Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2023) 47: 1595 – 1615

doi:10.55730/1300-0098.3452

Adjunction greatest element to ordered hypersemigroups

Niovi KEHAYOPULU*
TBAE, The Scientific and Technological Research Council of Türkiye, Turkiye

Received: 10.07.2023 • **Accepted/Published Online:** 08.08.2023 • **Final Version:** 25.09.2023

Abstract: As a continuation of the paper "Adjunction Identity to Hypersemigroup" in Turk J Math 2022; 46 (7): 2834–2853, it has been proved here that the adjunction of a greatest element to an ordered hypersemigroup is actually an embedding problem. The concept of pseudoideal has been introduced and has been proved that for each ordered hypersemigroup S an ordered hypersemigroup V having a greatest element (poe-hypersemigroup) can be constructed in such a way that there exists a pseudoideal T of S such that S is isomorphic to T. If S does not have a greatest element, then this can be regarded as the embedding of an ordered hypersemigroup in an ordered semigroup with greatest element.

Key words: *poe*-hypersemigroup, pseudoideal, embedding, semisimple, ideal, bi-ideal

1. Introduction

Fuchs and Halperin have shown that every regular ring can be embedded in a regular ring with identity [1]. The problem of adjunction identity to semigroups, greatest element to ordered sets has been considered in [2]. In both cases the adjunction has the same meaning: If S (P) is a semigroup (ordered set) without identity (greatest element), the adjunction of an identity to S (P) means that we construct a semigroup (ordered set) V with identity (greatest element) in such a way that there exists an ideal I of V such that $S \cong I$ ($P \cong I$). Later, it has been proved that each ordered semigroup S can be embedded in an ordered semigroup having a greatest element. If S does not have a greatest element, then this is a problem of adjunction greatest element to S [3]. The problem of adjunction identity to hypersemigroups has been considered [7]. As a continuation of the paper in [7], we discuss here the problem of adjunction of a greatest element to an ordered hypersemigroup.

2. Main result

A hypersemigroup is a nonempty set S with an "operation" $\circ$ assigning to each couple (a,b) of S a nonempty subset $a \circ b$ (called hyperoperation as the $a \circ b$ is a subset and not element of S) and an operation $*$ between the nonempty subsets A, B of S such that $A * B = \bigcup\{a \circ b \mid a \in A,\, b \in B\}$ satisfying the relation $\{a\} * (b \circ c) = (a \circ b) * \{c\}$ for all $a,b,c \in S$ [4]. A hypersemigroup $(S,\circ)$ is called an ordered hypersemigroup if there exists an order relation $\leq$ on S such that $a \leq b$ implies $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$ for every $c \in S$; in the sense that for every $u \in a \circ c$ there exists $v \in b \circ c$ such that $u \leq v$ and for every $u \in c \circ a$ there exists $v \in c \circ b$ such that $u \leq v$ [5].

*Correspondence: niovi.kehayopulu-tbae@tubitak.gov.tr
2010 *AMS Mathematics Subject Classification:* 06F99, 08A99, 20M75.

Definition 2.1 *Let* $(S, \circ, \leq)$ *be an ordered hypersemigroup. A nonempty subset* T *of* S *is called pseudoideal of* $(S, \circ, \leq)$ *if*

(1) $T * T \subseteq T$ *and*

(2) *if* $a \in T$ *and* $S \ni b \leq a$ *then* $b \in T$.

Definition 2.2 *Two ordered hypersemigroups* $(S, \circ, \leq)$ *and* $(T, \overline{\circ}, \preceq)$ *are called isomorphic if there exists a (1-1) mapping* f *of* S *onto* T *such that, for every* $a, b \in S$, *we have*

(1) $f(a \circ b) \subseteq f(a) \overline{\circ} f(b)$; *in the sense that if* $u \in a \circ b$, *then* $f(u) \in f(a) \overline{\circ} f(b)$

(2) *if* $a \leq b$, *then* $f(a) \preceq f(b)$

(3) *if* $a, b \in S$ *such that* $f(a) \preceq f(b)$, *the* $a \leq b$.

Theorem 2.3 *Let* $(S, \circ, \leq)$ *be an ordered hypersemigroup. Then there exists an ordered hypersemigroup* V *having a greatest element (poe-hypersemigroup) and a pseudoideal* T *of* V *such that* $S \cong T$.

Proof For an element e not containing in S ((x, x) is, for example, such an element), we consider the set $S \cup \{e\}$. We define an hyperoperation "$\overline{\circ}$" on $S \cup \{e\}$ and an operation "$\overline{*}$" on the set $\mathcal{P}^*\big(S \cup \{e\}\big)$ of all nonempty subsets of S as follows:

$$\overline{\circ} : \big(S \cup \{e\}\big) \times \big(S \cup \{e\}\big) \to \mathcal{P}^*\big(S \cup \{e\}\big) \mid (x, y) \to x \,\overline{\circ}\, y \text{ where}$$

$$x \,\overline{\circ}\, y = \begin{cases} x \circ y & \text{if } x, y \in S \\ \{e\} & \text{if } x \in S,\ y = e \\ \{e\} & \text{if } x = e,\ y \in S \\ \{e\} & \text{if } x = y = e \end{cases}$$

$$\overline{*} : \mathcal{P}^*\big(S \cup \{e\}\big) \times \mathcal{P}^*\big(S \cup \{e\}\big) \to \mathcal{P}^*\big(S \cup \{e\}\big) \mid (A, B) \to A \,\overline{*}\, B \text{ where}$$

$$A \,\overline{*}\, B = \bigcup_{a \in A,\, b \in B} a \,\overline{\circ}\, b$$

(For $A = \{x\}$, $B = \{y\}$, we clearly have $\{x\} \,\overline{*}\, \{y\} = \bigcup\limits_{u \in \{x\}, v \in \{y\}} u \,\overline{\circ}\, v = \bigcup\limits_{u = x, v = y} u \,\overline{\circ}\, v = x \,\overline{\circ}\, y$)

Then $\big(S \cup \{e\}, \overline{\circ}, \overline{*}\big)$ is a hypersemigroup. In fact:

(A) The operation $\overline{\circ}$ is well defined. Indeed: If $x, y \in S$, then $x \,\overline{\circ}\, y = x \circ y \subseteq S \subseteq S \cup \{e\}$. Otherwise, $x \,\overline{\circ}\, y = \{e\} \subseteq S \cup \{e\}$. Let $(x, y), (z, t) \in \big(S \cup \{e\}\big) \times \big(S \cup \{e\}\big)$ such that $(x, y) = (z, t)$. Then $x \,\overline{\circ}\, y = z \,\overline{\circ}\, t$. Indeed: If $x, y \in S$, then $z, t \in S$, $x \,\overline{\circ}\, y = x \circ y = z \circ t = z \,\overline{\circ}\, t$. If $x \in S$, $y = e$, then $z \in S$, $t = e$, $x \,\overline{\circ}\, y = \{e\}$ and $z \,\overline{\circ}\, t = \{e\}$ and so $x \,\overline{\circ}\, y = z \,\overline{\circ}\, t$. If $x = e$, $y \in S$, then $z = e$, $t \in S$, $x \,\overline{\circ}\, y = \{e\}$, $z \,\overline{\circ}\, t = \{e\}$ and so $x \,\overline{\circ}\, y = z \,\overline{\circ}\, t = \{e\}$. If $x = y = e$, then $z = t = e$ and $x \,\overline{\circ}\, y = \{e\} = z \,\overline{\circ}\, t$.

(B) The operation $\overline{*}$ is well defined. Indeed: Let $A, B \in \mathcal{P}^*\big(S \cup \{e\}\big)$. Since $\emptyset \neq a \,\overline{\circ}\, b \subseteq S \cup \{e\}$ for every $a \in A$ and every $b \in B$, we have $\emptyset \neq A \,\overline{*}\, B \subseteq S \cup \{e\}$. Let $(A, B), (C, D) \in \mathcal{P}^*\big(S \cup \{e\}\big) \times \mathcal{P}^*\big(S \cup \{e\}\big)$ such that $(A, B) = (C, D)$. Then $A \,\overline{*}\, B = \bigcup\limits_{a \in A,\, b \in B} a \,\overline{\circ}\, b = \bigcup\limits_{a \in C,\, b \in D} a \,\overline{\circ}\, b = C \,\overline{*}\, D$.

(C) $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$ for every $x, y, z \in S \cup \{e\}$. Indeed:
We have to check the following two cases:

(a) $x \in S$, $\big(y \in S$ or $y = e\big)$, $\big(z \in S$ or $z = e\big)$ and

(b) $x = e$, $\big(y \in S$ or $y = e\big)$, $\big(z \in S$ or $z = e\big)$.

(1) If $x, y, z \in S$, then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$; its proof is the same with the proof in [7, p. 2838].

(2) Let $x, y \in S$, $z = e$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$. Indeed: We have

$\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{x\} \mathbin{\overline{*}} \{e\} = x \mathbin{\overline{\circ}} e = \{e\}$ and $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{e\}$.

On the other hand, $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{e\} = \{e\}$. Indeed: If $t \in (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{e\}$, then $t \in u \mathbin{\overline{\circ}} e$ for some $u \in x \mathbin{\overline{\circ}} y$. Since $x, y \in S$, we have $x \mathbin{\overline{\circ}} y = x \circ y$, then $u \in x \circ y \subseteq S$. Since $u \in S$, we have $u \mathbin{\overline{\circ}} e = \{e\}$, then $t = e$ and so $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{e\} \subseteq \{e\}$. Let now $t = e$. Take an element $u \in x \mathbin{\overline{\circ}} y$ $(x \mathbin{\overline{\circ}} y \neq \emptyset)$. Since $x, y \in S$, we have $x \mathbin{\overline{\circ}} y = x \circ y \subseteq S$. Since $u \in S$, we have $u \mathbin{\overline{\circ}} e = \{e\}$. Then $t = e \in u \mathbin{\overline{\circ}} e = \{u\} \mathbin{\overline{*}} \{e\} \subseteq (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{e\}$ and so $\{e\} \subseteq (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{e\}$.

