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An inverse problem of finding a time-dependent coefficient in a fractional diffusion equation

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Abstract: This article is concerned with the study of unique solvability of an inverse coefficient problem of determining the coefficient at the lower term of a fractional diffusion equation. The direct problem is the initial-boundary problem for this equation with usual initial and homogeneous Dirichlet conditions. To determine the unknown coefficient, an overdetermination condition is given as the Neumann condition at the left end of the spatial interval. The theorems of existence and uniqueness of inverse problem solution are obtained. Furthermore, we propose a numerical algorithm based on a finite-difference scheme to accurately compute the inverse problem of simultaneously determining a time-dependent coefficient in a fractional diffusion equation, together with its solution. A test example using the developed numerical algorithm is presented herein.

Key words: Inverse problem, fractional diffusion equation, finite difference method, overdetermination condition

1. Introduction

Fractional differential equations (FDEs) have attracted the attention of many researchers. This is due to the fact that various models using fractional partial differential equations are successfully applied to describe a range of problems in mechanical engineering, viscoelasticity, electron transport, heat conduction, and high-frequency financial data.

The time-fractional diffusion equation was derived by replacing the standard time derivative with a time-fractional derivative. Direct problems, such as initial value problems and initial-boundary value problems for the time-fractional diffusion equation, have been studied extensively in recent years (see [17, 20, 21, 26, 32]). The author in [27] used the Green function method to obtain a general representation of solutions of the two-dimensional diffusion equation and constructed Green functions of the first, second, and mixed boundary value problems. Evolution equations with the regularized fractional derivative of order a with respect to the time variable and elliptic operator with constant and variable coefficients acting on the spatial variables were investigated. A fundamental solution to the Cauchy problem for these equations was constructed and investigated (see [36]).

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Inverse problems for classical integro-differential equations of heat conduction have been extensively studied. In the literature, the nonlinear inverse problems with different types of overdetermination conditions are the most frequently found ones (see, for example, [4, 6–9, 11–13] and references therein). In these studies, the authors discussed the unique solvability and stability estimates of the solution, as well as a numerical approach for solving such problems.

Inverse problems for fractional differential heat equations have not yet been fully investigated. In the literature, linear source determination problems and nonlinear coefficient inverse problems for the Cauchy problem with various types of overdetermination conditions are encountered most often (e.g., [10, 14–16, 34, 37–39] and references therein). The main results of these studies comprise the existence and uniqueness theorems, as well as a stability estimate for the solution of the problem of determining the reaction coefficient in a time-fractional diffusion equation.

Recent advances in numerical simulations for determining unknown coefficients in parabolic equations have led to many interesting results. In heat conduction, attention has been paid to the unique solvability of one-dimensional inverse problems for the heat equation when the unknown thermal coefficients are constant [3], time-dependent [24, 25], space-dependent [1], or temperature-dependent [19, 23, 29, 35]. Most of these simulations were performed using the finite difference method [18, 22, 28]. However, in our recent work [5], we have already demonstrated the application of finite difference and Fourier spectral numerical methods to solve the inverse problem of simultaneously determining a time-dependent unknown coefficient in a parabolic equation.

In this study, we investigate the local existence and uniqueness of an inverse problem of determining the time-dependent reaction coefficient in the time–fractional diffusion equation with initial boundary and overdetermination conditions. Furthermore, we employ the finite difference method for the numerical evolution of a time-dependent unknown coefficient in the one-dimensional fractional diffusion equation. The remainder of this paper is organized as follows. In the next section, Section 2, we present the mathematical formulations of the inverse problem. The numerical setup and finite-difference discretization for the inverse problem are presented in Section 3. In Section 4, we provide numerical results and discussion. Finally, conclusions are presented in Section 5.

2. Mathematical formulation of the problem and its investigation

Consider the linear one-dimensional fractional parabolic equation with a time-dependent coefficient

$$\partial_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - q(t)u(x,t) + f(x,t), \quad (x,t) \in (0,l) \times (0,T] =: \Omega_{lT}, \tag{2.1}$$

where $u(x,t)$ represents the temperature in a finite slab of length $l > 0$ over time interval $(0,T]$ with $T > 0$, $q(t)$ describes the coefficient of heat capacity, $f(x,t)$ is a source function; $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1]$:

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(\tau,x)}{(t-\tau)^\alpha} d\tau, \quad \alpha \in (0,1), \quad \text{and} \quad \partial_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}, \tag{2.2}$$

$\Gamma(\cdot)$ is the Euler's gamma function.

We study the inverse problem to find the coefficient q together with a solution u of Eq. (2.1) under the following initial condition:

$$u(x,0) = \varphi(x), \quad x \in [0,l], \tag{2.3}$$

boundary and overdetermination conditions:

$$u(0,t) = u(l,t) = 0, \tag{2.4}$$

$$\frac{\partial u(0,t)}{\partial x} = h(t), \quad t \in [0,T], \tag{2.5}$$

where $\varphi(x)$ and $h(t)$ are given functions. The conditions (2.4) represent the specification of the boundary temperature. (2.5) corresponds to a given heat flow at the left end of a rod of length l. Similar inverse problem with this type overdetermination condition is studied in [32].

Definition 2.1 *A function $u(x,t)$ is called a classical solution to the initial-boundary problem (2.1)-(2.4) if:*

1. *$u(x,t)$ is twice continuously differentiable in x for each $t > 0$;*
2. *for each $x \in (0,l)$ function $u(x,t)$ is continuous in t on $[0,T]$, and its fractional integral*

$$(I_{0+}^{1-\alpha}u)(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(x,\tau)d\tau}{(t-\tau)^\alpha} \tag{2.6}$$

is continuously differentiable in t for $t > 0$;

3. *$u(x,t)$ satisfies (2.1)-(2.4).*

The class of functions satisfying conditions 1), 2) of Definition 2.1 denote by $C^{2,\alpha}(\Omega_{lT})$.

