

Effect of fractional analysis on some special curves

Aykut HAS* , Beyhan YILMAZ 

Department of Mathematics, Faculty of Science, Kahramanmaraş Sütcü İmam University, Kahramanmaraş, Turkey

Received: 13.10.2022

Accepted/Published Online: 14.04.2023

Final Version: 18.07.2023

Abstract: In this study, the effect of fractional derivatives, whose application area is increasing day by day, on curves is investigated. As it is known, there are not many studies on a geometric interpretation of fractional calculus. When examining the effect of fractional analysis on a curve, the Caputo fractional analysis that fits the algebraic structure of differential geometry is used. This is because the Caputo fractional derivative of the constant function is zero. This is an important advantage and allows a variety of fractional physical problems to be based on a geometric basis. This effect is examined with the help of examples consistent with the theory and visualized for different values of the Caputo fractional analysis. The difference of this study from others is the use of Caputo fractional derivatives and integrals in calculations. Fractional calculus has applications in many fields such as physics, engineering, mathematical biology, fluid mechanics, signal processing, etc. Fractional derivatives and integrals have become extremely important as they give more numerical results than classical solutions in solving various problems in many fields. In addition, many problems that cannot be answered by classical analysis have been solved by Caputo fractional analysis. In this context, the curvatures of a curve are calculated by Caputo fractional analysis and obtained differently from the classical result. It is aimed to characterize the curve more accurately with the numerically more accurate calculation of the curvatures.

Key words: Fractional derivative, Caputo fractional analysis, special curves, curvatures, Frenet frame

1. Introduction

Fractional derivative and integral are generalized version of classical derivative and integral concepts that emerged as a result of studies by Leibniz and Newton. What is meant to be expressed in fractional derivatives is actually derivative of any rank. The concepts of fractional derivatives and integrals are as old as integer derivative and integral concepts, and fractionally derivative expression is mentioned in many sources. From this topic first mentioned in Leibniz's letter to L'Hospital in 1695 [11]. In addition to Leibniz, many mathematicians studied the same subject, including Liouville, Riemann, Weyl, Lagrange, Laplace, Fourier, Euler, and Abel [22]. Various definitions are given in the literature for fractional derivatives. Some of these fractional derivatives are Riemann Liouville (R-L), Caputo, Grünwald-Letnikov, Wely, and Riesz [18, 24, 25]. The definition of Caputo fractional analysis proposed by the Italian mathematician M. Caputo in the 1960s to eliminate the problem of calculating or measuring the initial values that arise in Laplace conversion applications of the Riemann-Liouville fractional analysis. The main advantage of the Caputo approach is that the initial conditions defined for Caputo-derived fractional differential equations are the same as the initial conditions defined for integer-level differential equations. Therefore, in recent studies in the literature, Caputo fractional derivative operator is preferred over

*Correspondence: ahas@ksu.edu.tr

2010 *AMS Mathematics Subject Classification*:26A33, 53A045, 53A5

Riemann-Liouville fractional derivative operator in analytical and numerical solutions of fractional differential equations.

One of the biggest problems with fractional calculus is that a geometric or physical interpretation is not yet complete. In fact, fractional derivatives are expected to give more detailed geometric and physical results since they give more numerical results than integer derivatives. Therefore, this topic has been studied by many authors recently [3, 4, 8, 12]. Especially recently, the focus has been on whether any geometric approximation can be made for fractional derivatives with differential geometry and vector analysis [13, 21, 27]. In addition, some special curves and characterizations of curve pairs are also investigated with fractional analysis [15, 16]. Moreover, the effects of fractional analysis on magnetic curves, which is an important application area in physics, are investigated [14, 30]. Although there are many types of fractional derivatives, the geometry of curves and surfaces can be explained more clearly thanks to the Leibnitz rule in the Caputo fractional derivative. Also, since the Caputo fractional derivative of the constant is zero, the Caputo fractional derivative is more open to geometric interpretation.

