbb{Z}_pC_4$, and proved that if $p \equiv 1(mod\ 4)$, then the group ring $\mathbb{Z}_pC_4$ is not $*$-clean; however, Theorem 5.5$(2b)$ demonstrates that it is $*$-semiclean. Furthermore, in Theorem 5.5$(2a)$, we relaxed the requirement that RC_4 be clean, allowing us to broaden the class of rings (rings that are $*$-semiclean but not $*$-clean are obtained). One such example is $\mathbb{Z}_{(5)}C_4$, which is explained below.

Theorem 5.5 *Let R be a commutative local ring and $G=C_4=\langle x\rangle$ be a cyclic group of order 4.*

1. *If $2 \notin U(R)$, then RC_4 is $*$-semiclean.*
2. *If $2 \in U(R)$, then RC_4 is $*$-semiclean if any of the condition given below is satisfied.*
   (a) The equation $z^2+1=0$ has no solutions in R.
   (b) RC_4 is clean and $U(RC_4)$ is torsion group.

Proof

1. Since $2 \in J(R)$, by Proposition 5.2, RC_4 is local. Hence, RC_4 is a $*$-semiclean.
2. (a) According to [10, Theorem 2.7], the ring RC_4 is a semiclean ring. By [6, Theorem 2.10], if the equation $z^2+1=0$ has no solution in R, then every idempotent of the ring RC_4 is a projection. Hence, by Proposition 3.2, the ring RC_4 is a $*$-semiclean ring.
   (b) The proof is similar to the proof of Theorem 5.3(3).

□

The examples given below are the direct consequences of Theorem 5.5.

Example 5.6 *1. The ring $\mathbb{Z}_{(5)}C_4$ is $*$-semiclean because the equation $z^2+1=0$ has no solution in $\mathbb{Z}_{(5)}$, but it is not $*$-clean because, according to [14], $\mathbb{Z}_{(5)}C_4$ is not clean.*

2. *By [22, Corollary 19], we can say that* $\mathbb{Z}_pC_4$*, where* $p > 2$ *is prime, is clean. Also, as* $2 \in U(\mathbb{Z}_pC_4)$*, by Theorem 5.5*(2b)*, we conclude* $\mathbb{Z}_pC_4$ *is* $*$*-semiclean, but by [6, Corollary 2.11], for* $p \equiv 1 \pmod 4$*,* $\mathbb{Z}_pC_4$ *is not* $*$*-clean.*

By using Theorem 5.7 and Theorem 5.8, we can find various other examples of $*$-semiclean rings that are not $*$-clean. Some of them are listed in Example 5.9.

Theorem 5.7 *Let* R *be a finite commutative local ring.*

1. *If* $2 \in U(R)$ *and* $C_r = \langle x \rangle$ *is a cyclic group of order* r*, then* RC_r *is* $*$*-semiclean.*

2. *If* $2 \in J(R)$*,* $C_r = \langle x \rangle$ *is a cyclic group of order* $r = 2^s t$ $(s \geq 0)$*, where* $2 \not| t$*, and* γ *is the cyclic permutation on the set* $J = \{1, 2, \cdots, t-1\}$ *defined as* $\gamma : J \to J$ *by* $j \to 2j (mod\ t)$*, then* RC_r *is* $*$*-semiclean.*

Proof

1. Let $x \in RC_r$. The group ring RC_r is periodic because it is finite. Thus, according to [21, Lemma 5.1], RC_r is clean. Furthermore, $2 \in U(R)$. Thus, by [20, Proposition 2.5], RC_r is a 2-*good* ring, i.e. $x = u_1 + u_2$, where $u_1, u_2 \in U(RC_r)$. As RC_r is periodic, according to [2, Proposition 2.3], $U(RC_r)$ is a torsion group. Because $u_1 \in U(RC_r)$, there exists $n \in \mathbb{N}$ such that $u_1^n = 1 = 1^*$. Thus, $u_1 \in Pri^*(RC_r)$ and $u_2 \in U(RC_r)$. As a result, an element x meets the condition of being $*$-semiclean. Hence, RC_r is $*$-semiclean.