Let $u(x,t)$ be a classical solution to the problem (2.1)-(2.4) and f, φ, h be enough smooth functions. We carry out the next converting of the inverse problem (2.1)-(2.5). Denote for this purpose the second derivative of $u(x,t)$ with respect to x, by $\vartheta(x,t)$, i.e. $\vartheta(x,t) := u_{xx}(x,t)$. Differentiating (2.1) and (2.3) twice in x, we get

$$\partial_t^\alpha \vartheta - \vartheta_{xx} + q(t)\vartheta(x,t) = f_{xx}(x,t), \quad (x,t) \in \Omega_{lT}, \tag{2.7}$$

$$\vartheta(x,0) = \varphi''(x), \quad x \in [0,l]. \tag{2.8}$$

To obtain boundary conditions for the function $\vartheta(x,t)$, we note that the second term of (2.1) is $\vartheta(x,t)$. Setting $x = 0, x = l$ in (2.1), we use (2.4). Then, assuming $f(0,t) = f(l,t) = 0$, we obtain

$$\vartheta(0,t) = \vartheta(l,t) = 0, \quad x \in [0,l]. \tag{2.9}$$

In order to obtain an overdetermination condition for the function $\vartheta(x,t)$, we differentiate equation (2.1) once with respect to x and using equality $u_{xx}(x,t) = \vartheta(x,t)$ and condition (2.5), we get

$$\vartheta_x(0,t) = q(t)h(t) + \partial_t^\alpha h(t) - f_x(0,t). \tag{2.10}$$

When the matching condition $\varphi'(0) = h(0)$ is satisfied, it is easy to derive from (2.7)-(2.10) the equations (2.1)-(2.5).

We assume that the data of the problem (2.1)-(2.5) satisfy the following conditions:

(A1) $\varphi(x) \in C^4[0,l]$, $\varphi^{(5)}(x) \in L_2[0,l]$, $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \varphi^{(4)}(0) = \varphi^{(4)}(l) = 0$,

(A2) $f(x,t) \in C(\overline{D}_T)\bigcap C^{4,1}_{x,t}(D_T), f^{(5)}_{xxxxx}(x,t) \in L_2(D_T), f(0,t) = f(l,t) = f''(0,t) = f''(l,t) = f^{(4)}(0,t) = f^{(4)}(l,t) = 0,$

(A3) $h(t) \in C^1[0,T]$ *and* $|h(t)| \geq h_0 = const > 0, h_0$ *is a given number,* $\varphi'(0) = h(0), \varphi'''(0) = q(0)h(0) + \partial_t^\alpha h(t) - f_x(0,0).$

The next section is devoted to the study of the direct problem (2.7)-(2.9).

2.1. Investigation of direct problem (2.7)-(2.9)

First, we show that the direct problem has a unique solution. We will seek the solution of problem (2.7)-(2.9) in the form

$$\vartheta(x,t) = \sum_{n=1}^{\infty} \vartheta_n(t)\sin(\lambda_n x),\ \ \lambda_n = \frac{\pi n}{l},\ \ \vartheta_n(t) = \sqrt{\frac{2}{l}}\int_0^l \vartheta(x,t)\sin(\lambda_n x)dx. \tag{2.11}$$

In view of equalities (2.11), from (2.7), we obtain the following equation:

$$\left(\partial_t^\alpha \vartheta_n\right)(t) + \lambda_n^2\vartheta_n(t) = F_n(t;\vartheta,q,f), \tag{2.12}$$

where

$$F_n(t;\vartheta,q,f) := f_n(t) - q(t)\vartheta_n(t),\ \ f_n(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xx}(x,t)\sin(\lambda_n x)dx.$$

The initial condition (2.8) gives:

$$\vartheta_n(0) = \sqrt{\frac{2}{l}}\int\limits_0^l \vartheta(x,0)\sin(\lambda_n x)dx = \sqrt{\frac{2}{l}}\int\limits_0^l \varphi''(x)\sin(\lambda_n x)dx =: \varphi_n. \tag{2.13}$$

We note that the initial-value problem (2.12), (2.13) is equivalent in the space $C[0,T]$ to the following Volterra integral equation of the second kind (see, for example, [30], p. 323):

$$\vartheta_n(t) = \varphi_n E_\alpha(-\lambda_n^2 t^\alpha) +$$

$$\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)f_n(\tau)d\tau + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)q(\tau)\vartheta_n(\tau)d\tau, \tag{2.14}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by the following series (see, for example, [30] p. 42):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

for $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$ denotes the real part of the complex number α and $E_{\alpha,1}(z) =: E_\alpha(z)$.

Further, to evaluate expressions with the Mittag-Leffler function, we use the following three statements from the book [30]. The proof of these assertions come from the definition of Caputo fractional derivative and differentiation of the Mittag-Leffler function.

Proposition 2.2 *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \kappa \leq |arg(z)| \leq \pi.$$

Proposition 2.3 *For $0 < \alpha < 1$, $t > 0$, we have $0 < E_{\alpha,1}(-t) < 1$. Moreover, $E_{\alpha,1}(-t)$ is completely monotonic, that is*

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t) \geq 0, \quad \forall n \in \mathbb{N}.$$

Proposition 2.4 *For $0 < \alpha < 1$, $\eta > 0$, we have $0 \leq E_{\alpha,\alpha}(-\eta) \leq \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

Using Propositions 2.3 and 2.4, we estimate the expression (2.14). Then, for $t \in [0;T]$, we get the following integral inequality:

$$|\vartheta_n(t)| \leq |\varphi_n| + \frac{t^\alpha \|f_n\|}{\alpha\Gamma(\alpha)} + \frac{||q||}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\vartheta_n(\tau)| d\tau, \tag{2.15}$$

where $\|f_n\| = \max_{t\in[0,T]} |f_n(t)|$, $\|q\| = \max_{t\in[0,T]} |q(t)|$. Applying a Gronwall-type inequality to (2.15) with a weakly singular kernel, we have the following estimate for $|\vartheta_n(t)|$ (see [31, 33]):

$$|\vartheta_n(t)| \leq \Big(|\varphi_n| + \frac{t^\alpha \|f_n\|}{\alpha\Gamma(\alpha)}\Big) E_\alpha(||q|| t^\alpha). \tag{2.16}$$

Using equality (2.12), from (2.16), we obtain an estimate for $|\partial^\alpha \vartheta_n(t)|$:

$$|\partial^\alpha \vartheta_n(t)| \leq (\lambda_n^2 + ||q||)\Big(|\varphi_n| + \frac{t^\alpha \|f_n\|}{\alpha\Gamma(\alpha)}\Big) E_\alpha(||q|| t^\alpha) + \|f_n\|.$$

From here, based on Proposition 2.2-2.4, we have the following lemma:

Lemma 2.5 *For any $t \in [0,T]$ and for sufficiently large n, the following estimates are valid:*

$$|\vartheta_n(t)| \leq \overline{C}_1\Big(|\varphi_n| + \|f_n\|\Big), \quad |\partial^\alpha \vartheta_n(t)| \leq \overline{C}_2\Big(n^2|\varphi_n| + n^2\|f_n\|\Big),$$

where $\overline{C}_i$, $i = 1, 2$, are positive constants, depending only on α, T, and $\|q\|$.