(3) Let $x \in S$, $y = e$, $z \in S$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$. Indeed: We have

$\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{x\} \mathbin{\overline{*}} (e \mathbin{\overline{\circ}} z) = \{x\} \mathbin{\overline{*}} \{e\} = x \mathbin{\overline{\circ}} e = \{e\}$ and

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = (x \mathbin{\overline{\circ}} e) \mathbin{\overline{*}} \{z\} = \{e\} \mathbin{\overline{*}} \{z\} = e \mathbin{\overline{\circ}} z = \{e\}$.

(4) Let $x \in S$, $y = z = e$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$. Indeed: We have

$\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{x\} \mathbin{\overline{*}} (e \mathbin{\overline{\circ}} e) = \{x\} \mathbin{\overline{*}} \{e\} = x \mathbin{\overline{\circ}} e = \{e\}$ and

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = (x \mathbin{\overline{\circ}} e) \mathbin{\overline{*}} \{e\} = \{e\} \mathbin{\overline{*}} \{e\} = e \mathbin{\overline{\circ}} e = \{e\}$.

(5) Let $x = e$, $y \in S$, $z \in S$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$. Indeed: We have

$\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$ and

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = (e \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{e\} \mathbin{\overline{*}} \{z\} = e \mathbin{\overline{\circ}} z = \{e\}$.

On the other hand, $\{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{e\}$. Indeed: If $t \in \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$, then $t \in e \mathbin{\overline{\circ}} u$ for some $u \in y \mathbin{\overline{\circ}} z$. Since $y, z \in S$, we have $y \mathbin{\overline{\circ}} z = y \circ z \subseteq S$. Since $u \in S$, we have $e \mathbin{\overline{\circ}} u = \{e\}$ and so $t = e$. Let now $t = e$. Take an element $u \in y \mathbin{\overline{\circ}} z$ $(y \mathbin{\overline{\circ}} z \neq \emptyset)$. Since $y, z \in S$, we have $y \mathbin{\overline{\circ}} z = y \circ z \subseteq S$. Since $u \in S$, we have $e \mathbin{\overline{\circ}} u = \{e\}$. Then we have $t = e \in e \mathbin{\overline{\circ}} u = \{e\} \mathbin{\overline{*}} \{u\} \subseteq \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$ and so $\{e\} \subseteq \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$.

(6) Let $x = e$, $y \in S$, $z = e$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$. Indeed: We have

$\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} e) = \{e\} \mathbin{\overline{*}} \{e\} = e \mathbin{\overline{\circ}} e = \{e\}$ and

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = (e \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{e\} = \{e\} \mathbin{\overline{*}} \{e\} = \{e\}$.

(7) Let $x = e$, $y = e$, $z \in S$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$. Indeed: We have

$\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{e\} \mathbin{\overline{*}} (e \mathbin{\overline{\circ}} z) = \{e\} \mathbin{\overline{*}} \{e\} = e \mathbin{\overline{\circ}} e = \{e\}$ and

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = (e \mathbin{\overline{\circ}} e) \mathbin{\overline{*}} \{z\} = \{e\} \mathbin{\overline{*}} \{z\} = e \mathbin{\overline{\circ}} z = \{e\}$.

(8) Let $x = y = z = e$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = (x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\}$. Indeed: We have

$\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{e\} \mathbin{\overline{*}} (e \mathbin{\overline{\circ}} e) = \{e\} \mathbin{\overline{*}} \{e\} = e \mathbin{\overline{\circ}} e = \{e\}$ and

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = (e \mathbin{\overline{\circ}} e) \mathbin{\overline{*}} \{e\} = \{e\} \mathbin{\overline{*}} \{e\} = \{e\}$.

We endow $S \cup \{e\}$ with the relation $\preceq$ defined by

$$\preceq := \leq \cup \{(x, e) \mid x \in S \cup \{e\}\}.$$

(D) The relation $\preceq$ is an order on $S\cup\{e\}$. Indeed:

It is reflexive: Let $a\in S\cup\{e\}$. If $a\in S$, then $(a,a)\in\leq\subseteq\preceq$; if $a=e$, then $(a,a)\in\{(x,e)\mid x\in S\cup\{e\}\}\subseteq\preceq$. Thus, we have $(a,a)\in\preceq$ for every $a\in S$ and the relation $\preceq$ is reflexive.

The relation $\preceq$ is symmetric. Indeed: Let $(a,b)\in\preceq$ and $(b,a)\in\preceq$. Then

$(a,b)\in\leq$ or $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and

$(b,a)\in\leq$ or $(b,a)=(y,e)$ for some $y\in S\cup\{e\}$.

We consider the cases:

(1) $(a,b)\in\leq$ and $(b,a)\in\leq$

(2) $(a,b)\in\leq$ and $(b,a)=(y,e)$ for some $y\in S\cup\{e\}$

(3) $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and $(b,a)\in\leq$

(4) $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and $(b,a)=(y,e)$ for some $y\in S\cup\{e\}$.

(1) If $a\leq b$ and $b\leq a$, then $a=b$.

(2) Let $(a,b)\in\leq$ and $(b,a)=(y,e)$ for some $y\in S\cup\{e\}$. Since $(a,b)\in\leq$, we have $a,b\in S$. Since $(y,a)=(y,e)$ for some $y\in S\cup\{e\}$, we have $a=e$. Thus, we have $S\ni a=e$. The case is impossible.

(3) Let $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and $(b,a)\in\leq$. Then we have $S\ni b=e$. The case is impossible.

(4) Let $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and $(b,a)=(y,e)$ for some $y\in S\cup\{e\}$. Then we have $b=e=a$ and so $a=b$.

The relation $\preceq$ is transitive. Indeed: Let $(a,b)\in\preceq$ and $(b,c)\in\preceq$. Then

$(a,b)\in\leq$ or $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and

$(b,c)\in\leq$ or $(b,c)=(y,e)$ for some $y\in S\cup\{e\}$. We consider the cases:

(1) $(a,b)\in\leq$ and $(b,c)\in\leq$. Then $(a,c)\in\leq\subseteq\preceq$.

(2) $(a,b)\in\leq$ and $(b,c)=(y,e)$ for some $y\in S\cup\{e\}$. Since $(b,c)=(y,e)$; $y\in S\cup\{e\}$, we have $c=e$. Then we have $(a,c)=(a,e)\in\{(x,e)\mid x\in S\cup\{e\}\}\subseteq\preceq$.

(3) Let $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and $(b,c)\in\leq$. Then we have $S\ni b=e$. The case is impossible.

(4) $(a,b)=(x,e)$ for some $x\in S\cup\{e\}$ and $(b,c)=(y,e)$ for some $y\in S\cup\{e\}$. Then we have $(a,c)=(a,e)\in\{(x,e)\mid x\in S\cup\{e\}\}\subseteq\preceq$.

(E) The element e is the greatest element of $S\cup\{e\}$. Indeed: Let $a\in S\cup\{e\}$. Then $(a,e)\in\{(x,e)\mid x\in S\cup\{e\}\}\subseteq\preceq$ and so $(a,e)\in\preceq$ i.e. $a\preceq e$.

(F) $(S\cup\{e\},\bar{\circ},\preceq)$ is a *poe*-hypersemigroup.

Let $a,b\in S\cup\{e\}$ such that $a\preceq b$. Then $a\,\bar{\circ}\,c\preceq b\,\bar{\circ}\,c$ and $c\,\bar{\circ}\,a\preceq c\,\bar{\circ}\,b$ for every $c\in S\cup\{e\}$. Let us prove the first one. The proof of the second is similar.

We have $\big(a\in S$ or $a=e\big)$, $a\preceq b$, $\big(c\in S$ or $c=e\big)$. Thus, we have

$a\in S,\ a\preceq b,\ c\in S$

$a\in S,\ a\preceq b,\ c=e$

$a=e,\ a\preceq b,\ c\in S$

$a=e,\ a\preceq b,\ c=e$.

Thus, we have to check the following cases:

(1) $a\in S,\ a\leq b,\ c\in S$

(2) $a\in S,\ (a,b)=(x,e)$ for some $x\in S\cup\{e\},\ c\in S$

(3) $a \in S,\ a \leq b,\ c = e$

(4) $a \in S,\ (a,b) = (x,e)$ for some $x \in S \cup \{e\},\ c = e$

(5) $a = e,\ a \leq b,\ c \in S$

(6) $a = e,\ (a,b) = (x,e)$ for some $x \in S \cup \{e\},\ c \in S$

(7) $a = e,\ a \leq b,\ c = e$

(8) $a = e,\ (a,b) = (x,e)$ for some $x \in S \cup \{e\},\ c = e$.

(1) Let $a \in S,\ a \leq b,\ c \in S$ and $u \in a \mathbin{\bar{\circ}} c$. Then there exists $v \in b \mathbin{\bar{\circ}} c$ such that $u \preceq v$. Indeed: Since $a,c \in S$, we have $a \mathbin{\bar{\circ}} c = a \circ c$. Since $a \leq b$, we have $a \circ c \preceq b \circ c$. Since $u \in a \circ c$, there exists $v \in b \circ c$ such that $u \leq v$. Since $b,c \in S$, we have $b \circ c = b \mathbin{\bar{\circ}} c$. Since $u \leq v$, we have $(u,v) \in \leq \subseteq \preceq$. Thus, we have $v \in b \mathbin{\bar{\circ}} c$ and $u \preceq v$.

(2) Let $a \in S,\ (a,b) = (x,e)$ for some $x \in S \cup \{e\},\ c \in S$ and $u \in a \mathbin{\bar{\circ}} c$. Then there exists $v \in b \mathbin{\bar{\circ}} c$ such that $u \preceq v$. Indeed: Since $a,c \in S$, we have $u \in a \mathbin{\bar{\circ}} c = a \circ c \subseteq S$. Since $u \in S$, we have $(u,e) \in \{(x,e) \mid x \in S \cup \{e\}\} \subseteq \preceq$ and so $u \preceq e$. Since $b = e$, we have $b \mathbin{\bar{\circ}} c = e \mathbin{\bar{\circ}} c = \{e\}$. For the element $v := e \in b \mathbin{\bar{\circ}} c$, we have $u \preceq v$.