2. Let $s \geq 1$. Then $C_r \cong C_{2^s} \times C_t$. Thus, $RC_r \cong (RC_{2^s})C_t$, where $C_t = \langle x \rangle$ is a cyclic group of order t. By [13, Theorem], $R' = RC_{2^s}$ is the local ring. Since $(R/J(R))$ is a field of char $= 2$ and $(R/J(R))C_{2^s} \to (R'/J(R'))$ is ring epimorphism, therefore, $(R'/J(R'))$ is also a field of char $= 2$. Let $a = a_0 + a_1 x + a_2 x^2 + \cdots + a_{t-1} x^{t-1}$ be an idempotent element of $(R'/J(R'))C_t$. Because $2 = 0$ and $x^t = 1$, it follows that $a^2 = a_0^2 + a_{\gamma(1)} x^{\gamma(1)} + \cdots a_{\gamma(t-1)} x^{\gamma(t-1)}$. Because γ is the cyclic permutation on the set $J = \{1, 2, \cdots, t-1\}$, therefore, $a_0^2 = a_0$ and $a_1^2 = a_1 = a_2 = \cdots = a_{t-1}$. So the idempotents of $(R'/J(R'))C_t$ are $0,\ 1,\ 1 + x + \cdots + x^{t-1}$, and $x + x^2 + \cdots + x^{t-1}$. Because $0^* = 0$, $1^* = 1$, $(1 + x + \cdots + x^{t-1})^* = 1 + x + \cdots + x^{t-1}$ and $(x + x^2 + \cdots + x^{t-1})^* = x + x^2 + \cdots + x^{t-1}$, implying that $(R'/J(R'))C_t$ has four idempotents, all of which are projections. Now, because C_t is a locally finite group, $J(R')C_t \subseteq J(R'C_t)$. As the $(\text{char}(R'/J(R')), t) = 1$, therefore, $(R'/J(R'))C_t$ is semisimple, implying that $R'J(C_t) = J(R'C_t)$. Therefore, we get $(R'/J(R'))C_t \cong R'C_t/J(R')C_t = R'C_t/J(R'C_t)$. Thus, the factor ring $R'C_t/J(R'C_t) = \overline{R'C_t}$ will also have only four idempotents : $\overline{0},\ \overline{1},\ \overline{1} + \overline{x} + \cdots + \overline{x}^{t-1}$, and $\overline{x} + \overline{x}^2 + \cdots + \overline{x}^{t-1}$, all of which are projections. Since the order of the ring $\overline{R'C_t}$ is finite, $\overline{R'C_t}$ is clean. Thus, $\overline{R'C_t}$ is $*$-clean, i.e. for each $\overline{a} \in \overline{R'C_t}$, there exist $\overline{p} \in P(\overline{R'C_t})$ and $\overline{u} \in U(\overline{R'C_t})$, such that $\overline{a} = \overline{p} + \overline{u}$. Moreover, in $R'C_t$ the elements $m_1 = 0$, $m_2 = 1$, $m_3 = t^{-1}(1 + x + \cdots + x^{t-1})$, and $m_4 = t^{-1}((t-1) - x - x^2 - \cdots - x^{t-1})$ are projections such that $\overline{m_1} = \overline{0}$, $\overline{m_2} = \overline{1}$, $\overline{m_3} = \overline{1} + \overline{x} + \cdots + \overline{x}^{t-1}$, and $\overline{m_4} = \overline{x} + \overline{x^2} + \cdots + \overline{x}^{t-1}$ which implies there exists a $n_1 = p \in P(R'C_t)$ such that $\overline{n_1} = \overline{p}$ for $\overline{p} \in P(\overline{R'C_t})$. There is also $n_2 = u \in U(R'C_t)$ such that $\overline{n_2} = \overline{u}$ for $\overline{u} \in U(\overline{R'C_t})$. Thus, there exists an element $n_3 = p + u \in R'C_t$ such that $\overline{n_3} = \overline{p} + \overline{u}$ for $\overline{p} + \overline{u} \in \overline{R'C_t}$. Then $\overline{n_3} = \overline{a}$, i.e. $a - n_3 \in J(R'C_t)$.