Formally, from (2.11) by term-by-term differentiation, we compose the series

$$\partial_t^\alpha \vartheta(x,t) = \sum_{n=1}^{\infty} \partial^\alpha \vartheta_n(t) \sin(\lambda_n x), \quad \vartheta_{xx}(x,t) = \sum_{n=1}^{\infty} \lambda_n^2 \vartheta_n(t) \sin(\lambda_n x), \tag{2.17}$$

Based on Lemma 2.5, we note the series (2.11), (2.17) for any $(x,t) \in D_T$ are estimated by

$$\overline{C}_3 \sum_{n=1}^{\infty} \Big(n^2 |\varphi_n| + n^2 \|f_n\| \Big), \tag{2.18}$$

where the constant $\overline{C}_3 > 0$ depends only on α, T, and $\|q\|$.

The following assertion is true:

Lemma 2.6 *If the conditions (A1), (A2) take place, then there are equalities*

$$\varphi_n = \frac{1}{\lambda_n^3} \varphi_n^{(3)}, \quad f_n(t) = \frac{1}{\lambda_n^3} f_n^{(3)}(t), \tag{2.19}$$

where

$$\varphi_n^{(3)} = \sqrt{\frac{2}{l}} \int_0^l \varphi^{(5)}(x) \cos(\lambda_n x) dx, f_n^{(3)}(t) = \sqrt{\frac{2}{l}} \int_0^l f^{(5)}_{xxxxx}(x,t) \cos(\lambda_n x) dx,$$

with the following estimates:

$$\sum_{n=1}^{\infty} |\varphi_n^{(3)}|^2 \le \|\varphi_n^{(3)}\|_{L_2[0,l]}, \quad \sum_{n=1}^{\infty} |f_n^{(3)}(t)|^2 \le \|f_n^{(3)}(t)\|_{L_2[0,l] \times C[0,T]}. \tag{2.20}$$

If the functions $\varphi(x)$ and $f(x,t)$ satisfy the conditions of Lemma 3.2, then due to representations (2.19) and (2.20) series (2.11), (2.17) converge uniformly in the rectangle Ω_{lT}; therefore, function $u(x,t)$ satisfies relations (2.7)–(2.9).

Using the above results, we obtain the following assertion:

Lemma 2.7 *Let* $q(t) \in C[0,T]$, *(A1), (A2) be satisfied, then there exists a unique solution of the direct problem* (2.7)-(2.9) *such that* $\vartheta(x,t) \in C^{2,\alpha}(\Omega_{lT})$.

Now we derive an estimate for the norm of the difference between the solution of the original integral equation (2.14) and the solution of this equation with perturbed functions $\widetilde{q}, \widetilde{\varphi}_n, \widetilde{f_n}$. Let $\widetilde{\vartheta}_n(t)$ be solution of the integral equation (2.14) corresponding to the functions $\widetilde{q}, \widetilde{\varphi}_n, \widetilde{f_n}$; i.e.

$$\widetilde{\vartheta}_n(t) = \widetilde{\varphi}_n E_\alpha(-\lambda_n^2 t^\alpha) +$$

$$+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n^2 (t-\tau)^\alpha\right) \widetilde{f}_n(\tau) d\tau + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n^2 (t-\tau)^\alpha\right) \widetilde{q}(\tau) \widetilde{\vartheta}_n(\tau) d\tau. \tag{2.21}$$

Composing the difference $\vartheta - \widetilde{\vartheta}$ with the help of the equations (2.14), (2.21) and introducing the notations $\vartheta - \widetilde{\vartheta} = \overline{\vartheta}_n, q - \widetilde{q} = \overline{q}, f_n - \widetilde{f}_n = \overline{f}_n$, we obtain the integral equation

$$\overline{\vartheta}_n(t) = \overline{\varphi}_n E_\alpha(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n^2 (t-\tau)^\alpha\right) \overline{f}_n(\tau) d\tau +$$

$$+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)\overline{q}(\tau)\vartheta_n(\tau)d\tau++\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)\widetilde{q}(\tau)\overline{\vartheta}_n(\tau)d\tau, \quad (2.22)$$

from which the following linear integral inequality for $|\overline{\vartheta}_n(t)|$ is derived:

$$|\overline{\vartheta}_n(t)| \leq |\overline{\varphi}_n| + \frac{t^\alpha\|\overline{f}_n\|}{\alpha\Gamma(\alpha)} + \frac{\|\overline{q}\|t^\alpha}{\alpha\Gamma(\alpha)}\Big(|\varphi_n| + \frac{t^\alpha\|f_n\|}{\alpha\Gamma(\alpha)}\Big)E_\alpha(\|q\|t^\alpha) + \frac{\|\widetilde{q}\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|\overline{\vartheta}_n(\tau)|d\tau.$$

Using Lemma 2.5, from the last inequality, we arrive at the estimate:

$$|\overline{\vartheta}_n(t)| \leq \Big\{|\overline{\varphi}_n| + \frac{t^\alpha\|\overline{f}_n\|}{\alpha\Gamma(\alpha)} + \frac{\|\overline{q}\|t^\alpha}{\alpha\Gamma(\alpha)}\Big(|\varphi_n| + \frac{t^\alpha\|f_n\|}{\alpha\Gamma(\alpha)}\Big)E_\alpha(\|q\|t^\alpha)\Big\}E_\alpha(\|\widetilde{q}\|t^\alpha), \ \ t\in[0,T]. \quad (2.23)$$

Indeed, the expression (2.23) is stability estimate for Fourier coefficients of the solution to the problem (2.7)-(2.9). The uniqueness of the solution to the problem (2.7)-(2.9) follows from (2.11) and (2.23).