In this study, some special curves which are very important for differential geometry are recharacterized with the help of Caputo fractional analysis. The main purpose of this study is to determine the effect of fractional analysis on curves and to reveal a certain geometric interpretation. In addition, it is desired to determine the different aspects of classical analysis and fractional analysis based on curves. As mentioned in the conclusion of the article, fractional analysis does not give a different result from classical analysis on the Frenet frame at any point of the curve. However, as it can be easily seen in Theorem 2.7, the difference of fractional analysis from classical analysis on the curve is to make the curvature and torsion dependent on an α variable. As is known, from the definition of curvature of the curve, curvature measures the deviation from the tangent. Therefore, since fractional analysis gives the curvature of a curve dependent on the variable α , we can analyze the deviation from the tangent more numerically and interpret it geometrically more easily. Torsion of a curve measures how much the curve deviates from the plane. Similarly, since Caputo fractional analysis gives the torsion of a curve dependent on the variable α , we can analyze the deviation from the plane more numerically and interpret it geometrically more easily.

2. Preliminaries

2.1. Some basic definitions and theorems of fractional calculus

This section provides definitions and properties of fractional calculus. In the context of fractional calculus, the description the definition of Caputo is widely used as a fractional operator.

Caputo fractional derivative of order $0 < \alpha \leq 1$ for a smooth function $f(t)$ is given by [5, 9, 17];

$$({}^C D_{0+}^\alpha f)(t) \equiv (I_{0+}^{1-\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{df(\tau)}{d\tau} d\tau. \quad (2.1)$$

The relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative is as follows;

$$({}^C D_{0+}^\alpha f)(t) = (D_{0+}^\alpha f)(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0). \quad (2.2)$$

The Leibniz product rule for the Caputo fractional derivative is written by [6]:

$$({}^C D_{0+}^\alpha f g)(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{d^k f}{dt^k} (D_{0+}^{\alpha-k} g)(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0)g(0). \tag{2.3}$$

By using relation (2.3), the fractional derivative of composite function $f(t(s))$ is given by [6]:

$$({}^C D_{0+}^\alpha f)(t(s)) = \frac{f(t(s)) - f(t(0))}{\Gamma(1-\alpha)} s^{-\alpha} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{s^{k-\alpha}}{\Gamma(k-\alpha+1)} \frac{d^k f(t(s))}{ds^k}. \tag{2.4}$$

In the equation above, it is seen that the Leibnitz product rule and the fractional derivative of the composite function are given by the infinite series.

In terms of physical applications, the use of the Caputo fraction variant has two advantages, as can be seen from equation (2.1). The first is that the initial conditions of Caputo fractional derivatives and fractional differential equations are in the same form as integer-order differential equations. The other advantage is that the derivative of the constant is zero in the Caputo fractional derivative as mentioned earlier. Especially as mentioned in the second feature, the fact that the constant derivative of the Caputo derivative is zero is very important for its use in differential geometry. For this reason, Caputo derivative is used in this study. Finally, the following notation will be used for the representation of Caputo fractional derivative, as

$$({}^C D_{0+}^\alpha f)(t) \equiv \frac{d^\alpha f(t)}{dt^\alpha}.$$

2.2. Some basic definitions and theorems of differential geometry of curves

In this section, the curves in \mathbb{R}^3 will be introduced in a nutshell.

Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively, be scalar product and norm in the n -dimensional Euclidean space \mathbb{R}^n . If the curve $y : \tilde{I} \rightarrow \mathbb{R}^n$ verifies $\|\dot{y}\| \neq 0$ for each $t \in \tilde{I}$ and y is called the regular curve. Then there exists a unit reparameterization x of curve y , $x = y \circ u^{-1} : I \rightarrow \mathbb{R}^n$, such that

$$u(t) = \int_{t_0}^t \left\| \frac{dy}{d\sigma} \right\| d\sigma, \quad t_0 \in \tilde{I}, \tag{2.5}$$

and $\left\| \frac{dx}{du} \right\| = 1$. The parameter u is said to be arc-length.