(3) Let $a \in S,\ a \leq b,\ c = e$ and $u \in a \mathbin{\bar{\circ}} c$. Then there exists $v \in b \mathbin{\bar{\circ}} c$ such that $u \preceq v$. Indeed: We have $u \in a \mathbin{\bar{\circ}} c = a \mathbin{\bar{\circ}} e = \{e\}$ and so $u = e$. We also have $b \mathbin{\bar{\circ}} c = b \mathbin{\bar{\circ}} e = \{e\}$ and $e \preceq e$ (since $\preceq$ is reflexive). So, for the element $v := e \in b \mathbin{\bar{\circ}} c$, we have $u \preceq v$.

(4) Let $a \in S,\ (a,b) = (x,e)$ for some $x \in S \cup \{e\},\ c = e$ and $u \in a \mathbin{\bar{\circ}} c$. Then there exists $v \in b \mathbin{\bar{\circ}} c$ such that $u \preceq v$. Indeed: We have $u \in a \mathbin{\bar{\circ}} c = a \mathbin{\bar{\circ}} e = \{e\}$ and so $u = e$. We also have $b \mathbin{\bar{\circ}} c = e \mathbin{\bar{\circ}} c = \{e\}$ and $e \preceq e$ (as $\preceq$ is reflexive). For the element $v := e \in b \mathbin{\bar{\circ}} c$, we have $u \preceq v$.

(5) Let $a = e,\ a \leq b,\ c \in S$ and $u \in a \mathbin{\bar{\circ}} c$. Since $a \leq b$, we have $a \in S$. Since $a = e$, we have $e \in S$. The case is impossible.

(6) Let $a = e,\ (a,b) = (x,e)$ for some $x \in S \cup \{e\},\ c \in S$ and $u \in a \mathbin{\bar{\circ}} c$. Then there exists $v \in b \mathbin{\bar{\circ}} c$ such that $u \preceq v$. Indeed: We have $u \in a \mathbin{\bar{\circ}} c = e \mathbin{\bar{\circ}} c = \{e\}$ and so $u = e$. We also have $b \mathbin{\bar{\circ}} c = e \mathbin{\bar{\circ}} c = \{e\}$. For the element $v := e \in b \mathbin{\bar{\circ}} c$, we have $u \preceq v$.

(7) Let $a = e,\ a \leq b,\ c = e$ and $u \in a \mathbin{\bar{\circ}} c$. Then there exists $v \in b \mathbin{\bar{\circ}} c$ such that $u \preceq v$. Indeed: We have $u \in a \mathbin{\bar{\circ}} c = e \mathbin{\bar{\circ}} e = \{e\}$ and so $u = e$. We also have $b \mathbin{\bar{\circ}} c = b \mathbin{\bar{\circ}} e = \{e\}$. So, for the element $v := e \in b \mathbin{\bar{\circ}} c$, we have $u \preceq v$.

(8) Let $a = e,\ (a,b) = (x,e)$ for some $x \in S \cup \{e\},\ c = e$ and $u \in a \mathbin{\bar{\circ}} c$. Then there exists $v \in b \mathbin{\bar{\circ}} c$ such that $u \preceq v$. Indeed: We have $u \in a \mathbin{\bar{\circ}} c = e \mathbin{\bar{\circ}} e = \{e\}$ and so $u = e$. We also have $b \mathbin{\bar{\circ}} c = e \mathbin{\bar{\circ}} e = \{e\}$. So, for the element $v := e \in b \mathbin{\bar{\circ}} c$, we have $u \preceq v$.

(G) The ordered hypersemigroups $(S,\circ,\leq)$ and $(S,\bar{\circ},\preceq)$ are isomorphic under the identity mapping. Indeed, for the one to one and onto mapping

$$i : (S,\circ,\leq) \to (S,\bar{\circ},\preceq) \mid a \to i(a) := a$$

and, any $a,b \in S$, we have

(1) $i(a \circ b) \subseteq i(a) \mathbin{\bar{\circ}} i(b)$; that is if $u \in a \circ b$, then $u \in a \mathbin{\bar{\circ}} b$. This is clear, as $a,b \in S$ implies $a \circ b = a \mathbin{\bar{\circ}} b$.

(2) $a \le b$ implies $a \preceq b$. Indeed, if $a \le b$, then $(a,b) \in \le \subseteq \le \cup \{(x,e) \mid x \in S \cup \{e\}\} = \preceq$ i.e. $(a,b) \in \preceq$ and so $a \preceq b$.

(3) if $a,b \in S$ such that $i(a) \preceq i(b)$, then $a \le b$. Indeed: if $i(a) \preceq i(b)$, then $a \preceq b$ i.e. $(a,b) \in \le \cup \{(x,e) \mid c \in S \cup \{e\}\}$. If $(a,b) \in \le$, then $a \le b$ and the proof is complete. If $(a,b) \in \{(x,e) \mid x \in S \cup \{e\}\}$, then $(a,b) = (x,e)$ for some $x \in S \cup \{e\}$. Then we have $S \ni b = e$ i.e. $e \in S$ and the case is impossible.

(G) S is a pseudoideal of $(S \cup \{e\}, \overline{\circ}, \preceq)$. Indeed, $\emptyset \ne S \subseteq S \cup \{e\}$, $S * S \subseteq S$ and if $a \in S$ and $S \cup \{e\} \ni b \preceq a$, then $b \in S$ (as $b = e$ implies $e = a \in S$ that is impossible). □

3. Some further results

A *poe*-semigroup $(S, \cdot, \le)$ is called regular if $a \le aea$ for every $a \in S$; intra-regular if $a \le ea^2e$ for every $a \in S$. It is called right (resp. left) regular if $a \le a^2e$ (resp. $a \le ea^2$) for every $a \in S$. A *poe*-semigroup $(S, \cdot, \le)$ is called right (resp. left) quasi-regular if $a \le aeae$ (resp. $a \le eaea$) for every $a \in S$. It is called semisimple if $a \le eaeae$ for every $a \in S$.

These concepts can be extended for a *poe*-hypersemigroup $(S, \circ, \le)$ in the way indicated below.

Definition 3.1 *A poe-hypersemigroup $(S, \circ, \le)$ is called regular if $\{a\} \le (a \circ e) * \{a\}$ for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (a \circ e) * \{a\}$ and $a \le t$.*

To see that Definition 3.1 is correct, we have to prove that it coincides with the definition of a regular *poe*-hypersemigroup. A *poe*-hypersemigroup $(S, \circ, \le)$ is called regular if for every $a \in S$ there exists $x \in S$ such that $\{a\} \le (a \circ x) * \{a\}$ (in the sense that for every $a \in S$ there exist $x, t \in S$ such that $t \in (a \circ x) * \{a\}$ and $a \le t$) [6].
In this respect, the following proposition holds.

Proposition 3.2 *Let $(S, \circ, \le)$ is a poe-hypersemigroup. The following are equivalent:*

(1) *S is regular.*

(2) *$\{a\} \le (a \circ e) * \{a\}$ for every $a \in S$.*

Proof First of all, for any nonempty subsets A, B, C of S, $A \preceq B$ implies $A * C \subseteq (B * C]$. Indeed: Let $x \in A * C$. Then $x \in a \circ c$ for some $a \in A$, $c \in C$. Since $a \in A$, there exists $b \in B$ such that $a \le b$. Then $a \circ c \preceq b \circ c$. Since $x \in a \circ c$, there exists $y \in b \circ c$ such that $x \le y \in B * C$ and so $x \in (B * C]$.
$(1) \Longrightarrow (2)$. Let $a \in S$. Since S is regular, there exist $x, t \in S$ such that $t \in (a \circ x) * \{a\}$ and $a \le t$. Since $x \le e$, we have $a \circ x \preceq a \circ e$, then $t \in (a \circ x) * \{a\} \subseteq \Big((a \circ e) * \{a\}\Big]$. Then $t \le y$ for some $y \in (a \circ e) * \{a\} \, (\subseteq (S * S) * S \subseteq S * S \subseteq S)$, $y \in (a \circ e) * \{a\}$ and $a \le y$ and property (2) is satisfied.
$(2) \Longrightarrow (1)$. Let $a \in S$. By (2), there exists $t \in S$ such that $t \in (a \circ e) * \{a\}$ and $a \le t$. We put $x := e$. Then $x, t \in S$, $t \in (a \circ x) * \{a\}$ and $a \le t$ and property (1) holds. □

In a similar way, the following definitions are true.

Definition 3.3 *A poe-hypersemigroup $(S, \circ, \le)$ is called intra-regular if $\{a\} \le (e \circ a) * (a \circ e)$ for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (e \circ a) * (a \circ e)$ and $a \le t$.*

Definition 3.4 *A poe-hypersemigroup* $(S, \circ, \leq)$ *is called right regular if* $\{a\} \leq \{a\} * (a \circ e)$ $(= (a \circ a) * \{e\})$ *for any* $a \in S$*; in the sense that for any* $a \in S$ *there exists* $t \in S$ *such that* $t \in \{a\} * (a \circ e)$ *and* $a \leq t$*. It is called left regular if* $\{a\} \leq (e \circ a) * \{a\}$ $(= \{e\} * (a \circ a))$ *for any* $a \in S$*; in the sense that for any* $a \in S$ *there exists* $t \in S$ *such that* $t \in (e \circ a) * \{a\}$ *and* $a \leq t$*.*

Definition 3.5 *A poe-hypersemigroup* $(S, \circ, \leq)$ *is called right quasi-regular if* $\{a\} \leq (a \circ e) * (a \circ e)$ *for every* $a \in S$*; in the sense that for any* $a \in S$ *there exists* $t \in S$ *such that* $t \in (a \circ e) * (a \circ e)$ *and* $a \leq t$*. It is called left quasi-regular if* $\{a\} \leq (e \circ a) * (e \circ a)$ *for any* $a \in S$*; in the sense that for any* $a \in S$ *there exists* $t \in S$ *such that* $t \in (e \circ a) * (e \circ a)$ *and* $a \leq t$*.*

Definition 3.6 *A poe-hypersemigroup* $(S, \circ, \leq)$ *is called semisimple if* $\{a\} \leq (e \circ a) * (e \circ a) * \{e\}$ *for any* $a \in S$*; in the sense that for any* $a \in S$ *there exists* $t \in S$ *such that* $t \in (e \circ a) * (e \circ a) * \{e\}$ *and* $a \leq t$*.*

Proposition 3.7 *The poe-hypersemigroup* $\big(S \cup \{e\}, \bar{\circ}, \preceq\big)$ *constructed in Theorem 2.3 is regular and intra-regular.*

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (a \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} \{a\}$ and $a \preceq t$. Indeed: If $a \in S$, then $a \mathbin{\bar{\circ}} e = \{e\}$, $(a \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} \{a\} = \{e\} \mathbin{\bar{*}} \{a\} = e \mathbin{\bar{\circ}} a = \{e\}$. If $a = e$, then $a \mathbin{\bar{\circ}} e = e \mathbin{\bar{\circ}} e = \{e\}$, $(a \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} \{a\} = \{e\} \mathbin{\bar{*}} \{a\} = \{e\}$. In each case, we have $e \in (a \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} \{a\}$ and $a \preceq e$.

Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$ and $a \preceq t$. Indeed, for the element $t := e \in S \cup \{e\}$, we have $e \in \{e\} = e \mathbin{\bar{\circ}} e = \{e\} \mathbin{\bar{*}} \{e\} = (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$; thus, we have $e \in (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$ and $a \preceq e$ and so $\big(S \cup \{e\}, \bar{\circ}, \preceq\big)$ is intra-regular. □

Proposition 3.8 *The poe-hypersemigroup* $(S \cup \{e\}, \bar{\circ}, \preceq)$ *is right regular and left regular.*

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in \{a\} \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$ and $a \preceq t$. Indeed: for the element $t := e \in S \cup \{e\}$, we have $e \in \{e\} = a \mathbin{\bar{\circ}} e = \{a\} \mathbin{\bar{*}} \{e\} = \{a\} \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$; thus, we have $e \in \{a\} \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$ and $a \preceq e$ and so $\big(S \cup \{e\}, \bar{\circ}, \preceq\big)$ is right regular. We also have $e \in \{e\} = e \mathbin{\bar{\circ}} a = \{e\} \mathbin{\bar{*}} \{a\} = (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} \{a\}$; thus, we have $e \in (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} \{a\}$ and $a \preceq e$ and so $\big(S \cup \{e\}, \bar{\circ}, \preceq\big)$ is left regular. □

Proposition 3.9 *The poe-semigroup* $(S \cup \{e\}, \bar{\circ}, \preceq)$ *is right quasi-regular and left quasi-regular.*

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (a \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$ and $a \preceq t$. In fact, for the element $t := e \in S \cup \{e\}$, we have $e \in \{e\} = e \mathbin{\bar{\circ}} e = \{e\} \mathbin{\bar{*}} \{e\} = (a \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} (a \mathbin{\bar{\circ}} e)$ and $a \preceq e$ and so $(S \cup \{e\}, \bar{\circ}, \preceq)$ is right quasi-regular. We also have $e \in \{e\} = e \mathbin{\bar{\circ}} e = \{e\} \mathbin{\bar{*}} \{e\} = (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} a)$ and $a \preceq e$ and so $(S \cup \{e\}, \bar{\circ}, \preceq)$ is left quasi-regular. □

Proposition 3.10 *The poe-semigroup* $(S \cup \{e\}, \bar{\circ}, \preceq)$ *is semisimple.*

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} \{e\}$ and $a \preceq t$. Indeed, for the element $t := e \in S \cup \{e\}$, we have $e \in (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} \{e\}$ and $a \preceq e$. □

According to Proposition 3.7, $\big(S \cup \{e\}, \bar{\circ}, \preceq\big)$ is intra-regular. This can be also obtained as corollary to the next proposition. To prove it, we need the following lemma.

If $(S,\circ,*,\leq)$ is an ordered hypersemigroup and A,B nonempty subsets of S we write $A\leq B$ if for any $a\in A$ there exists $b\in B$ such that $a\leq b$.

Lemma 3.11 *Let $(S,\circ,\leq)$ be an ordered hypersemigroup. Then we have the following:*

(a) *For any nonempty subsets A,B,C of S such that $A\leq B$, we have $A*C\leq B*C$ and $C*A\leq C*B$*

(b) *The operation $*$ is associative (see, for example [4]) .*

(c) *If $A\leq B\leq C$, then $A\leq C$.*

Proof (a) Let $A\leq B$ and $x\in A*C$. Then there exists $y\in B*C$ such that $x\leq y$. Indeed: Since $x\in A*C$, we have $x\in a\circ c$ for some $a\in A$, $c\in C$. Since $a\in A$, there exists $b\in B$ such that $a\leq b$. Then $a\circ c\leq b\circ c$. Since $x\in a\circ c$, there exists $y\in b\circ c=B*C$ such that $x\leq y$.
(c) If $a\in A$, then there exists $b\in B$ such that $a\leq b$. Since $b\in B$, there exists $c\in C$ such that $b\leq c$. Hence, for any $a\in A$ there exists $c\in C$ such that $a\leq c$ and so $A\leq C$.

□

Proposition 3.12 *A poe-hypersemigroup $(S,\circ,\leq)$ that is right regular or left regular is intra-regular.*

Proof Let $(S,\circ,\leq)$ be right regular and $a\in S$. Then we have

$$\begin{aligned}
\{a\} &\leq (a\circ a)*\{e\}=\{a\}*\{a\}*\{e\}\leq\{e\}*\{a\}*\{e\}\ (\text{since } a\leq e \text{ implies } \{a\}\leq\{e\})\\
&\leq \{e\}*\Big(\{a\}*\{a\}*\{e\}\Big)*\{e\}=\{e\}*\{a\}*\{a\}*\Big(\{e\}*\{e\}\Big)\\
&\leq \{e\}*\{a\}*\{a\}*\{e\}\ (\text{as } \{e\}*\{e\}=e\circ e\leq\{e\})\\
&= (e\circ a)*(a\circ e))
\end{aligned}$$

and so S is intra-regular. □

By Propositions 3.7, 3.8, 3.9, and 3.10, we have the following corollary.

Corollary 3.13 *Each ordered hypersemigroup can be embedded*

(1) *in a regular poe-hypersemigroup.*

(2) *in an intra-regular poe-hypersemigroup.*

(3) *in a right regular (or left regular) poe-hypersemigroup.*

(4) *in right quasi-regular (or left quasi-regular) poe-hypersemigroup.*

(5) *in a semisimple poe-hypersemigroup.*

4. Examples

We apply the above results to the following examples.

Remark 4.1 Theorem 2.3 can be also applied to a *poe*-hypersemigroup and we have the following: If $(S,\circ,\leq)$ is a *poe*-hypersemigroup, t an element not included in S, $\overline{\circ}$ the hyperoperation and $\preceq$ the order $S\cup\{t\}$ defined in Theorem 2.3, then the set $V:=\big(S\cup\{t\},\overline{\circ},\preceq\big)$ is still a *poe*-hypersemigroup and S is a pseudoideal of V.

Let us give an example based on the remark.

Example 4.2 We consider the ordered semigroup $S=\{a,b,c\}$ given by Table 1 and Figure 1. From this, the ordered hypersemigroup given by Table 2 and the same figure (Figure 1) can be obtained. Take an element t not included in S and consider the ordered hypersemigroup $S\cup\{t\}$. Then $(S\cup\{t\},\overline{\circ},\preceq)$ is a *poe*-hypersemigroup having the S as a pseudoideal. According to Section 2, the ordered hypersemigroup $(S\cup\{e\},\overline{\circ},\preceq)$ given by Table 3 and Figure 2 is regular, intra-regular, right regular, left regular, right quasi-regular, left quasi-regular, and semisimple. Independently,
$(S\cup\{e\},\overline{\circ},\preceq)$ is regular, that is, $\{a\}\preceq(a\,\overline{\circ}\,e)\,\overline{\circ}\,\{a\}$ for every $a\in S\cup\{e\}$; in other words, for every $a\in S\cup\{e\}$ there exists $t\in(a\,\overline{\circ}\,e)\,\overline{*}\,\{a\}$ such that $a\preceq t$. In fact,

$e\in(a\,\overline{\circ}\,e)\,\overline{*}\,\{a\}=\{e\}\,\overline{*}\,\{a\}=e\,\overline{\circ}\,a=\{e\}$ and $a\preceq e$;
$e\in(b\,\overline{\circ}\,e)\,\overline{*}\,\{b\}=\{e\}\,\overline{*}\,\{b\}=e\,\overline{\circ}\,b=\{e\}$ and $b\preceq e$;
$e\in(c\,\overline{\circ}\,e)\,\overline{*}\,\{c\}=\{e\}\,\overline{*}\,\{c\}=e\,\overline{\circ}\,c=\{e\}$ and $c\preceq e$;
$e\in(e\,\overline{\circ}\,e)\,\overline{*}\,\{e\}=\{e\}\,\overline{*}\,\{e\}=e\,\overline{\circ}\,e=\{e\}$ and $e\preceq e$.

$(S\cup\{e\},\overline{\circ},\preceq)$ is intra-regular as

$e\in(e\,\overline{\circ}\,a)\,\overline{*}\,(a\,\overline{\circ}\,e)=\{e\}\,\overline{*}\,\{e\}=e\,\overline{\circ}\,e=\{e\}$ and $a\preceq e$;
$e\in(e\,\overline{\circ}\,b)\,\overline{*}\,(b\,\overline{\circ}\,e)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $b\preceq e$;
$e\in(e\,\overline{\circ}\,c)\,\overline{*}\,(c\,\overline{\circ}\,e)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $c\preceq e$;
$e\in(e\,\overline{\circ}\,e)\,\overline{*}\,(e\,\overline{\circ}\,e)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $e\preceq e$.

$(S\cup\{e\},\overline{\circ},\preceq)$ is right regular as

$e\in(a\,\overline{\circ}\,a)\,\overline{*}\,\{e\}=\{a\}\,\overline{*}\,\{e\}=a\,\overline{\circ}\,e=\{e\}$ and $a\preceq e$;
$e\in(b\,\overline{\circ}\,b)\,\overline{*}\,\{e\}=\{b\}\,\overline{*}\,\{e\}=b\,\overline{\circ}\,e=\{e\}$ and $b\preceq e$;
$e\in(c\,\overline{\circ}\,c)\,\overline{*}\,\{e\}=\{a,b,c\}\,\overline{*}\,\{e\}=b\,\overline{\circ}\,e=\{e\}$ and $c\preceq e$;
$e\in(e\,\overline{\circ}\,e)\,\overline{*}\,\{e\}=\{e\}\,\overline{*}\,\{e\}=e\,\overline{\circ}\,e=\{e\}$ and $e\preceq e$.