2.2. Investigation of the inverse problem

In this section, we study the inverse problem as the problem of determining of functions $q(t)$ from relations (2.7)-(2.10), using the contraction mapping principle.

Firstly, by differentiating (2.11) with respect to x, we get the following equality:

$$\vartheta_x(x,t) = \sum_{n=1}^{\infty} \lambda_n\vartheta_n(t)\cos(\lambda_n x). \quad (2.24)$$

Setting in (2.24) $x=0$ and using additional condition (2.10), after simple converting, we obtain the integral equation for determining $q(t)$:

$$q(t) = q_0(t) - \frac{1}{h(t)}\sum_{n=1}^{\infty}\lambda_n\vartheta_n(t;q), \quad (2.25)$$

where

$$q_0(t) = \frac{1}{h(t)}\Big[f_x(0,t) - \partial^\alpha h(t)\Big],$$

and $\vartheta_n(t;q)$ means that the solution of integral equation (2.14) depends on $q(t)$.

We introduce an operator B, defining it by the right hand side of (2.25)

$$B[q](t) = q_0(t) - \frac{1}{h(t)}\sum_{n=1}^{\infty}\lambda_n\vartheta_n(t;q). \quad (2.26)$$

Then, the equation (2.25) is written in more convenient form

$$q(t) = B[q](t). \quad (2.27)$$

Let $q_{00} := \max_{t\in[0,T]}|q_0(t)|$. Fix a number $\rho>0$ and consider the ball

$$\Phi^T(q_0,\rho) := \{q(t) : q(t)\in C[0,T], \|q-q_0\|_{C[0,T]} \leq \rho\}.$$

Theorem 2.8 *Let (A1)-(A3) be satisfied. Then there exists a number $T^* \in (0,T)$, such that there exists a unique solution $q(t) \in C[0,T^*]$ of the inverse problem (2.7)-(2.10).*

Proof Let us first prove that for an enough small $T>0$ the operator B maps the ball $\Phi^T(q_0,\rho)$ into itself. Indeed, for any function $q(t) \in C[0,T]$, the function $B[q](t)$ calculated using formula (2.27) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\|B[q](t)-q_0(t)\| \leq \frac{1}{h_0}E_{\alpha,1}(\|q\|T^\alpha)\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_n|+\frac{T^\alpha\|f_n\|}{\alpha\Gamma(\alpha)}\Big).$$

Here we have used the estimate (2.16). In view of Lemmas 2.5 and 2.6, last series is convergent. Note that the function occurring on the right-hand side in this inequality is monotone increasing with T, and the fact that the function $q(t)$ belongs to the ball $\Phi^T(q_0,\rho)$ implies the inequality

$$\|q\| \leq \rho + \|q_0\|. \tag{2.28}$$

Therefore, we only strengthen the inequality if we replace $\|q\|$ in this inequality with the expression $\rho+\|q_0\|$. Performing these replacements, we obtain the estimate

$$\|B[q](t)-q_0(t)\| \leq \frac{1}{h_0}E_{\alpha,1}((\rho+\|q_0\|)T^\alpha)\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_n|+\frac{T^\alpha\|f_n\|}{\alpha\Gamma(\alpha)}\Big).$$

Let T_1 be a smallest positive root of the equation

$$m_1(T)=\frac{1}{h_0}E_{\alpha,1}((\rho+\|q_0\|)T^\alpha)\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_n|+\frac{T^\alpha\|f_n\|}{\alpha\Gamma(\alpha)}\Big)=\rho.$$

Then for $T\in[0,T_1]$ we have $B[q](t)\in\Phi^T(q_0,\rho)$.

Now consider two functions $q(t)$ and $\widetilde{q}(t)$ belonging to the ball $\Phi^T(q_0,\rho)$ and estimate the distance between $B[q](t)$ and $B[\widetilde{q}](t)$ in the space $C[0,T]$. The function $\widetilde{\vartheta}_n(t)$ corresponding to $\widetilde{q}(t)$ satisfies the integral equation (2.21) with the functions $\varphi_n=\widetilde{\varphi}_n$ and $f_n=\widetilde{f}_n$. Composing the difference $B[q](t)-B[\widetilde{q}](t)$ with the help of equations (2.14), (2.21) and then estimating its norm, we obtain

$$\|B[q](t)-B[\widetilde{q}](t)\| \leq \frac{1}{h_0}\sum_{n=1}^{\infty}\lambda_n\|\overline{\vartheta}_n\|.$$

Using inequality (2.16) and the estimate (2.23) with $\varphi_n=\widetilde{\varphi}_n$ and $f_n=\widetilde{f}_n$, we continue the previous inequality in the following form:

$$\|B[q](t)-B[\widetilde{q}](t)\| \leq \frac{1}{h_0}E_{\alpha,1}(\|\widetilde{q}\|T^\alpha)E_{\alpha,1}(\|q\|T^\alpha)\sum_{n=1}^{\infty}\frac{\lambda_n\|\overline{q}\|T^\alpha}{\alpha\Gamma(\alpha)}\Big(|\varphi_n|+\frac{T^\alpha\|f_n\|}{\alpha\Gamma(\alpha)}\Big). \tag{2.29}$$

The functions $q(t)$ and $\widetilde{q}(t)$ belong to the ball $\Phi^T(q_0,\rho)$, and hence for each of these functions one has inequality (2.28). Note that the function on the right-hand side in inequality (2.29) at the factor $\|\overline{q}\|$ is monotone increasing with $\|q\|, \|\widetilde{q}\|$, and T. Consequently, replacing $\|q\|$ and $\|\widetilde{q}\|$ in inequality (2.29) with $\rho+\|q_0\|$ will only strengthen the inequality. This, we have

$$\|B[q](t)-B[\widetilde{q}](t)\| \leq \frac{1}{h_0}\Big(E_{\alpha,1}((\rho+\|q_0\|)T^{\alpha})\Big)^2 \sum_{n=1}^{\infty} \frac{\lambda_n T^{\alpha}}{\alpha\Gamma(\alpha)}\Big(|\varphi_n| + \frac{T^{\alpha}\|f_n\|}{\alpha\Gamma(\alpha)}\Big)\|\overline{q}\| \leq m_2(T)\|\overline{q}\|.$$