Let $x = x(u)$ be a regular unit speed curve in the Euclidean 3-space where s measures its arc length. Also, let $t = x'$ be its unit tangent vector, $n = \frac{t'}{\|t'\|}$ be its principal normal vector and $b = t \times n$ be its binormal vector. The triple $\{t, n, b\}$ be the Frenet frame of the curve x . Then the Frenet formula of the curve is given by

$$\begin{pmatrix} t'(u) \\ n'(u) \\ b'(u) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(u) & 0 \\ -\kappa(u) & 0 & \tau(u) \\ 0 & -\tau(u) & 0 \end{pmatrix} \begin{pmatrix} t(u) \\ n(u) \\ b(u) \end{pmatrix} \tag{2.6}$$

where $\kappa(u) = \left\| \frac{d^2x}{du^2} \right\|$ and $\tau(u) = \langle \frac{dn}{du}, b \rangle$ are curvature and torsion of x , respectively [26]. For an arbitrary parameter s , κ and τ are calculated by

$$\kappa(s) = \frac{\|\dot{x}(s) \times \ddot{x}(s)\|}{\|\dot{x}(s)\|^3}, \quad \tau(s) = \frac{\langle \dot{x}(s) \times \ddot{x}(s), \ddot{\ddot{x}}(s) \rangle}{\|\dot{x}(s) \times \ddot{x}(s)\|^2}. \tag{2.7}$$

Definition 2.1 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If any U fixed direction with the unit tangent vector of the curve x makes a fixed angle, the curve x is called the general helix [7]. The most well-known characterization of the helix curve is $\frac{\tau}{\kappa} = \text{constant}$ for each $u \in I$ (Lancret theorem)[26].

Definition 2.2 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If any U fixed direction with the unit principal normal vector of the curve x makes a fixed angle, the curve x is called the slant helix. Izumiya and Takeuchi obtain a necessary and sufficient condition to be become slant helix for a curve: a curve is an oblique propeller if its geodetic curvature and the principal normal satisfy the expression

$$\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'$$

is a constant function [19].

Definition 2.3 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . The curve x is called rectifying curves for all $u \in I$ if the orthogonal complement of $n(u)$ contains a fixed point. Since the orthogonal complement of n is $n^\perp = \{v \in T_\alpha E^3 \mid \langle v, N \rangle = 0\}$ the position vector of rectifying curve x in E^3 can be written as

$$x(u) = \lambda t(u) + \mu b(u),$$

where λ, μ are differentiable functions [10].

Definition 2.4 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . Then it is said that x is an osculating curve if its position vector is in the orthogonal complement of the binormal vector for all $u \in I$ such as,

$$x(u) = \lambda t(u) + \mu n(u),$$

where λ, μ are differentiable functions [20].

Definition 2.5 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . The curve x for which the position vector always lie in their normal plane, is for simplicity called normal curve. The following equation is provided in normal curves for each $u \in I$,

$$x(u) = \lambda n(u) + \mu b(u),$$

where λ, μ are differentiable functions [20].

2.3. Curvatures of curves in Euclidean 3-space with fractional-order

In this section, some concepts for differential geometry, which have been rebuilt with fractional derivatives, will be summarized.

Let $x : I \rightarrow \mathbb{R}^3$ be parameterized by the arc-length u . Consider another parameter s given by

$$u \rightarrow s = \left[\frac{\alpha^2}{\Gamma(2 - \alpha)} u \right]^{\frac{1}{\alpha}}, \tag{2.8}$$

where Γ is the Euler gamma function and $0 < \alpha \leq 1$. Due to (2.5), s is also a function of t . Let us put $s = h(t)$. Then (2.8) gives,

$$h(t) = \left(\frac{\alpha^2}{\Gamma(2-\alpha)} \int_{t_0}^t \left\| \frac{dx}{d\sigma} \right\| d\sigma \right)^{\frac{1}{\alpha}}. \tag{2.9}$$

From equation (2.9), we get

$$\dot{h} = \frac{dh}{dt} = \frac{\alpha h^{1-\alpha}(t)}{\Gamma(2-\alpha)} \|\dot{x}\|, \tag{2.10}$$

which implies that \dot{h} is positive for each t and so its inverse function $t = h^{-1}(s)$ exists. If a fractional derivative of the composite function is applied here, taking into account equation (2.4), the following equation is obtained as [28]

$$\frac{d^\alpha x(h(t))}{ds^\alpha} = \frac{\alpha h^{1-\alpha}(t)}{\Gamma(2-\alpha)} (h^{-1})' \dot{x}. \tag{2.11}$$

If equations (2.9) and (2.10) are replaced in equation (2.11), we get $\left\| \frac{d^\alpha x}{ds^\alpha} \right\| = 1$ for each s . This result shows us that the curve x is unit-speed.