Since $R^{'}$ is finite, $R^{'}$ is an artinian ring, which implies $J(R^{'})$ is nilpotent. Thus, $J(R^{'})C_t$ is nil-ideal. By [11, Corollary 4.3], $J(R^{'})C_t$ is nilpotent. Since $J(R^{'}C_t) = J(R^{'})C_t$, the ideal $J(R^{'}C_t)$ is also nilpotent. Since $a - n_3 \in J(R^{'}C_t)$, therefore, $a - n_3 = a - (p + u) = k$ for some $k \in J(R^{'}C_t)$. Simplifying it, we get $a = p + u + k$, where $p \in P(R^{'}C_t)$, $u \in U(R^{'}C_t)$, and $k \in J(R^{'}C_t)$. Thus, $a = p + v$, where $p \in P(R^{'}C_t)$ and $v = (u + k) \in U(R^{'}C_t)$. As a result, an element a meets the condition of being $*$-clean. Hence, $RC_r = R^{'}C_t$ is $*$-clean. Thus, RC_r is $*$-semiclean.

□

Theorem 5.8 *Let R be a finite commutative local ring and G be a locally finite abelian group.*

1. *If $2 \in U(R)$, then RG is $*$-semiclean.*
2. *If $2 \in J(R)$ and G is a locally finite 2-group, then RG is $*$-semiclean.*
3. *If $2 \in J(R)$ with $R/J(R) \cong \mathbb{F}_2$ and exponent of G is r, where r is an odd positive integer, and a $q \in \mathbb{N}$ exists such that $2^q \equiv -1(mod\ r)$, then RG is $*$-semiclean.*

Proof

1. Let $x \in RG$. Since G is a locally finite abelian group, there exists a finite subgroup H such that $x \in RH$. The rest of the proof is similar to that of Theorem 5.7(1).
2. Since $2 \in J(R)$, by Proposition 5.2, RG is local. Hence, RG is $*$-semiclean.
3. We will first show that the group ring $\overline{RG'}$ is $*$-clean for any arbitary finite abelian group, say $G^{'}$ (with odd exponent say r) such that $2^q \equiv -1(mod\ \text{r})$ for some $q \in \mathbb{N}$. Let $a = x_1 + x_2 + \cdots + x_t$ be the idempotent element of $(R/J(R))G^{'}$, where $x_i \in G^{'}$ for $i = 1$ to t. Then $(x_1 + x_2 + \cdots + x_t)^2 = x_1^2 + x_2^2 + \cdots + x_t^2 = x_1 + x_2 + \cdots + x_t$. Thus, $\{x_1, x_2, \cdots, x_t\} = \{x_1^2, x_2^2, \cdots, x_t^2\}$. Furthermore, if $x \in \{x_1, x_2, \cdots, x_t\}$, then $x^{2^k} \in \{x_1, x_2, \cdots, x_t\}$ for some $k \in \mathbb{N}$. Thus, an element x can be rewritten as $x = (x_{k_1} + x_{k_1}^2 + \cdots + x_{k_1}^{2^{m_1}}) + \cdots + (x_{k_j} + x_{k_j}^2 + \cdots + x_{k_j}^{2^{m_j}})$. Here the elements x_{k_i} are distinct and m_i's are the smallest positive integers such that $x_{k_i}^{2^{m_i+1}} = x_{k_i}$. Evaluating x^*, we have $x^* = (x_{k_1}^{-1} + x_{k_1}^{-2} + \cdots + x_{k_1}^{-2^{m_1}}) + \cdots + (x_{k_j}^{-1} + x_{k_j}^{-2} + \cdots + x_{k_j}^{-2^{m_j}})$. Since, for some $q \in \mathbb{N}$, we have $2^q \equiv -1(mod\ \text{p})$, thus, clearly $a^* = a$, i.e. every idempotent of $(R/J(R))G^{'}$ is a projection. Now, as the order of $(R/J(R))G^{'}$ is finite, it is a clean ring. As a result, the ring $(R/J(R))G^{'}$ is $*$-clean. Now, as G is a locally finite group, therefore, $J(R)G^{'} \subseteq J(RG^{'})$. Since order of every element of $G^{'}$ is invertible in $(R/J(R))$, therefore, $(R/J(R))G^{'}$ is semisimple. Thus, $J(R)G^{'} = J(RG^{'})$. Therefore, we get $(R/J(R))G^{'} \cong RG^{'}/J(RG^{'})$. Thus, every idempotent of $RG^{'}/J(RG^{'})$ is a projection. Being the ring $RG^{'}/J(RG^{'}) = \overline{RG'}$ of finite order, it is a clean ring. Thus, it is a $*$-clean ring.
   Let $z \in RG$. Since G is a locally finite abelian group, there exists a finite abelian subgroup H such that $z \in RH$. For $l_1 = z \in RH$, there exists a $\overline{z} \in \overline{RH}$ such that $\overline{l_1} = \overline{z}$. Because $\overline{z} \in \overline{RH}$, and because, as explained above, the group ring $\overline{RH}$ is a $*$-clean, there exists $\overline{p} \in P(\overline{RH})$ and $\overline{u} \in U(\overline{RH})$, such that