Let T_2 be a smallest positive root of the equation

$$m_2(T) = \frac{1}{h_0}\Big(E_{\alpha,1}((\rho+\|q_0\|)T^{\alpha})\Big)^2 \sum_{n=1}^{\infty} \frac{\lambda_n T^{\alpha}}{\alpha\Gamma(\alpha)}\Big(|\varphi_n| + \frac{T^{\alpha}\|f_n\|}{\alpha\Gamma(\alpha)}\Big) = 1.$$

Then, for $T \in [0, T_2)$, we have that the distance between the functions $B[q](t)$ and $B[\widetilde{q}](t)$ in $C[0,T]$ is not greater than the distance between the functions $\|q\|$ and $\|\widetilde{q}\|$. Consequently, if we choose $T^* < \min(T_1, T_2)$, then the operator B is a contraction in the ball $\Phi^T(q_0, \rho)$. However, in accordance with the Banach theorem, the operator B has unique fixed point in the ball $\Phi^T(q_0, \rho)$; i.e. there exists a unique solution of equation (2.27). Theorem 2.8 is proven. □

Let T, l be positive fixed numbers. Consider the set $K(\chi_0)(\chi_0 > 0$ is some fixed number) of the given functions (φ, f, h), for which all conditions (A1)-(A3) are fulfilled and

$$\max\Big\{\|\varphi\|_{C^4[0,T]}, \|h\|_{C^1[0,T]}, \|h^{-1}\|_{C[0,T]}, \|f\|_{C^{4,1}(D_{Tl})}\Big\} \leq \chi_0.$$

We denote by $Q(\chi_1)$ the set of functions $q(t)$, satisfying the following condition $\|q\|_{C[0,T]} \leq \chi_1, \ \ \chi_1 > 0$.

Theorem 2.9 *Let $(\varphi, f, h) \in K(\chi_0)$, $(\widetilde{\varphi}, \widetilde{f}, \widetilde{h}) \in K(\chi_0)$ and $q \in Q(\chi_1)$, $\widetilde{q} \in Q(\chi_1)$. Then, for solution of the inverse problem (2.7)-(2.10), the following stability estimate holds:*

$$\|q-\tilde{q}\|_{C[0,T]} \leq d\Big(\|\varphi-\tilde{\varphi}\|_{C^4[0,l]} + \|f-\tilde{f}\|_{C^{4,1}(\Omega_{lT})} + \|h-\tilde{h}\|_{C^1[0,T]}\Big), \tag{2.30}$$

where the constant d depends only on T, l, χ_0, χ_1.

Proof To prove this theorem, using (2.25), we write down the equations for $\widetilde{q}(t)$ and compose the difference $\overline{q}(t) = q(t) - \widetilde{q}(t)$. Then, after estimating the resulting relation and using (2.16),(2.23), we obtain the following estimates:

$$|q(t)-\widetilde{q}(t)| \leq d_0\Big(\|\overline{f}\| + \|\overline{\varphi}\| + \|\overline{h}\|\Big) + d_1\int_0^t (t-\tau)^{\alpha-1}|q(\tau)-\widetilde{q}(\tau)|d\tau, t \in [0,T], \tag{2.31}$$

where d_0 d_1 depends only on $\chi_0, \chi_1, T, \alpha$. Applying a Gronwall-type inequality to (2.31) with a weakly singular kernel, we have the following estimate (see [31, 33]):

$$|q(t)-\widetilde{q}(t)| \leq d_0\Big(\|\overline{f}\| + \|\overline{\varphi}\| + \|\overline{h}\|\Big)E_{\alpha,1}\Big(d_1\Gamma(\alpha)t^{\alpha}\Big), t \in [0,T], \tag{2.32}$$

This inequality implies the estimate (2.30), if we set $d = d_0E_{\alpha,1}\Big(d_1\Gamma(\alpha)T^{\alpha}\Big)$. □

From Theorem 2.9 follows also the next assertion on uniqueness in whole for solution to the inverse problem.

Theorem 2.10 *Let the functions φ, f, h and $\widetilde{\varphi}, \widetilde{f}, \widetilde{h}$ have the same meaning as in Theorem 2.9 and conditions (A1)-(A3). Moreover, if $\varphi = \widetilde{\varphi}, f = \widetilde{f}, h = \widetilde{h}$ for $t \in [0,T]$, then $q(t) = \widetilde{q}(t), t \in [0,T]$.*

3. Numerical procedure

The objective of this section is to present a numerical solution of the inverse problem for a fractional diffusion equation (2.1) with initial boundary (2.3) and overdetermination (2.4) conditions using the finite difference method. Consider a one-dimensional domain $\Omega \in (0,l)$. Let N_x be the total number of discretization points. Hence, we define $\Delta x = l/N_x$ as the spatial step size and denote the discretized points as, $x_i = i\Delta x$ where $0 \le i \le N_x$ is a positive integer. Let u_i^n be an approximation of $u(x_i, t_n)$, where $t_n = n\Delta t$ and $\Delta t = T/N_t$ is the temporal step size, and N_t is the total number of time steps.