Theorem 2.6 *Let $x = x(s)$ be a regular unit speed curve in the Euclidean 3-space where s measures its arc length. Also, let $t_\alpha = \frac{d^\alpha x}{ds^\alpha}$ be its unit tangent vector, $n_\alpha = \frac{t'_\alpha}{\|t'_\alpha\|}$ be principal normal vector and $b_\alpha = t_\alpha \times n_\alpha$ be its binormal vector of x at s . The triple $\{t_\alpha, n_\alpha, b_\alpha\}$ be the Frenet frame of the curve x . Then the Frenet formula of the curve is given by*

$$\begin{pmatrix} t'_\alpha(s) \\ n'_\alpha(s) \\ b'_\alpha(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} t_\alpha(s) \\ n_\alpha(s) \\ b_\alpha(s) \end{pmatrix}, \tag{2.12}$$

where $\kappa_\alpha(s) = \|t'_\alpha\|$ and $\tau_\alpha(s) = \langle n'_\alpha, b_\alpha \rangle$ are curvature and torsion of x with fractional order, respectively [1].

Theorem 2.7 *Let $x = x(s)$ be a regular unit speed curve in the Euclidean 3-space where s measures its arc length. $\kappa_\alpha(s) = \|t'_\alpha\|$ and $\tau(s) = \langle n'_\alpha, b_\alpha \rangle$ are curvature and torsion of x , respectively. Accordingly, the following equations of curvatures with fractional order α are available [1],*

$$\kappa_\alpha(s) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} \kappa(s), \tag{2.13}$$

$$\tau_\alpha(s) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} \tau(s). \tag{2.14}$$

Conclusion 2.8 *Let $x = x(s)$ be a unit speed curve where s measures the arc length. As can be seen from the equation (2.12), the Caputo derivative has no effect on the Frenet frame, so the Frenet elements do not undergo any change. However, considering the equations (2.13) and (2.14), the curvature and torsion of the*

x curve changed under the Caputo fractional derivative. As can be seen, curvature and torsion are calculated based on the numerical value of α . As a result, the curvature and torsion of the curve can be calculated much more numerically with Caputo fractional analysis.

3. Main results

In this section, some special curves previously obtained with classical derivative will be obtained with Caputo fractional derivative.

Definition 3.1 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the unit tangent vector of the curve x according to Caputo fractional derivative makes a fixed angle with any u fixed direction, the curve x is called a fractional general helix (F-general helix).

Theorem 3.2 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is the fractional general helix according to the Frenet frame defined by the Caputo fractional derivative, the following equation exists:

$$\frac{\kappa_\alpha}{\tau_\alpha} = \text{constant}. \tag{3.1}$$

Proof Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If any u fixed direction with the unit tangent vector of the curve x makes a fixed angle, the x curve is called the general helix. Accordingly, the following equation can be written,

$$\langle t_\alpha, u \rangle = \cos \theta.$$

If derivative of the above equation is taken according to s ,

$$\langle t'_\alpha, u \rangle + \langle t_\alpha, u' \rangle = 0.$$

Using the u -direction is fixed, we can see

$$\langle t'_\alpha, u \rangle = 0.$$

If equation (2.12) is used in above equation, we get

$$\kappa_\alpha \langle n_\alpha, u \rangle = 0. \tag{3.2}$$

When $\kappa_\alpha = 0$, Eq. (3.1) will be constant as can be easily seen. For $\kappa_\alpha \neq 0$, it is clear that $\langle n_\alpha, u \rangle = 0$. If we take the derivative of equation (3.2) according to s again, we obtain the following

$$\kappa'_\alpha \langle n_\alpha, u \rangle + \kappa_\alpha \langle n'_\alpha, u \rangle + \kappa_\alpha \langle n_\alpha, u' \rangle = 0,$$

$$\kappa_\alpha \langle n'_\alpha, u \rangle = 0.$$

If the Frenet formulas in equation (2.12) are used in this equation, we can write

$$\kappa_\alpha \langle (-\kappa_\alpha t_\alpha + \tau_\alpha b_\alpha), u \rangle = 0,$$

and

$$-\kappa_\alpha^2 \langle t_\alpha, u \rangle + \kappa_\alpha \tau_\alpha \langle b_\alpha, u \rangle = 0.$$