$\overline{z} = \overline{p} + \overline{u}$. Because $J(RH)$ is the $*$-invariant nil ideal of a $*$-ring RH, there exists a $n_1 = p \in P(RH)$ such that $\overline{n_1} = \overline{p}$ for $\overline{p} \in P(\overline{RH})$. There is also $n_2 = u \in U(RH)$ such that $\overline{n_2} = \overline{u}$ for $\overline{u} \in U(\overline{RH})$. Thus, there exists an element $n_3 = p + u \in RH$ such that $\overline{n_3} = \overline{p} + \overline{u}$ for $\overline{p} + \overline{u} \in \overline{RH}$. Thus, $\overline{n_3} = \overline{z}$, i.e. $z - n_3 \in J(RH)$. Also, the ideal $J(RH)$ is nilpotent. Since $z - n_3 \in J(RH)$, $z - n_3 = z - (p + u) = k$ for some $k \in J(RH)$. Simplifying it, we get $z = p + u + k$, where $p \in P(RH)$, $u \in U(RH)$, and $k \in J(RH)$. Thus, $z = p + v$, where $p \in P(RH)$, and $v = (u + k) \in U(RH)$. As a result, element z meets the condition of being $*$-clean. Hence, RH is $*$-clean. Thus, RH is $*$-semiclean, which implies RG is $*$-semiclean.

□

The examples given below are the direct consequences of Theorem 5.7 and Theorem 5.8. These are $*$-semiclean but not $*$-clean group rings.

Example 5.9 *1. The ring F_3C_8 is $*$-semiclean, but by [16, Example 3.12], it is not $*$-clean.*

2. The ring $F_7(C_4 \times C_8)$ is $*$-semiclean, but by [16, Example 3.10(1)], it is not $*$-clean.

3. The ring F_3C_{35} is $*$-semiclean, but by [8, Example 3.3], it is not $*$-clean.

5.2. Non-abelian group rings

In this section, we investigate when a non-abelian group ring RG is $*$-semi-clean, where R is a commutative local ring and G is Q_8, Q_{2n}, D_{2n}, and D_6.

5.2.1. Quaternion group Q_8

The group ring $\mathbb{Z}_pQ_8$ was studied by Gao in [6], and it was shown that it is not $*$-clean; however, by Theorem 5.10, we obtain that it is $*$-semiclean.

Theorem 5.10 *Let R be a commutative local ring and $G = Q_8 = \langle x, y | x^4 = 1, x^2 = y^2, yx = x^{-1}y \rangle$ be a quaternion group of order 8.*

1. If $2 \notin U(R)$, then RQ_8 is $*$-semiclean.

2. If $2 \in U(R)$, RQ_8 is clean and $U(RQ_8)$ is a torsion group, then RQ_8 is $*$-semiclean.

Proof

1. As R is local, Q_8 is a finite 2-group, and $2 \in J(R)$, therefore, by Proposition 5.2, RQ_8 is local. Thus, RQ_8 is a $*$-semiclean ring.

2. The proof is similar to the proof of Theorem 5.3(3).

□

The example given below is the direct consequence of Theorem 5.10.