$(S\cup\{e\},\overline{\circ},\preceq)$ is left regular as

$e\in(e\,\overline{\circ}\,a)\,\overline{*}\,\{a\}=\{e\}\,\overline{*}\,\{a\}=e\,\overline{\circ}\,a=\{e\}$ and $a\preceq e$;
$e\in(e\,\overline{\circ}\,b)\,\overline{*}\,\{b\}=\{a,b,c\}\,\overline{*}\,\{b\}=\{a,b,c\}$ and $b\preceq e$;
$e\in(e\,\overline{\circ}\,c)\,\overline{*}\,\{c\}=\{a,b,c\}\,\overline{*}\,\{c\}=\{a,b,c\}$ and $c\preceq e$;
$e\in(e\,\overline{\circ}\,e)\,\overline{*}\,\{e\}=\{e\}\,\overline{*}\,\{e\}=e\,\overline{\circ}\,e=\{e\}$ and $e\preceq e$.

$(S\cup\{e\},\overline{\circ},\preceq)$ is right quasi-regular as

$e\in(a\,\overline{\circ}\,e)\,\overline{*}\,(a\,\overline{\circ}\,e)=\{e\}\,\overline{*}\,\{e\}=e\,\overline{\circ}\,e=\{e\}$ and $a\preceq e$;
$e\in(b\,\overline{\circ}\,e)\,\overline{*}\,(b\,\overline{\circ}\,e)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $b\preceq e$;
$e\in(c\,\overline{\circ}\,e)\,\overline{*}\,(c\,\overline{\circ}\,e)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $c\preceq e$;
$e\in(e\,\overline{\circ}\,e)\,\overline{*}\,(e\,\overline{\circ}\,e)=\{e\}$ and $e\preceq e$.

$(S\cup\{e\},\overline{\circ},\preceq)$ is left quasi-regular as

$e\in(e\,\overline{\circ}\,a)\,\overline{*}\,(e\,\overline{\circ}\,a)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $a\preceq e$.
$e\in(e\,\overline{\circ}\,b)\,\overline{*}\,(e\,\overline{\circ}\,b)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $b\preceq e$.
$e\in(e\,\overline{\circ}\,c)\,\overline{*}\,(e\,\overline{\circ}\,c)=\{e\}\,\overline{*}\,\{e\}=\{e\}$ and $c\preceq e$.

$e \in (e \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} e) = \{e\} \mathbin{\bar{*}} \{e\} = \{e\}$ and $e \preceq e$.

$(S \cup \{e\}, \bar{\circ}, \preceq)$ is semisimple as

$e \in (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} a) \mathbin{\bar{*}} \{e\} = \{e\} \mathbin{\bar{*}} \{e\} = \{e\}$ and $a \preceq e$.

$e \in (e \mathbin{\bar{\circ}} b) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} b) \mathbin{\bar{*}} \{e\} = \{e\} \mathbin{\bar{*}} \{e\} = \{e\}$ and $b \preceq e$.

$e \in (e \mathbin{\bar{\circ}} c) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} c) \mathbin{\bar{*}} \{e\} = \{e\} \mathbin{\bar{*}} \{e\} = \{e\}$ and $c \preceq e$.

$e \in (e \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} (e \mathbin{\bar{\circ}} e) \mathbin{\bar{*}} \{e\} = \{e\} \mathbin{\bar{*}} \{e\} = \{e\}$ and $e \preceq e$.

S is a pseudoideal of $S \cup \{e\}$. Indeed:

$$\begin{aligned} S * S &= \{a,b,c\} * \{a,b,c\} = a \circ a \cup a \circ b \cup a \circ c \cup b \circ a \cup b \circ b \cup b \circ c \cup c \circ a \cup c \circ b \cup c \circ c \\ &= \{a,b,c\} \subseteq S \end{aligned}$$

and if $a \in S$ and $S \cup \{e\} \ni b \preceq a$, then $b \in S$ (as $b = e$ implies $e \preceq a$ and so $e = a \in S$ that is impossible).

Table 1: The multiplication of the ordered semigroup of Example 4.2.

$\cdot$	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

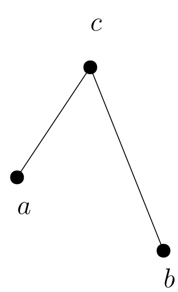

Figure 1: The order of Example 4.2.

Table 2: The hyperoperation of $(S, \circ, \leq)$ of Example 4.2.

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a,b,c\}$
b	$\{a\}$	$\{b\}$	$\{a,b,c\}$
c	$\{a\}$	$\{a,b,c\}$	$\{a,b,c\}$

Example 4.3 (see also [3]) We consider the ordered semigroup $S = \{a,b,c,d,e,f,g\}$ given by Table 4 and Figure 3. From this, in the way indicated in [5], the ordered hypersemigroup defined by Table 5 and the same figure (Figure 3) can be obtained. If h is an element not containing in S, then the Table 6 and Figure 4 define a *poe*-hypersemigroup that is regular, intra-regular, right (left) regular, right (left) quasi-regular, and semisimple. In Theorem 2.3, we assume that S is an ordered hypersemigroup. In case of *poe*-hypersemigroups, one can continue the process given in Theorem 2.3 for countable many steps as it is shown in Figure 5.

Table 3: The hyperoperation of $(S \cup \{e\}, \overline{\circ}, \preceq)$ of Example 4.2.

$\overline{\circ}$	a	b	c	e
a	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{e\}$
b	$\{a\}$	$\{b\}$	$\{a, b, c\}$	$\{e\}$
c	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{e\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

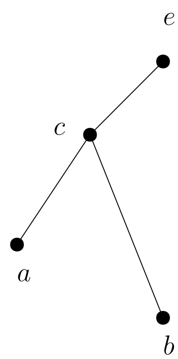

Figure 2: The order $S \cup \{e\}$ of Example 4.2.

Independently,
$(S \cup \{h\}, \overline{\circ}, \preceq)$ is regular, that is $\{a\} \preceq (a \,\overline{\circ}\, h) \,\overline{*}\, \{a\}$ for every $a \in S \cup \{h\}$; in other words, for every $a \in S \cup \{h\}$ there exists $t \in (a \,\overline{\circ}\, h) \,\overline{*}\, \{a\}$ such that $a \preceq t$. Indeed, we have

$h \in (a \,\overline{\circ}\, h) \,\overline{*}\, \{a\} = \{h\} \,\overline{*}\, \{a\} = h \,\overline{\circ}\, a = \{h\}$ and $a \preceq h$
$h \in (b \,\overline{\circ}\, h) \,\overline{*}\, \{b\} = \{h\} \,\overline{*}\, \{b\} = h \,\overline{\circ}\, b = \{h\}$ and $b \preceq h$
$h \in (c \,\overline{\circ}\, h) \,\overline{*}\, \{c\} = \{h\} \,\overline{*}\, \{c\} = h \,\overline{\circ}\, c = \{h\}$ and $c \preceq h$
$h \in (d \,\overline{\circ}\, h) \,\overline{*}\, \{d\} = \{h\} \,\overline{*}\, \{d\} = h \,\overline{\circ}\, d = \{h\}$ and $d \preceq h$
$h \in (h \,\overline{\circ}\, h) \,\overline{*}\, \{e\} = \{h\} \,\overline{*}\, \{e\} = h \,\overline{\circ}\, e = \{h\}$ and $e \preceq h$
$h \in (f \,\overline{\circ}\, h) \,\overline{*}\, \{f\} = \{h\} \,\overline{*}\, \{f\} = h \,\overline{\circ}\, f = \{h\}$ and $f \preceq h$
$h \in (g \,\overline{\circ}\, h) \,\overline{*}\, \{g\} = \{h\} \,\overline{*}\, \{g\} = h \,\overline{\circ}\, g = \{h\}$ and $g \preceq h$
$h \in (h \,\overline{\circ}\, h) \,\overline{*}\, \{h\} = \{h\} \,\overline{*}\, \{h\} = h \,\overline{\circ}\, h = \{h\}$ and $h \preceq h$.

$(S \cup \{h\}, \overline{\circ}, \preceq)$ is intraregular, that is $\{a\} \preceq (h \,\overline{\circ}\, a) \,\overline{*}\, (a \,\overline{\circ}\, h)$ for every $a \in S \cup \{h\}$; in other words, for every $a \in S \cup \{h\}$ there exists $t \in (h \,\overline{\circ}\, a) \,\overline{*}\, (a \,\overline{\circ}\, h)$ such that $a \preceq t$. Indeed, we have

$h \in (h \,\overline{\circ}\, a) \,\overline{*}\, (a \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = h \,\overline{\circ}\, h = \{h\}$ and $a \preceq h$
$h \in (h \,\overline{\circ}\, b) \,\overline{*}\, (b \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = h \,\overline{\circ}\, h = \{h\}$ and $b \preceq h$
$h \in (h \,\overline{\circ}\, c) \,\overline{*}\, (c \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = h \,\overline{\circ}\, h = \{h\}$ and $c \preceq h$
$h \in (h \,\overline{\circ}\, d) \,\overline{*}\, (d \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = \{h\}$ and $d \preceq h$
$h \in (h \,\overline{\circ}\, e) \,\overline{*}\, (e \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = \{h\}$ and $e \preceq h$
$h \in (h \,\overline{\circ}\, f) \,\overline{*}\, (f \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = \{h\}$ and $f \preceq h$
$h \in (h \,\overline{\circ}\, g) \,\overline{*}\, (g \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = \{h\}$ and $g \preceq h$
$h \in (h \,\overline{\circ}\, h) \,\overline{*}\, (h \,\overline{\circ}\, h) = \{h\} \,\overline{\circ}\, \{h\} = \{h\}$ and $h \preceq h$.