If the equality $\langle t, u \rangle = \cos \theta$ is used in above equation, we have

$$\langle b_\alpha, u \rangle = \frac{\kappa_\alpha}{\tau_\alpha} \cos \theta.$$

If derivative of this expression is taken according to s again, the following equation is obtained

$$-\tau_\alpha \langle n_\alpha, u \rangle = \left(\frac{\kappa_\alpha}{\tau_\alpha} \right)' \cos \theta.$$

Because of $\langle n_\alpha, u \rangle = 0$ as previously stated, we can write

$$\left(\frac{\kappa_\alpha}{\tau_\alpha} \right)' = 0,$$

and

$$\frac{\kappa_\alpha}{\tau_\alpha} = c, \quad c \in \mathbb{R}.$$

That way, the proof is complete. □

Definition 3.3 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the unit principal normal vector of x according to Caputo fractional derivative makes a fixed angle with any fixed direction u , the curve x is called a fractional slant helix (F-slant helix).

Theorem 3.4 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is the fractional slant helix according to the Frenet frame defined by the Caputo fractional derivative, the following equation exists:

$$\left(\frac{\tau_\alpha}{\kappa_\alpha} \right)' \frac{1}{\kappa_\alpha} = \text{constant}.$$

Proof Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If any u fixed direction with the unit principal normal vector of the curve x makes a fixed angle, the x curve is called the slant helix. Accordingly, the following equation can be written, as

$$\langle n_\alpha, u \rangle = \cos \theta.$$

If both sides of the above equation are derivated according to s ,

$$\langle n'_\alpha, u \rangle + \langle n_\alpha, u' \rangle = 0.$$

Since u is a fixed direction, we have

$$\langle n'_\alpha, u \rangle = 0.$$

If Frenet formulas of x is used in above equation, we get

$$\langle (-\kappa_\alpha t_\alpha + \tau_\alpha b_\alpha), u \rangle = 0$$

and

$$-\kappa_\alpha \langle t_\alpha, u \rangle + \tau_\alpha \langle b_\alpha, u \rangle = 0.$$

As can be seen from the rotation matrix, $\langle n_\alpha, u \rangle = \cos \theta$ and $\langle b_\alpha, u \rangle = \sin \theta$. If this equation is used in the above equation, then we have

$$\langle t_\alpha, u \rangle = \frac{\tau_\alpha}{\kappa_\alpha} \sin \theta.$$

If once again derivative is taken in this equation according to s , we can write

$$\kappa_\alpha \langle n_\alpha, u \rangle = \left(\frac{\tau_\alpha}{\kappa_\alpha} \right)' \sin \theta.$$

If the equality $\langle n_\alpha, u \rangle = \cos \theta$ is written here in place,

$$\left(\frac{\tau_\alpha}{\kappa_\alpha} \right)' \frac{1}{\kappa_\alpha} = \cot \theta$$

is obtained. Because θ is selected as a constant, following equation can be written,

$$\left(\frac{\tau_\alpha}{\kappa_\alpha} \right)' \frac{1}{\kappa_\alpha} = c, \quad c \in \mathbb{R}.$$

□

Definition 3.5 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the position vector of x always lies in its rectifying plane according to the Frenet frame defined by the Caputo fractional derivative, the curve x is called the a fractional rectifying curve (F-rectifying curve).

Theorem 3.6 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is the fractional rectifying curve according to Frenet frame defined by the Caputo fractional derivative, the following equation exists,

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} (\lambda \kappa_\alpha - \mu \tau_\alpha)$$

where λ, μ are differentiable functions.