The fractional derivative of the function $u(x,t)$ in Eq. (2.1) defined by (2.2) can be approximated as,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \approx \frac{1}{\Gamma(2-\alpha)\Delta t^\alpha} \sum_{j=0}^{n-1} a_j (u_i^{n-j+1} - u_i^{n-j}), \quad n \in (0, N_t) \tag{3.1}$$

with order $\alpha > 0$, where $\Gamma(\cdot)$ is the Euler's gamma function and $a_j = (1+j)^{1-\alpha} - j^{1-\alpha}$, for details see [2]. By utilizing the approximation (3.1) for the time derivative and centered second-order finite difference for spatial derivative, Eq. (2.1) can be rewritten in the following form:

$$\frac{1}{\Gamma(2-\alpha)\Delta t^\alpha} \sum_{j=0}^{n-1} a_j (u_i^{n-j+1} - u_i^{n-j}) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - q^n u_i^n + f_i^n. \tag{3.2}$$

Then, u_i^{n+1} at grid point i for the time step $n+1$ results in

$$u_i^{n+1} = u_i^n + \frac{1}{a_0}\Big(\Delta t^\alpha \Gamma(2-\alpha)(A_i^n - q^n u_i^n) - \sum_{j=1}^{n} a_j (u_i^{n+1-j} - u_i^{n-j})\Big), \tag{3.3}$$

where $A_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + f_i^n$ has been used. Furthermore, we denote $B_i^n = \sum_{j=1}^n a_j (u_i^{n+1-j} - u_i^{n-j})$ to simplify the expression (3.3). Note that at time $n = 0$ the last term $\sum_{j=1}^n a_j (u_i^{n+1-j} - u_i^{n-j})$ of (3.3) leads to u_j^{-1}, which is undefined; hence, this summation has been neglected for $n = 0$. The unknown coefficient q^n can be determined by applying the forward finite difference to the overdetermination condition (2.4) at the grid point $i = 0$, [5],

$$\frac{u_1^{n+1} - u_0^{n+1}}{\Delta x} = h^{n+1}. \tag{3.4}$$

Components u_0^{n+1} and u_1^{n+1} can be obtained at the discretized points $i = 0$ and $i = 1$ from Eq. (3.3). Then, by substituting u_0^{n+1} and u_1^{n+1} the unknown coefficient q^n can be evaluated as,

$$q^n = \frac{u_1^n - u_0^n + \dfrac{1}{a_0}\Delta t^\alpha \Gamma(2-\alpha)(A_1^n - A_0^n) - \dfrac{1}{a_0}(B_1^n - B_0^n) - \Delta x h^{n+1}}{\dfrac{1}{a_0}\Delta t^\alpha \Gamma(2-\alpha)(u_1^n - u_0^n)}. \tag{3.5}$$

Eq. (3.5) can be reduced to a compact form using the boundary condition (2.4), note that $a_0 = 1$,

$$q^n = \frac{u_1^n + \Delta t^\alpha \Gamma(2-\alpha) A_1^n - B_1^n - \Delta x h^{n+1}}{\Delta t^\alpha \Gamma(2-\alpha) u_1^n}. \tag{3.6}$$

Based on these equations, Algorithm 1 has been constructed for the simultaneous determination of $u(x,t)$ and the unknown coefficient $q(t)$, and implemented in open-source Python.

Algorithm 1 Finite difference scheme for the evolution of $u(x,t)$ and the unknown coefficient $q(t)$

Require: α, $\Omega = [0,l]$, N_x, T, N_t
Ensure: $u_i^0 = \varphi(x_i)$, f_i^n, h^n, $\Gamma(2-\alpha)$ ▷ For clarity $u_i^n = u(x_i, t_n)$, $f_i^n = f(x_i, t_n)$, $h^n = h(t_n)$
$\Delta x = l/N_x$, $x_i = i\Delta x$, $N_x \leftarrow i$
$\Delta t = T/N_t$, $t_n = n\Delta t$, $N_t \leftarrow n$
while $n \le N_t$ **do**
  while $i \le N_x - 1$ **do**
    $A_i^n = (u_{i+1}^n - 2u_i^n + u_{i-1}^n)/\Delta x^2 + f_i^n$ ▷ in Eq. (3.3)
    while $j < n$ **do**
      $a_j = (2+j)^{1-\alpha} - (j+1)^{1-\alpha}$ ▷ Note that $a_0 = 1$
      $B_i^n = \sum_{j=1}^n a_j (u_i^{n+1-j} - u_i^{n-j})$ ▷ in Eq. (3.5)
    end while
  end while
  evaluate q^n using Eq. (3.5)
  while $i \le N_x - 1$ **do**
    $u_i^{n+1} = u_i^n + 1/a_0 \Big(\Delta t^\alpha \Gamma(2-\alpha)(A_i^n - q^n u_i^n) - B_i^n \Big)$ ▷ Opdate u_i
  end while
  $u_0^{n+1} = u_{N_x-1}^{n+1} = 0$ ▷ Enforce the boundary conditions
end while

4. Numerical example

In this section, the numerical results obtained using Algorithm 1 are presented for the test example (4.1). The results are presented for three different values of α: 0.5, 0.75, and 1. In this example, we consider $l = 2\pi$ and $T = 1$. The computational details are provided in Section 3. The results have been analyzed by calculating the relative error between the exact and estimated solutions, defined as,

$$\eta(u) = max_{1 \le i \le N_x} |u_i^{numerical} - u_i^{exact}|$$
$$\eta(q_n) = |q_n^{numerical} - q_n^{exact}|$$