$(S \cup \{h\}, \overline{\circ}, \preceq)$ is right regular, that is $\{a\} \preceq (a \,\overline{\circ}\, a) \,\overline{*}\, \{h\}$ for every $a \in S \cup \{h\}$; in other words, for every

$a \in S \cup \{h\}$ there exists $t \in (a \,\bar{\circ}\, a) \,\bar{*}\, \{h\}$ such that $a \preceq t$. Indeed, we have

$h \in (a \,\bar{\circ}\, a) \,\bar{*}\, \{h\} = \{a\} \,\bar{*}\, \{h\} = a \,\bar{\circ}\, h = \{h\}$ and $a \preceq h$
$h \in (b \,\bar{\circ}\, b) \,\bar{*}\, \{h\} = \{a\} \,\bar{*}\, \{h\} = a \,\bar{\circ}\, h = \{h\}$ and $b \preceq h$
$h \in (c \,\bar{\circ}\, c) \,\bar{*}\, \{h\} = \{a\} \,\bar{*}\, \{h\} = \{h\}$ and $c \preceq h$
$h \in (d \,\bar{\circ}\, d) \,\bar{*}\, \{h\} = \{a\} \,\bar{*}\, \{h\} = \{h\}$ and $d \preceq d$
$h \in (e \,\bar{\circ}\, e) \,\bar{*}\, \{h\} = \{a\} \,\bar{*}\, \{h\} = \{h\}$ and $e \preceq h$
$h \in (f \,\bar{\circ}\, f) \,\bar{*}\, \{h\} = \{f\} \,\bar{*}\, \{h\} = f \,\bar{\circ}\, h = \{h\}$ and $f \preceq h$
$h \in (g \,\bar{\circ}\, g) \,\bar{*}\, \{h\} = \{a\} \,\bar{*}\, \{h\} = a \,\bar{\circ}\, h = \{h\}$ and $g \preceq h$
$h \in (h \,\bar{\circ}\, h) \,\bar{*}\, \{h\} = \{h\} \,\bar{*}\, \{h\} = h \,\bar{\circ}\, h = \{h\}$ and $h \preceq h$.

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is left regular, that is $\{a\} \preceq \{h\} \,\bar{*}\, (a \,\bar{\circ}\, a)$ for every $a \in S \cup \{h\}$; that is for every $a \in S \cup \{h\}$ there exists $t \in \{h\} \,\bar{*}\, (a \,\bar{\circ}\, a)$ such that $a \preceq t$. Indeed, we have

$h \in \{h\} \,\bar{*}\, (a \,\bar{\circ}\, a) = \{h\} \,\bar{*}\, \{a\} = h \,\bar{\circ}\, a = \{h\}$ and $a \preceq h$
$h \in \{h\} \,\bar{*}\, (b \,\bar{\circ}\, b) = \{h\} \,\bar{*}\, \{a\} = \{h\}$ and $b \preceq h$
$c \in \{h\} \,\bar{*}\, (c \,\bar{\circ}\, c) = \{h\} \,\bar{*}\, \{a\} = \{h\}$ and $c \leq h$
$h \in \{h\} \,\bar{*}\, (d \,\bar{\circ}\, d) = \{h\} \,\bar{*}\, \{a\} = \{h\}$ and $d \preceq h$
$h \in \{h\} \,\bar{*}\, (e \,\bar{\circ}\, e) = \{h\} \,\bar{*}\, \{a\} = \{h\}$ and $e \preceq h$
$h \in \{h\} \,\bar{*}\, (f \,\bar{\circ}\, f) = \{h\} \,\bar{*}\, \{f\} = h \,\bar{\circ}\, f = h$ and $f \preceq h$
$h \in \{h\} \,\bar{*}\, (g \,\bar{\circ}\, g) = \{h\} \,\bar{*}\, \{a\} = \{h\}$ and $g \preceq h$
$h \in \{h\} \,\bar{*}\, (h \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = h \,\bar{\circ}\, h = \{h\}$ and $h \preceq h$.

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is right quasi-regular, that is $\{a\} \preceq (a \,\bar{\circ}\, h) \,\bar{*}\, (a \,\bar{*}\, h)$ for every $a \in S \cup \{h\}$. Indeed, we have

$\{a\} \preceq (a \,\bar{\circ}\, h) \,\bar{\circ}\, (a \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = h \,\bar{\circ}\, h = \{h\}$, $\{b\} \preceq (b \,\bar{\circ}\, h) \,\bar{\circ}\, (b \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = \{h\}$
$\{c\} \preceq (c \,\bar{\circ}\, h) \,\bar{\circ}\, (c \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = \{h\}$, $\{d\} \preceq (d \,\bar{\circ}\, h) \,\bar{\circ}\, (h \,\bar{\circ}\, d) = \{h\} \,\bar{*}\, \{h\} = \{h\}$
$\{e\} \preceq (e \,\bar{\circ}\, h) \,\bar{\circ}\, (e \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = \{h\}$, $\{f\} \preceq (f \,\bar{\circ}\, h) \,\bar{\circ}\, (f \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = \{h\}$
$\{g\} \preceq (g \,\bar{\circ}\, h) \,\bar{\circ}\, (g \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = \{h\}$, $\{h\} \preceq (h \,\bar{\circ}\, h) \,\bar{\circ}\, (h \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = \{h\}$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is left quasi-regular, that is $\{a\} \preceq (h \,\bar{\circ}\, a) \,\bar{*}\, (h \,\bar{\circ}\, a)$ for every $a \in S \cup \{h\}$. Indeed, we have

$\{a\} \preceq (h \,\bar{\circ}\, a) \,\bar{\circ}\, (h \,\bar{\circ}\, a) = \{h\} \,\bar{*}\, \{h\} = \{h\}$, $\{b\} \preceq (h \,\bar{\circ}\, b) \,\bar{\circ}\, (h \,\bar{\circ}\, b) = \{h\} \,\bar{*}\, \{h\} = \{h\}$
$\{c\} \preceq (h \,\bar{\circ}\, c) \,\bar{\circ}\, (h \,\bar{\circ}\, c) = \{h\} \,\bar{*}\, \{h\} = \{h\}$, $\{d\} \preceq (h \,\bar{\circ}\, d) \,\bar{\circ}\, (h \,\bar{\circ}\, d) = \{h\} \,\bar{*}\, \{h\} = \{h\}$
$\{e\} \preceq (h \,\bar{\circ}\, e) \,\bar{\circ}\, (h \,\bar{\circ}\, e) = \{h\} \,\bar{*}\, \{h\} = \{h\}$, $\{f\} \preceq (h \,\bar{\circ}\, f) \,\bar{\circ}\, (h \,\bar{\circ}\, f) = \{h\} \,\bar{*}\, \{h\} = \{h\}$
$\{g\} \preceq (h \,\bar{\circ}\, g) \,\bar{\circ}\, (h \,\bar{\circ}\, g) = \{h\} \,\bar{*}\, \{h\} = \{h\}$, $\{h\} \preceq (h \,\bar{\circ}\, h) \,\bar{\circ}\, (h \,\bar{\circ}\, h) = \{h\} \,\bar{*}\, \{h\} = \{h\}$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is semisimple, that is $\{a\} \preceq (h \,\bar{\circ}\, a) \,\bar{*}\, (h \,\bar{\circ}\, a)$ for every $a \in S \cup \{h\}$. Indeed, for every $x \in S$, we have

$(x \,\bar{\circ}\, h) \,\bar{*}\, (x \,\bar{\circ}\, h) = \{h\}$; and $a \leq h$, $b \leq h$, $c \leq h$, $d \leq h$, $e \leq h$, $f \leq h$, $g \leq h$, $h \leq h$.

For the definitions of intra-regular, right (left) regular, right (left) quasi-regular, and semisimple ordered hypersemigroups, we refer to [6]

Recall that the ordered hypersemigroup $(S, \cdot, \leq)$ given by Table 4 and Figure 3 is

(a) not regular as, for example, $\nexists\ x \in S$ such that $\{b\} \leq (b \circ x) * \{b\}$

(b) not intra-regular as, for example, $\nexists\ x, y \in S$ such that $\{c\} \leq (x \circ x) * (c \circ y)$

(c) not right regular, as $\nexists\ x \in S$ such that $\{f\} \le (f \circ f) * \{x\}$

(d) not left regular, as $\nexists\ x \in S$ such that $\{f\} \le \{x\} * (a \circ a)$

(e) not right quasi-regular, as $\nexists\ x, y \in S$ such that $\{e\} \le (e \circ x) * (e \circ y)$

(f) not left quasi-regular, as $\nexists\ x, y \in S$ such that $\{e\} \le (x \circ e) * (y \circ y)$

(g) not semisimple, as $\nexists\ x, y, z \in S$ such that $\{f\} \le (x \circ f) * (y \circ f) * \{z\}$.

Table 4: The multiplication of the ordered semigroup S of Example 4.3.

$\cdot$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	a	a	a	a	a	a
c	a	a	a	a	a	a	a
d	a	a	a	a	a	a	a
e	a	a	a	a	a	a	b
f	a	b	c	d	e	f	a
g	a	a	a	a	a	a	a

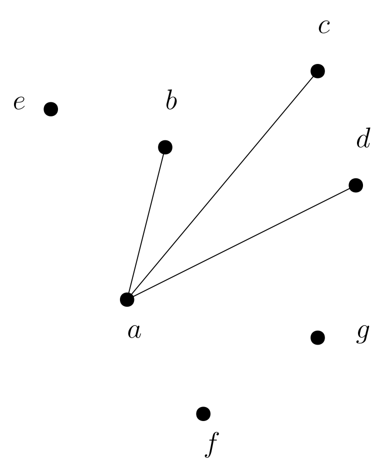

Figure 3: The order of S of Example 4.3.

Table 5: The hyperoperation of S of Example 4.3.

$\circ$	a	b	c	d	e	f	g
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
e	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$
f	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$	$\{e\}$	$\{f\}$	$\{a\}$
g	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$

According to Remark 4.1, we can continue this process for countable many steps, the resulting figure is the following:

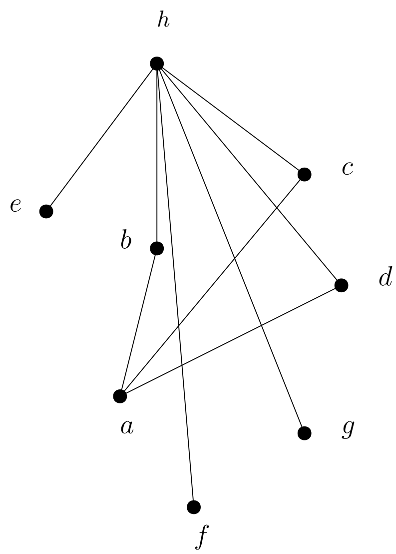

Figure 4: The order of $S \cup \{h\}$ of Example 4.3.