Example 5.11 *The ring $\mathbb{Z}_pQ_8$ (where $p > 2$ is prime) is clean. Furthermore, because $2 \in U(\mathbb{Z}_pQ_8)$, we can conclude from Theorem 5.10(2) that $\mathbb{Z}_pQ_8$ is $*$-semi-clean. However, according to [6, Example 3.9], $\mathbb{Z}_pQ_8$ is not $*$-clean.*

5.2.2. Generalized quaternion group Q_{2n} and Dihedral group D_{2n}

The group ring F_qQ_{2n} was studied by Hongdi Huang in [7] and it was shown that if $4|n$ and $gcd(q,2n)=1$, then it is not $*$-clean; however, by Theorem 5.12, we obtain that it is $*$-semiclean.

Theorem 5.12 *Let R be a finite commutative local ring and $G=Q_{2n}=\langle x,y|x^4=1, y^{\frac{n}{2}}=x^2, y^x=y^{-1}\rangle$ be the generalised quaternion group of order 2n or $G=D_{2n}=<x,y|y^n=x^2=1, xyx^{-1}=y^{-1}>$ be the dihedral group of order $2n$.*

1. *If $2\in U(R)$, then RQ_{2n} and RD_{2n} are $*$-semiclean.*
2. *If $2\in J(R)$, then RQ_{2n} and RD_{2n} (where n is a power of 2) are $*$-semiclean.*

Proof

1. The proof is similar to the proof of Theorem 5.7(1).
2. As R is local, Q_{2n} and D_{2n} are finite 2-groups, and $2\in J(R)$, therefore, by Proposition 5.2, RQ_{2n} and RD_{2n} are local. Thus, RQ_{2n} and RD_{2n} are $*$-semiclean rings.

□

The example given below is the direct consequence of Theorem 5.12.

Example 5.13 *The ring F_qQ_{2n} (where $gcd(q,2)=1$) is clean. Furthermore, because $2\in U(F_qQ_{2n})$, we can conclude from Theorem 5.12(1) that F_qQ_{2n} is $*$-semi-clean. However, according to [7, Theorem 4.7], F_qQ_{2n} is not $*$-clean if $4|n$ and $gcd(q,2n)=1$.*

In 2015 [6], Gao, Chen, and Li investigated the group ring $\mathbb{Z}_2D_6$, and proved that it is not $*$-clean; however, Example 5.14 demonstrates that it is $*$-semiclean. To prove $\mathbb{Z}_2D_6$ is $*$-semiclean, we have shown that every element is written as sum of a $*$-periodic element and a unit. To check this, we first represented every element of $\mathbb{Z}_2D_6$ in a matrix, and by using the SAGE [17] software obtain units, $*$-periodic elements. We then checked whether every element of $\mathbb{Z}_2D_6$ can be written as the sum of a $*$-periodic element and unit of it. By [9], the matrix representation $\sigma(\omega)$ of an element $\omega=\alpha_0+\alpha_1y+\alpha_2y^2+\alpha_3x+\alpha_4yx+\alpha_5y^2x\in RD_6$, where $D_6=\langle x,y|y^3=x^2=1, xyx^{-1}=y^{-1}\rangle$ is a dihedral group of order 6, as given by $\sigma(\omega)=\begin{bmatrix}A & B\\ B^T & A^T\end{bmatrix}$, where $A=circ\begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2\end{bmatrix}$ and $B=circ\begin{bmatrix}\alpha_3 & \alpha_4 & \alpha_5\end{bmatrix}$. The codes for this are given below.

Example 5.14 *Consider the ring $\mathbb{Z}_2D_6$. The group of all units of $\mathbb{Z}_2D_6$ is $U(\mathbb{Z}_2D_6)=\{x, yx, y^2x, 1, y+y^2+x+yx+y^2x, 1+y+y^2+x+yx, 1+y+y^2+x+y^2x, 1+y+y^2+yx+y^2x, y, y^2, 1+y+x+yx+y^2x, 1+y^2+x+yx+y^2x\}$. The set of all $*$-peridic elements of $\mathbb{Z}_2D_6$ is $Pri^*(\mathbb{Z}_2D_6)=\{0, x, yx, x+yx, y^2x, x+y^2x, yx+y^2x, x+yx+y^2x, 1, 1+x, 1+yx, 1+x+yx, 1+y^2x, 1+x+y^2x, 1+yx+y^2x, 1+x+yx+y^2x, y, y+x+yx+y^2x, 1+y, 1+y+x+yx+y^2x, y^2, y^2+x+yx+y^2x, 1+y^2, 1+y^2+x+yx+y^2x, y+y^2, y+y^2+x, y+y^2+yx, y+y^2+x+yx, y+y^2+y^2x, y+y^2+x+y^2x, y+y^2+yx+y^2x, y+y^2+x+yx+y^2x, 1+y+y^2, 1+y+y^2+x, 1+y+y^2+yx, 1+y+y^2+x+yx, 1+y+y^2+y^2x, 1+y+y^2+x+y^2x, 1+y+y^2+yx+y^2x, 1+y+y^2+x+yx+y^2x\}$. Every element of $\mathbb{Z}_2D_6$ can be written as the sum of a $*$-periodic element and a unit. Thus, we can say that the group ring $\mathbb{Z}_2D_6$ is $*$-semiclean, but by [6, Theorem 3.4], it is not $*$-clean.*