Proof Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is a rectifying curve, the following equation exists, as previously seen in the definition,

$$x = \lambda t_\alpha + \mu b_\alpha,$$

where λ, μ are differentiable functions. If the Caputo fractional derivative of the above equation is taken from the order α -th according to the s , we get the following equations

$$\frac{d^\alpha x}{ds^\alpha} = \frac{d^\alpha}{ds^\alpha} (\lambda t_\alpha + \mu b_\alpha),$$

$$t_\alpha = \frac{d^\alpha}{ds^\alpha} (\lambda t_\alpha + \mu b_\alpha).$$

If the classical derivative of both sides of the above equation is taken according to the s , the following equation is obtained as

$$t'_\alpha = \frac{d}{ds} \frac{d^\alpha}{ds^\alpha} (\lambda t_\alpha + \mu b_\alpha).$$

If the commutative property of the Caputo fractional derivative [23] and Frenet formulas are used

$$t'_\alpha = \frac{d^\alpha}{ds^\alpha} \frac{d}{ds} (\lambda t_\alpha + \mu b_\alpha),$$

and necessary arrangements are made, we can write

$$\kappa_\alpha n_\alpha = \frac{d^\alpha}{ds^\alpha} (\lambda' t_\alpha + (\lambda \kappa_\alpha - \mu \tau_\alpha) n_\alpha + \mu' b_\alpha).$$

From the mutual equalities of the above equation, we can easily see that

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} (\lambda \kappa_\alpha - \mu \tau_\alpha).$$

□

Definition 3.7 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the position vector of x always lies in its normal plane according to the Frenet frame defined by the Caputo fractional derivative, the curve x is called a fractional normal curve (F-normal curve).

Theorem 3.8 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is the fractional normal curve according to the Frenet frame defined by the Caputo fractional derivative, the following equation exists:

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} \left(\frac{(\lambda\mu)' - \mu c}{\lambda} \right), \quad c \in \mathbb{R},$$

where λ, μ are differentiable functions.

Proof Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is a F-normal curve, the following equation exists,

$$x = \lambda n_\alpha + \mu b_\alpha,$$

where λ, μ are differentiable functions. If the Caputo fractional derivative of the above equation is taken from the order α -th according to the s , we obtain

$$\frac{d^\alpha x}{ds^\alpha} = \frac{d^\alpha}{ds^\alpha} (\lambda n_\alpha + \mu b_\alpha),$$

and

$$t = \frac{d^\alpha}{ds^\alpha} (\lambda n_\alpha + \mu b_\alpha).$$

If the classical derivative of both sides of the above equation is taken according to the s , we get

$$t' = \frac{d}{ds} \frac{d^\alpha}{ds^\alpha} (\lambda n_\alpha + \mu b_\alpha).$$

If the commutative property of the Caputo fractional derivative and Frenet formulas are used, we can write

$$\kappa_\alpha n_\alpha = \frac{d^\alpha}{ds^\alpha} (-\lambda \kappa_\alpha t_\alpha + (\lambda' - \mu \tau_\alpha) n_\alpha + (\mu' + \lambda \tau_\alpha) b_\alpha).$$

From the mutual equalities of the above equation, we get

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} (\lambda' - \mu \tau_\alpha) \tag{3.3}$$

and

$$\frac{d^\alpha}{ds^\alpha} (\mu' + \lambda \tau_\alpha) = 0. \tag{3.4}$$

From equation (3.4), we have

$$\mu' + \lambda \tau_\alpha = c, \quad c \in \mathbb{R},$$

and

$$\tau_\alpha = \frac{c - \mu}{\lambda}. \tag{3.5}$$

If equation (3.5) is used in equation (3.3) and the necessary arrangements are made,

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} \left(\frac{(\lambda \mu)' - \mu c}{\lambda} \right), \quad c \in \mathbb{R}.$$

□

Definition 3.9 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the position vector of x always lies in its osculating plane according to the Frenet frame defined by the Caputo fractional derivative, the curve x is called a fractional osculating curve (F-osculating curve).

Theorem 3.10 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is the fractional osculating curve according to the Frenet frame defined by the Caputo fractional derivative, the following equation exists:

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} \left(\frac{(\lambda \mu)' - \lambda c}{\mu} \right), \quad c \in \mathbb{R},$$

where λ, μ are differentiable functions.