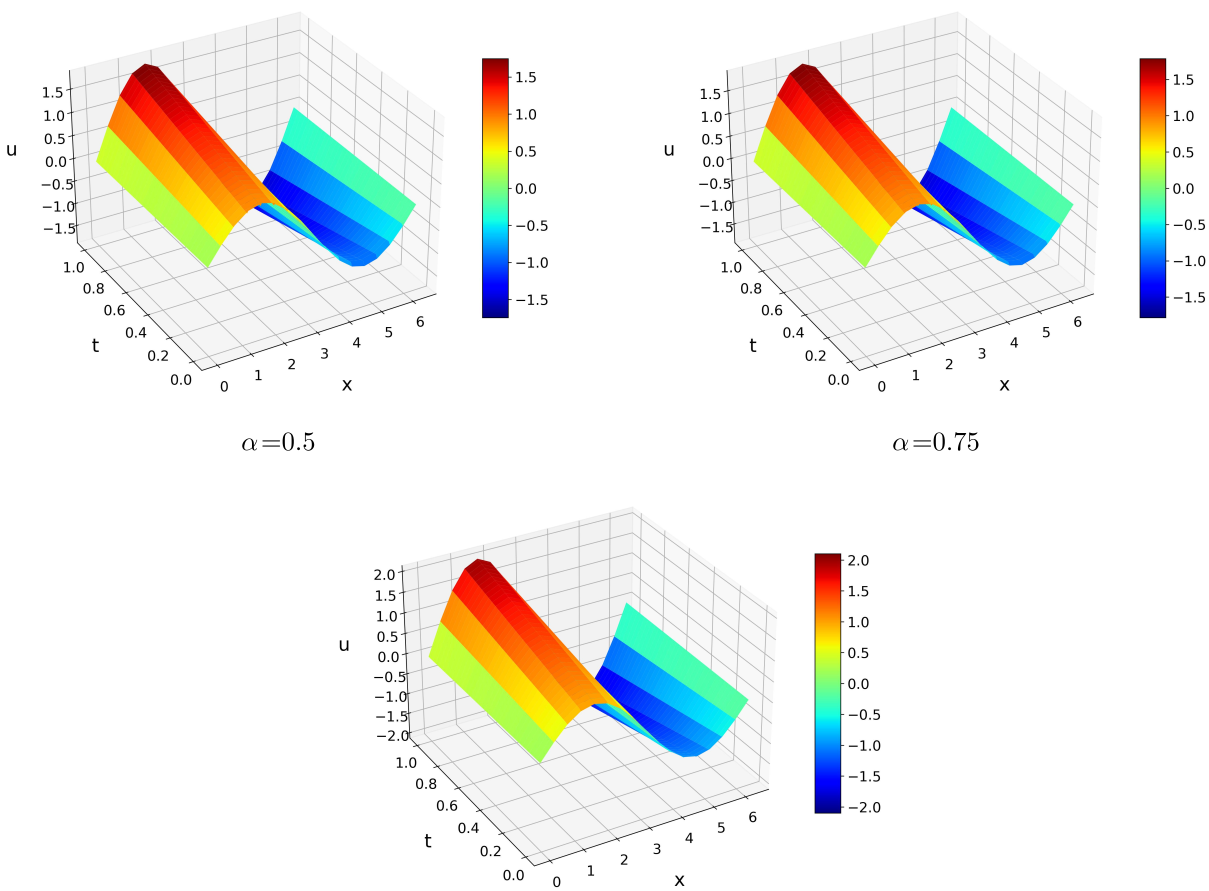
We solve the fractional inverse problem (2.1)-(2.4) with the following input data:

$$\varphi(x) = \Gamma(2-\alpha)\sin(x), \quad h(t) = \Gamma(2-\alpha)(1+t) \quad f(x,t) = \Big(t^{1-\alpha} + \Gamma(2-\alpha)(1+t) + 1\Big)\sin(x) \tag{4.1}$$

for $x \in (0, l = 2\pi)$ and $t \in (0, T = 1)$. The exact solution is given by

$$u(x,t) = \Gamma(2-\alpha)(1+t)\sin(x), \quad q(t) = \frac{1}{\Gamma(2-\alpha)(1+t)}.$$

The one-dimensional domain was discretized with N_x=16 grid points and a with grid spacing of $\Delta x = l/N_x$ in the program, where $l = 2\pi$. The time increment between the time steps Δt was taken as 10^{-4}. We choose a smaller N_x due to the computation time; however, we could use a finer time increment Δt and larger discretization points N_x for better accuracy in $q(t)$ and $u(x,t)$, respectively. Figure 1 presents numerical results for $u(x,t)$.



α=0.5

α=0.75

α=1

Figure 1. Numerical evolution of $u(x,t)$ over time for α=0.5, 0.75, 1.

For clarity, the comparison of numerical and analytical solutions are presented in Figure 2. As can be seen, as α increases, the maximum value of u increases slightly.

The values of the relative errors $\eta(u)$ and $\eta(q)$ between the exact and numerical solutions of $u(x,t)$ and $q(t)$, respectively, are shown in Tables 1 and 2.

Figure 3 compares the analytical and numerical solutions of $q(t)$. As can be seen, near time $t = 0$, the numerically evaluated values of q^n are relatively high compared to the analytical values for $\alpha = 0.5$ and $\alpha = 0.75$.