Table 6: The hyperoperation of $S \cup \{h\}$ of Example 4.3.

$\overline{\circ}$	a	b	c	d	e	f	g	h
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
e	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{h\}$
f	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{e\}$	$\{f\}$	$\{a\}$	$\{h\}$
g	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
h	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$

5. Pseudoideal and ideals of $S \cup \{e\}$

Proposition 5.1 *If $(S,\circ,\leq)$ is an ordered hypersemigroup and T is a pseudoideal of S, then T is a pseudoideal of $(S\cup\{e\},\overline{\circ},\preceq)$.*

Proof Since $T * T \subseteq T$, we have $T \overline{*} T \subseteq T$. Indeed: Let $x \in T \overline{*} T$. Then $x \in a \,\overline{\circ}\, b$ for some $a,b \in T$. Since $a,b \in T \subseteq S$, we have $a \,\overline{\circ}\, b = a \circ b$. Then $x \in a \circ b = \{a\} * \{b\} \subseteq T * T \subseteq T$ and so $x \in T$.
Let now $a \in T$ and $S \cup \{e\} \ni b \preceq a$. Then $b \in T$. Indeed: We have

$a \in T$, $\big(b \in S$ or $b = e\big)$, $b \preceq a$; that is we have the following two cases:

(a) $a \in T$, $b \in S$, $\Big(b \leq a$ or $(b,a) = (x,e)$ for some $x \in S \cup \{e\}\Big)$

(b) $a \in T$, $b = e$, $\Big(b \leq a$ or $(b,a) = (x,e)$ for some $x \in S \cup \{e\}\Big)$

So we have to check the following:

(1) $a \in T$, $b \in S$, $b \leq a$

(2) $a \in T$, $b \in S$, $(b,a) = (x,e)$ for some $x \in S$

(3) $a \in T$, $b \in S$, $(b,a) = (e,e)$

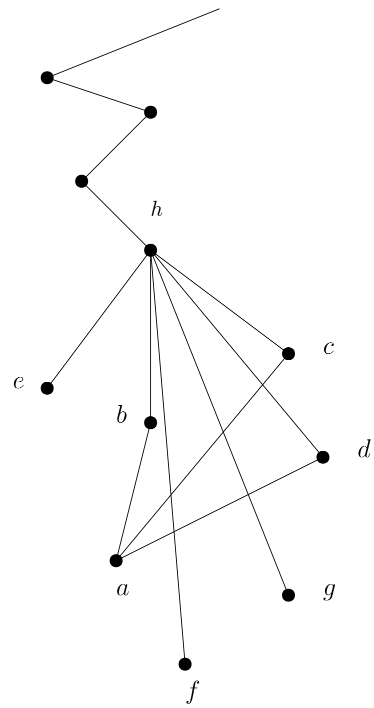

Figure 5: Theorem 2.3 holds for a *poe*-hypersemigroup as well and the process given in that theorem can be continued for countable many sets leading to the order of Figure 5.

(4) $a \in T$, $b = e$, $b \leq a$

(5) $a \in T$, $b = e$, $(b, a) = (x, e)$ for some $x \in S$

(6) $a \in T$, $b = e$, $(b, a) = (e, e)$.

(1) If $a \in T$, $b \in S$, $b \leq a$ then, since T is a pseudoideal of $(S, \circ, \leq)$, we have $b \in T$.

(2) Let $a \in T$, $b \in S$, $(b, a) = (x, e)$ for some $x \in S$. Since $T \subseteq S$, we have $a \in S$. Since $(b, a) = (x, e)$ for some $x \in S$, we have $a = e$. Thus, we have $e \in S$. The case is impossible.

(3) Let $a \in T$, $b \in S$, $(b, a) = (e, e)$. Then we have $S \ni b = e$. The case is impossible.

(4) Let $a \in T$, $b = e$, $b \leq a$. Then we have $S \ni b = e$. The case is impossible.

(5) Let $a \in T$, $b = e$, $(b, a) = (x, e)$ for some $x \in S$. Then $e = b = x \in S$. The case is impossible.

(6) Let $a \in T$, $b = e$, $(b, a) = (e, e)$. Then we have $T \ni a = e = b$ and so $b \in T$. □

Proposition 5.2 *Let $(S, \circ, \leq)$ be an ordered hypersemigroup. If A is an ideal of $(S, \circ, \leq)$, then $A \cup \{e\}$ is an ideal of $(S \cup \{e\}, \overline{\circ})$ but it is not an ideal of $(S \cup \{e\}, \overline{\circ} \preceq)$.*

Proof We have $\Big(A \cup \{e\}\Big) \overline{*} \Big(S \cup \{e\}\Big) \subseteq A \cup \{e\}$. Indeed: Let $t \in \Big(A \cup \{e\}\Big) \overline{*} \Big(S \cup \{e\}\Big)$. Then $t \in x \,\overline{\circ}\, y$ for some $x \in A \cup \{e\}$ and $y \in S \cup \{e\}$. We consider the cases:

(1) $x \in A$, $y \in S$

(2) $x \in A$, $y = e$

(3) $x = e$, $y \in S$

(4) $x = y = e$.

(1) Let $x \in A$, $y \in S$. Since $x, y \in S$, we have $x \,\bar{\circ}\, y = x \circ y$. We have $t \in x \circ y = \{x\} \,\bar{*}\, \{y\} \subseteq A * S \subseteq A$ and so $t \in A \subseteq A \cup \{e\}$.

(2) Let $x \in A$, $y = e$. Then $t \in x \,\bar{\circ}\, y = x \,\bar{\circ}\, e = \{e\}$ and so $t = e \in A \cup \{e\}$.

(3) Let $x = e$, $y \in S$. Then $t \in x \,\bar{\circ}\, y = e \,\bar{\circ}\, y = \{e\}$ and so $t \in A \cup \{e\}$.

(4) Let $x = y = e$. Then $t \in x \,\bar{\circ}\, y = e \,\bar{\circ}\, e = \{e\}$ and so $t \in A \cup \{e\}$.

Similarly, $\Big(S \cup \{e\}\Big) \,\bar{*}\, \Big(A \cup \{e\}\Big) \subseteq A \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a, b, e\}$ is an ideal of $S = \{a, b, c, d, e, f, g\}$. Indeed: We have

$$\begin{aligned}\{a,b,e\} * \{a,b,c,d,e,f,g\} &= (a \circ a) \cup (a \circ b) \cup (a \circ c) \cup (a \circ d) \cup (a \circ e) \cup (a \circ f) \cup (a \circ g)\\ &\quad \cup (b \circ a) \cup (b \circ b) \cup (b \circ c) \cup (b \circ d) \cup (b \circ e) \cup (b \circ f) \cup (b \circ g)\\ &\quad \cup (e \circ a) \cup (e \circ b) \cup e \circ c) \cup (e \circ d) \cup (e \circ e) \cup (e \circ f) \cup (e \circ g)\\ &= \{a\} \cup \{b\} = \{a, b\} \subseteq \{a, b, e\},\end{aligned}$$

similarly, $\{a, b, c, d, e, f, g\} * \{a, b, e\} = \{a\} \cup \{a, b\} \cup \{e\} \subseteq \{a, b, e\}$ and if $x \in \{a, b, e\}$ and $\{a, b, c, d, e, f, g\} \ni y \leq x$, then $y \in \{a, b, e\}$, but $\{a, b, e, h\}$ is not an ideal of $S \cup \{h\}$ as $f \in S \cup \{h\}$, $f \leq h$ and $f \notin \{a, b, e, h\}$. □

Proposition 5.3 *Let $(S, \circ, \leq)$ be an ordered hypersemigroup. If B is a bi-ideal of $(S, \circ, \leq)$, then $B \cup \{e\}$ is a bi-ideal of $(S \cup \{e\}, \bar{\circ})$, but it is not an ideal of $(S \cup \{e\}, \bar{\circ}, \preceq)$.*

Proof We have $\Big(B \cup \{e\}\Big) \,\bar{*}\, \Big(S \cup \{e\}\Big) \,\bar{*}\, \Big(B \cup \{e\}\Big) \subseteq B \cup \{e\}$. In fact:

Let $t \in \Big(B \cup \{e\}\Big) \,\bar{*}\, \Big(S \cup \{e\}\Big) \,\bar{*}\, \Big(B \cup \{e\}\Big)$. We have $t \in u \,\bar{\circ}\, v$ for some $u \in \Big(B \cup \{e\}\Big) \,\bar{*}\, \Big(S \cup \{e\}\Big)$, $v \in B \cup \{e\}$ and $u \in x \,\bar{\circ}\, y$ for sone $x \in B \cup \{e\}$, $y \in S \cup \{e\}$. We have the cases:

(a) $x \in B$, $\big(y \in S$ or $y = e\big)$, $\big(v \in B$ or $v = e\big)$

(b) $x = e$, $\big(y \in S$ or $y = e\big)$, $\big(v \in B$ or $v = e\big)$.

So we have to check the following:

(1) $x \in B$, $y \in S$, $v \in B$

(2) $x \in B$, $y \in S$, $v = e$

(3) $x \in B$, $y = e$, $v \in B$

(4) $x \in B$, $y = e$, $v = e$

(5) $x = e$, $y \in S$, $v \in B$

(6) $x = e$, $y \in S$, $v = e$

(7) $x = e$, $y = e$, $v \in B$

(8) $x = y = v = e$.