Code for the construction of a matrix representation of $\mathbb{Z}_2 D_6$.

```
Type = Integer(3)
Field = GF(Integer(2))
Vector = Field*Type

CM = [matrix.circulant(a) for a in Vector]
Length = len(CM)

Matrices_64 = []
for x in range(Length):
for y in range(Length):
CB = block_matrix(Integer(2),Integer(2),[CM[x],CM[y],CM[y].T,CM[x].T])
Matrices_64.append(CB)
```

Code to find the units of $\mathbb{Z}_2 D_6$.

```
Elements = Field*Integer(1)
Zero = Elements[Integer(0)][Integer(0)]
One = Elements[Integer(1)][Integer(0)]

Identity_row = [One,Zero,Zero,Zero,Zero,Zero]
Identity_Matrix = matrix.circulant(Identity_row)
Matrices_Unit = []
List_Matrices_64 = list(range(len(Matrices_64)))

for x in List_Matrices_64:
y = x
while y <=List_Matrices_64[len(List_Matrices_64)-Integer(1)]:
if y not in List_Matrices_64:
y = y+Integer(1)

else:
mul_r = Matrices_64[x]*Matrices_64[y]
if mul_r == Identity_Matrix:
mul_r_rev = Matrices_64[y]*Matrices_64[x]
if mul_r_rev == Identity_Matrix:
Matrices_Unit.append(x)
Matrices_Unit.append(y)
break
y = y+Integer(1)
```

Code to find the $*$-periodic element of $\mathbb{Z}_2D_6$.

```
Zero_row = [Zero for x in range(Integer(6))]
Zero_Matrix = matrix.circulant(Zero_row)

Zero_row_3 = [Zero for x in range(Integer(3))]
One_row_3 = [One for x in range(Integer(3))]
Combination_row_3 = [Zero,One,One]
Zero_matrix_3 = matrix.circulant(Zero_row_3)
One_matrix_3 = matrix.circulant(One_row_3)
Comb_matrix_3 = matrix.circulant(Combination_row_3)

Projection1 = block_matrix(2,2,[One_matrix_3,Zero_matrix_3,Zero_matrix_3,One_matrix_3])
Projection2 = block_matrix(2,2,[Comb_matrix_3,Zero_matrix_3,Zero_matrix_3,Comb_matrix_3])

Matrices_StrPeriodic = []
N = Integer(1000000)

for x in range(len(Matrices_64)):
res = Matrices_64[x]
i = Integer(1)
while i <=N:
res = res*Matrices_64[x]
if res == Identity_Matrix or res == Zero_Matrix or res == Projection1
or res == Projection2 :
Matrices_StrPeriodic.append([x,i+Integer(1)])
break
i = i+Integer(1)
```

Code to check whether every element of $\mathbb{Z}_2D_6$ can be written as the sum of $*$-periodic element and unit of it.

```
Matrices_Star_Semiclean = []
Star_Semiclean_map = []

StarPeriodic_Set = set(x[Integer(0)] for x in Matrices_StrPeriodic)
Unit_set = set(Matrices_Unit)

for x in StarPeriodic_Set:
for y in Unit_set:
res = Matrices_64[x]+Matrices_64[y]
if res in Matrices_64:
index = Matrices_64.index(res)
if index not in Matrices_Star_Semiclean:
Matrices_Star_Semiclean.append(index)
Star_Semiclean_map.append([x,y,index])
```

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