Proof Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space E^3 . If the curve x is a osculating curve, the following equation exists,

$$x = \lambda t_\alpha + \mu n_\alpha,$$

where λ, μ are differentiable functions. If the Caputo fractional derivative of the above equation is taken from the order α -th according to the s , we get

$$t_\alpha = \frac{d^\alpha}{ds^\alpha} (\lambda t_\alpha + \mu n_\alpha).$$

If the classical derivative of both sides of the above equation is taken according to the s ,

$$t'_\alpha = \frac{d}{ds} \frac{d^\alpha}{ds^\alpha} (\lambda t_\alpha + \mu n_\alpha).$$

If the commutative property of the Caputo fractional derivative and Frenet formulas are used, we can write

$$\kappa_\alpha n_\alpha = \frac{d^\alpha}{ds^\alpha} ((\lambda' - \mu\kappa_\alpha)t_\alpha + (\mu' + \lambda\kappa_\alpha)n_\alpha + \mu\tau_\alpha b_\alpha).$$

Using the mutual equalities, we have

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} (\mu' + \lambda\kappa_\alpha) \tag{3.6}$$

and

$$\frac{d^\alpha}{ds^\alpha} (\lambda' - \mu\kappa_\alpha) = 0. \tag{3.7}$$

If equation (3.7) is edited, we can say

$$\lambda' - \mu\kappa_\alpha = c, \quad c \in \mathbb{R},$$

$$\kappa_\alpha = \frac{\lambda' - c}{\mu}. \tag{3.8}$$

If equation (3.8) is used in equation (3.6) and the necessary arrangements are made, we get

$$\kappa_\alpha = \frac{d^\alpha}{ds^\alpha} \left(\frac{(\lambda\mu)' - \lambda c}{\mu} \right), \quad c \in \mathbb{R}.$$

□

Example 3.11 Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a regular with arbitrary speed fractional curve in \mathbb{R}^3 parameterized by

$$x(s) = \frac{\Gamma(2-\alpha)}{\alpha} \left(-\int s^{1-\alpha} \sin s, \int s^{1-\alpha} \cos s, 4s^{1-\alpha} \right).$$

The view of the curve for different α values are given in Figures 1 and 2.

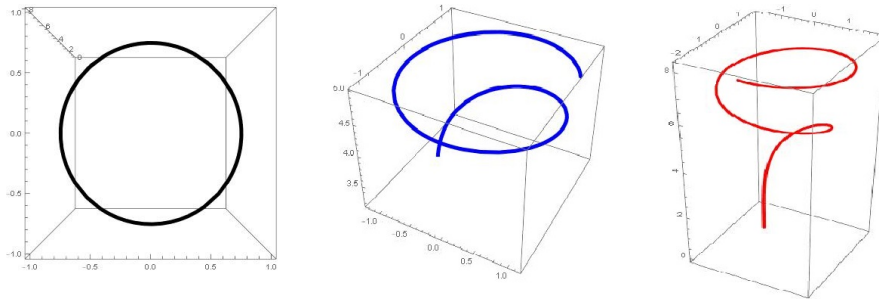


Figure 1. Fractional curve $x(s)$ for $\alpha \rightarrow 1$ (Black), $\alpha = 0.9$ (Blue) and $\alpha = 0.7$ (Red), respectively.

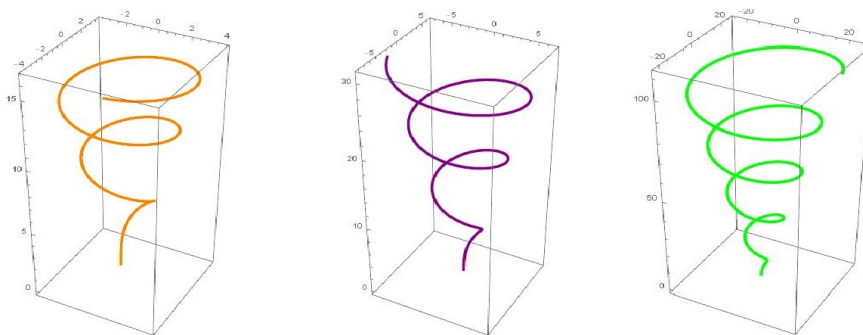


Figure 2. Fractional curve $x(s)$ for $\alpha = 0.5$ (Orange), $\alpha = 0.3$ (Purple) and $\alpha = 0.1$ (Green), respectively.