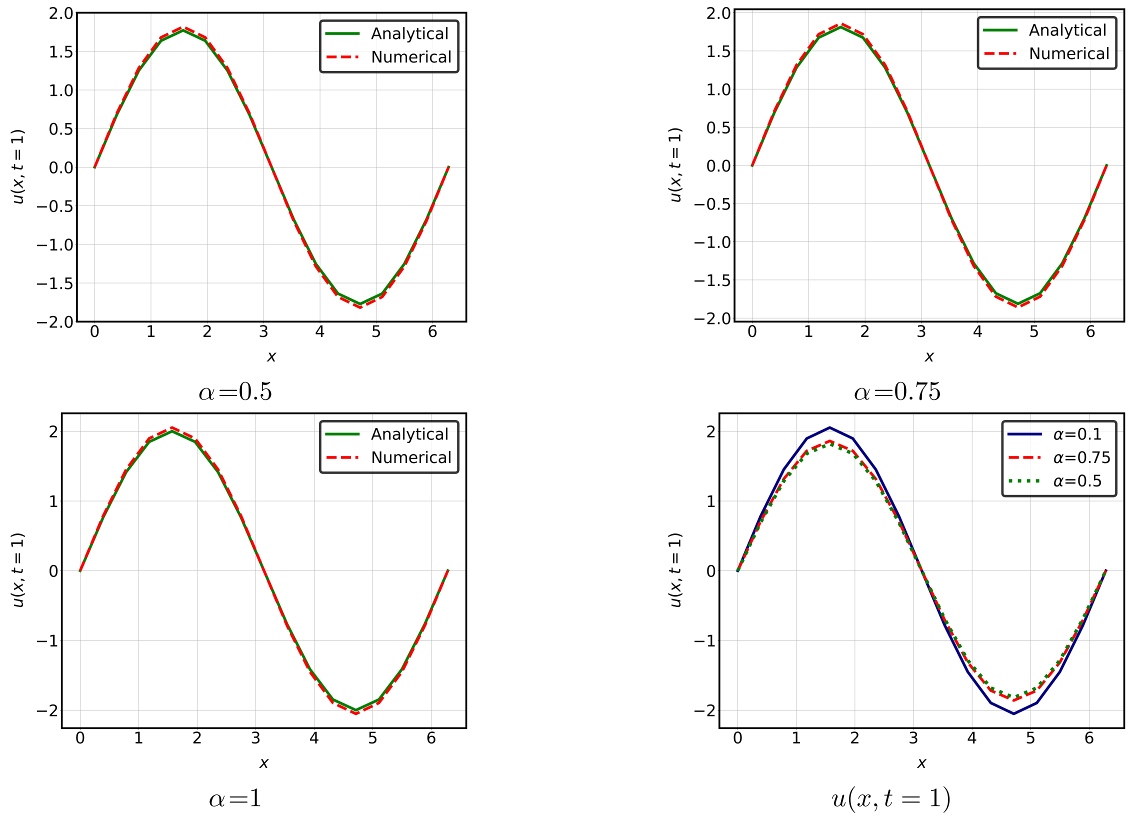


Figure 2. (a), (b), (c) - comparison between the numerical and analytical solutions of $u(x,t)$. (d) - numerical results of $u(x,t=1)$ for the final time step.

Table 1. The relative error $\eta(u)$ between the exact and numerical solutions of $u(x,t)$.

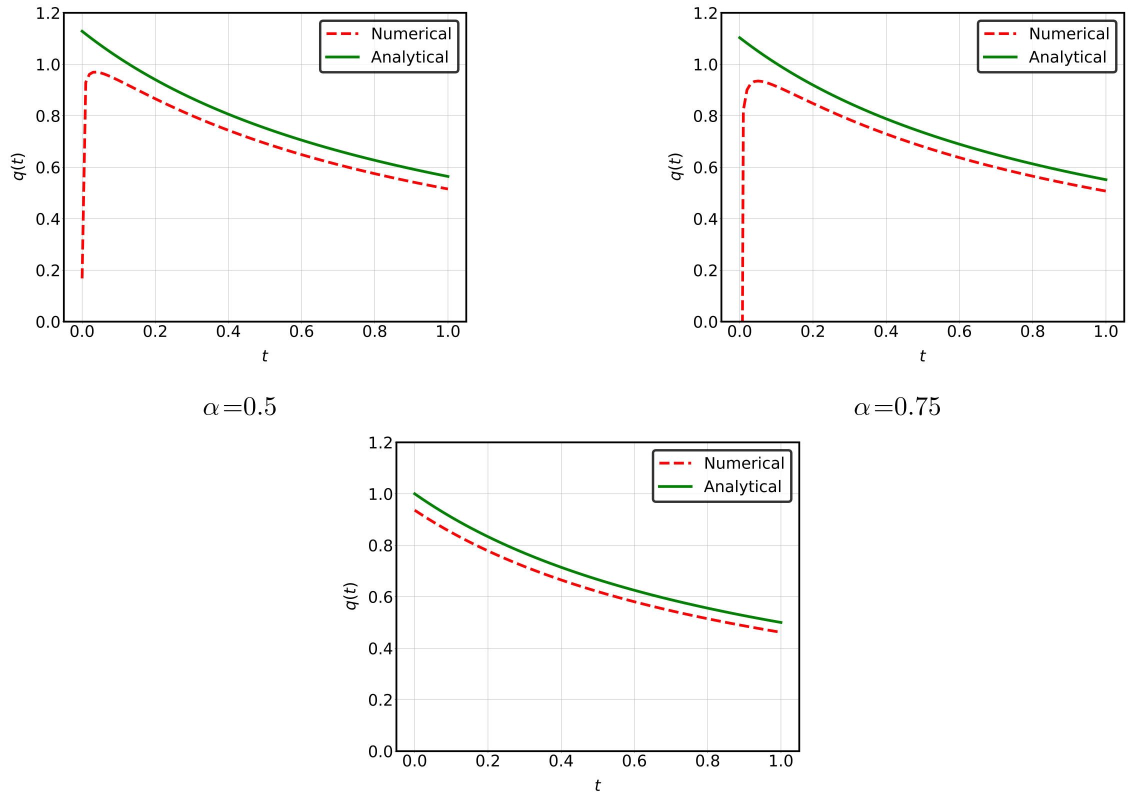
$\eta(u)$										
t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\alpha=1$	0.0288	0.0315	0.0341	0.0367	0.0393	0.0419	0.0445	0.0418	0.0441	0.0464
$\alpha=0.75$	0.0261	0.0285	0.0309	0.0333	0.0356	0.0380	0.0404	0.0427	0.0451	0.0475
$\alpha=0.5$	0.0256	0.0279	0.0302	0.0325	0.0348	0.0371	0.0395	0.0418	0.0441	0.0464

Table 2. The relative error $\eta(q)$ between the exact and numerical solutions of $q(t)$.

$\eta(q)$											
t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\alpha=1$	0.0639	0.0590	0.0552	0.0519	0.0491	0.0467	0.0446	0.0427	0.0410	0.0395	0.0382
$\alpha=0.75$	3.6275	0.0888	0.0715	0.0637	0.0588	0.0551	0.0522	0.0498	0.0477	0.0459	0.0443
$\alpha=0.5$	0.96	0.0886	0.0743	0.0672	0.0625	0.0591	0.0562	0.0539	0.0519	0.0502	0.0487

5. Conclusion

The inverse problem of determining the unknown coefficient in the fractional diffusion equation is considered. First, the inverse problem is analyzed for the existence and uniqueness of the solution. Second, a numerical



α=0.5 α=0.75

α=1

Figure 3. Comparison between the numerical and analytical solutions of $q(t)$.

procedure for the inverse problem was presented using the finite difference approach to simultaneously identify the time-dependent coefficient in the one-dimensional fractional diffusion equation. The resulting inverse problem was reformulated as a constrained regularized minimization problem, which was solved using the open-source programming language, Python. The numerical results for $u(x,t)$ are in good agreement with the analytical solution. However, the numerical results of q^0 at time $t=0$ are relatively high compared to the analytical solution for $\alpha=0.5$ and $\alpha=0.75$. Nevertheless, the numerically obtained values of q^n stabilized further over time.

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