Let $x : I \subset \mathbb{R} \rightarrow E^3$ be a regular with arbitrary speed fractional curve in \mathbb{R}^3 parameterized by

$$x(s) = \frac{\Gamma(2-\alpha)}{\alpha} \left(-\frac{3}{49} \int s^{1-\alpha} \sin s, \frac{4}{49} \int s^{1-\alpha} \cos s, \frac{5}{49} \int s^{1-\alpha} \right).$$

The view of the curve for different α values are given in Figures 3 and 4.

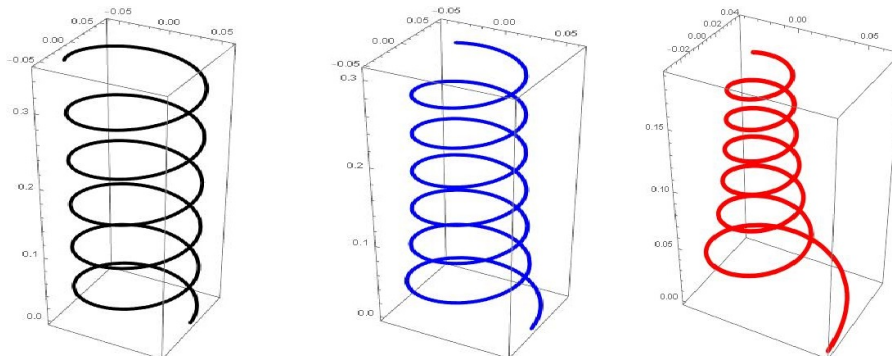


Figure 3. Fractional curve $x(s)$ for $\alpha \rightarrow 1$ (Black), $\alpha = 0.9$ (Blue) and $\alpha = 0.7$ (Red), respectively.

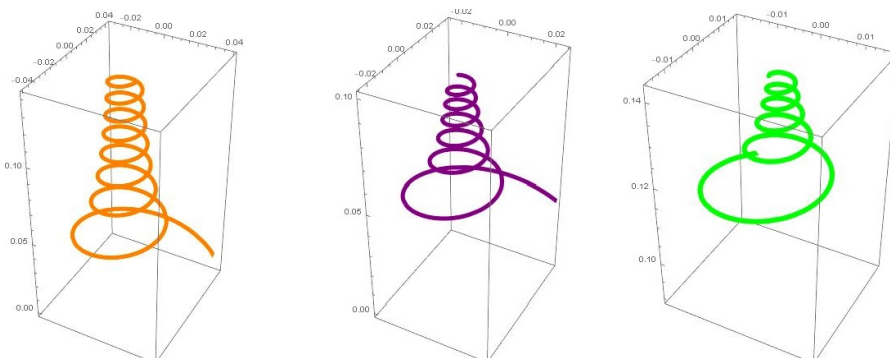


Figure 4. Fractional curve $x(s)$ for $\alpha = 0.5$ (Orange), $\alpha = 0.3$ (Purple) and $\alpha = 0.1$ (Green), respectively.

4. Conclusion

The study areas of fractional derivatives and integrals are increasing day by day. Today, fractional analysis is preferred in the solutions of differential equations encountered in many fields of physics, chemistry, biology, and engineering. The reason for this is the claim that the solutions obtained by fractional analysis give a more precise numerical solution. As mentioned above, no results have been obtained yet regarding the effect of fractional analysis, which has found its place in many fields, on geometry. Studies in this area are still very new. For example, one of these studies examined the effect of fractional analysis on geometry [27], while another study investigated the effect of fractional analysis on electromagnetic curves in fiber optic [29]. It should be taken into account that fractional analysis, which is claimed to give clearer solutions numerically, can also give clearer information geometrically. As seen in the article, Caputo fractional analysis helps us to calculate the curvature and torsion of a curve more numerically. Thus, as can be clearly seen from the curvature and torsion definitions, the tangent deviation and plane deviation amounts, respectively, can be calculated more numerically. In addition, its ability to be calculated more numerically provides us with a geometric interpretation. In this article, only the effect of Caputo fractional analysis on curvature and torsion and its effect on special curves are examined. However, many more geometric problems remain unanswered. For example, can the motion or characteristic of a curve be more clearly explained by fractional analysis? This work is a pioneer in geometric problems. Future studies in this field will make many contributions to geometry.

Acknowledgment

We would like to thank the editor and referees for their valuable and helpful suggestions that improved the paper radically.

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