(1) Let $x \in B$, $y \in S$, $v \in B$. We have $t \in u \,\bar{\circ}\, v = \{u\} \,\bar{*}\, \{v\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{v\}$. We also have $(x \,\bar{\circ}\, y) \,\bar{*}\, \{v\} \subseteq (x \circ y) * \{v\}$. Indeed: Let $t \in (x \,\bar{\circ}\, y) \,\bar{*}\, \{v\}$. Then $t \in a \,\bar{\circ}\, v$ for some $a \in x \,\bar{\circ}\, y$. Since $x, y \in S$, we have $a \in x \,\bar{\circ}\, y = x \circ y \subseteq S$. Since $a, v \in S$, we have $a \,\bar{\circ}\, v = a \circ v$. Thus, we have $t \in a \circ v = \{a\} * \{v\} \subseteq (x \circ y) * \{v\}$

and so $(x \,\bar{\circ}\, y) \,\bar{*}\, \{v\} \subseteq (x \circ y) * \{v\}$. Hence, we have $t \in (x \circ y) * \{v\} = \{x\} * \{y\} * \{v\} \subseteq B * S * B \subseteq B$ and so $t \in B \subseteq B \cup \{e\}$.
(2) Let $x \in B$, $y \in S$, $v = e$. Since $t \in u \,\bar{\circ}\, v = t \in u \,\bar{\circ}\, e = \{e\}$, we have $t = e \in B \cup \{e\}$.
(3) Let $x \in B$, $y = e$, $v \in B$. We have

$$t \in u \,\bar{\circ}\, v = \{u\} \,\bar{*}\, \{v\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{v\} = (x \,\bar{\circ}\, e) \,\bar{*}\, \{v\} = \{e\} \,\bar{*}\, \{v\} = e \,\bar{\circ}\, c = \{e\}$$

and so $t = e \in B \cup \{e\}$.
(4) Let $x \in B$, $y = e$, $v = e$. We have

$$t \in u \,\bar{\circ}\, v = \{u\} \,\bar{*}\, \{v\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{e\} = (x \,\bar{\circ}\, e) \,\bar{*}\, \{e\} = \{e\} \,\bar{*}\, \{e\} = e \,\bar{\circ}\, e = \{e\}$$

and so $t = e \in B \cup \{e\}$.
(5) Let $x = e$, $y \in S$, $v \in B$. We have

$$t \in u \,\bar{\circ}\, v = \{u\} \,\bar{*}\, \{v\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{v\} = (e \,\bar{\circ}\, y) \,\bar{*}\, \{v\} = \{e\} \,\bar{*}\, \{v\} = e \,\bar{\circ}\, v = \{e\}$$

and so $t \in B \cup \{e\}$.
(6) Let $x = e$, $y \in S$, $v = e$. We have

$$t \in u \,\bar{\circ}\, v = \{u\} \,\bar{*}\, \{v\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{v\} = (e \,\bar{\circ}\, y) \,\bar{*}\, \{e\} = \{e\} \,\bar{\circ}\, \{e\} = e \,\bar{\circ}\, e = \{e\}$$

and so $t \in B \cup \{e\}$.
(7) Let $x = e$, $y = e$, $v \in B$. We have

$$t \in u \,\bar{\circ}\, v = \{u\} \,\bar{*}\, \{v\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{v\} = (e \,\bar{\circ}\, e) \,\bar{*}\, \{v\} = \{e\} \,\bar{*}\, \{v\} = e \,\bar{\circ}\, v = \{e\}$$

and so $t \in B \cup \{e\}$.
(8) Let $t = y = v = e$. We have

$$t \in u \,\bar{\circ}\, v = \{u\} \,\bar{*}\, \{v\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{v\} = (e \,\bar{\circ}\, e) \,\bar{*}\, \{e\} = \{e\} \,\bar{*}\, \{e\} = e \,\bar{\circ}\, e = \{e\}$$

and so $t \in B \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a, b, e\}$ (as an ideal) is a bi-ideal of $S = \{a, b, c, d, e, f, g\}$, but, as we have already seen, $f \in S \cup \{h\}$, $f \leq h$ and $f \notin \{a, b, e, h\}$ and so $\{a, b, e, h\}$ is not a bi-ideal of $S \cup \{h\}$.

The set $\{a, d, e, g\}$ is a bi-ideal of $\{a, b, c, d, e, f, g\}$ as

$\{a, d, e, g\} * \{a, b, c, d, e, f, g\} * \{a, d, e, g\} = \{a, b\} * \{a, b, c, d, e, f, g\} = \{a\} \subseteq \{a, d, e, g\}$,

$x \in \{a, d, e, g\}$ and $\{a, b, c, d, e, f, g\} \ni y \leq x$ implies $x \in \{a, d, e, g\}$, but $\{a, d, e, g, h\}$ is not a bi-ideal of $\{a, b, c, d, e, f, g, h\}$ as $h \in \{a, d, e, g, h\}$ and $\{a, b, c, d, e, f, g, h\} \ni f \leq h$, but $f \notin \{a, d, e, g, h\}$. □

Proposition 5.4 *Let $(S, \circ, \leq)$ be an ordered hypersemigroup. If Q is a quasi-ideal of $(S, \circ, \leq)$, then $Q \cup \{e\}$ is a quasi-ideal of $(S \cup \{e\}, \bar{\circ})$, but it is not a quasi-ideal of $(S \cup \{e\}, \bar{\circ}, \preceq)$.*

Proof We have $\Big((Q\cup\{e\})\,\overline{*}\,(S\cup\{e\})\Big)\cap\Big((S\cup\{e\})\,\overline{*}\,(Q\cup\{e\})\Big)\subseteq Q\cup\{e\}$. Indeed:

Let $t\in\Big((Q\cup\{e\})\,\overline{*}\,(S\cup\{e\})\Big)\cap\Big((S\cup\{e\})\,\overline{*}\,(Q\cup\{e\})\Big)$. Then $t\in x\,\overline{\circ}\,y$ for some $x\in Q\cup\{e\}$, $y\in S\cup\{e\}$ and $t\in z\,\overline{\circ}\,h$ for some $z\in S\cup\{e\}$, $h\in S\cup\{e\}$. We have the cases:

(a) $x\in Q$, $\big(y\in S$ or $y=e\big)$, $\big(z\in S$ or $z=e\big)$, $\big(h\in Q$ or $h=e\big)$

(b) $x=e$, $\big(y\in S$ or $y=e\big)$, $\big(z\in S$ or $z=e\big)$, $\big(h\in Q$ or $h=e\big)$

So, we have to check the following:

(1) $x\in Q,\ y\in S,\ z\in S,\ h\in Q$ (2) $x\in Q,\ y\in S,\ z\in S,\ h=e$
(3) $x\in Q,\ y\in S,\ z=e,\ h\in Q$ (4) $x\in Q,\ y\in S,\ z=e,\ h=e$
(5) $x\in Q,\ y=e,\ z\in S,\ h\in Q$ (6) $x\in Q,\ y=e,\ z\in S,\ h=e$
(7) $x\in Q,\ y=e,\ z=e,\ h\in Q$ (8) $x\in Q,\ y=e,\ z=e,\ h=e$
(9) $x=e,\ y\in S,\ z\in S,\ h\in Q$ (10) $x=e,\ y\in S,\ z\in S,\ h=e$
(11) $x=e,\ y\in S,\ z=e,\ h\in Q$ (12) $x=e,\ y\in S,\ z=e,\ h=e$
(13) $x=e,\ y=e,\ z\in S,\ h\in Q$ (14) $x=e,\ y=e,\ z\in S,\ h=e$
(15) $x=e,\ y=e,\ z=e,\ h\in Q$ (16) $x=e,\ y=e,\ z=e,\ h=e$.

(1) Let $x\in Q$, $y\in S$, $z\in S$, $h\in Q$. Since $x,y\in S$, we have $t\in x\,\overline{\circ}\,y=x\circ y=\{x\}*\{y\}\subseteq Q*S$. Since $z,h\in S$, we have $t\in z\,\overline{\circ}\,h=z\circ h=\{z\}*\{h\}\subseteq S*Q$. Then we have $t\in(Q*S)\cap(S*Q)\subseteq(Q*S]\cap(S*Q]\subseteq Q$ and so $t\in Q\subseteq Q\cup\{e\}$.
(2) Let $x\in Q$, $y\in S$, $z\in S$, $h=e$. Since $x,y\in S$, we have $t\in x\,\overline{\circ}\,y=x\circ y=\{x\}*\{y\}\subseteq Q*S\subseteq S$. Since $z\in S$, $h=e$, we have $t\in z\,\overline{\circ}\,h=z\,\overline{\circ}\,e=\{e\}$. Then $S\ni t=e$. The case is impossible.
(3) Let $x\in Q$, $y\in S$, $z=e$, $h\in Q$. Then $t\in x\,\overline{\circ}\,y=x\circ y=\{x\}*\{y\}\subseteq Q*S\subseteq S$ and $t\in z\,\overline{\circ}\,h=z\,\overline{\circ}\,e=\{e\}$. Then we have $S\ni t=e$. The case is impossible.
(4) Let $x\in Q$, $y\in S$, $z=e$, $h=e$. Then $t\in x\,\overline{\circ}\,y=x\circ y=\{x\}*\{y\}\subseteq Q*S\subseteq S$ and $t\in z\,\overline{\circ}\,h=e\,\overline{\circ}\,e=\{e\}$. Then $S\ni t=e$, the case is impossible.
(5) Let $x\in Q$, $y=e$, $z\in S$, $h\in Q$. Then $t\in x\,\overline{\circ}\,y=x\,\overline{\circ}\,e=\{e\}$, $t\in z\,\overline{\circ}\,h=z\circ h=\{z\}*\{h\}\subseteq S*Q\subseteq S$. Then $S\ni t=e$, the case is impossible.
(6) Let $x\in Q$, $y=e$, $z\in S$, $h=e$. Then $t\in x\,\overline{\circ}\,y=x\,\overline{\circ}\,e=\{e\}$, $t\in z\,\overline{\circ}\,h=z\,\overline{\circ}\,e=\{e\}$. Then $t=e\in Q\cup\{e\}$.
(7) Let $x\in Q$, $y=e$, $z=e$, $h\in Q$. Then $t\in x\,\overline{\circ}\,y=x\,\overline{\circ}\,e=\{e\}$, $t\in z\,\overline{\circ}\,h=e\,\overline{\circ}\,h=\{e\}$. Then $t=e\in Q\cup\{e\}$.
(8) Let $x\in Q$, $y=e$, $z=e$, $h=e$. Then $t\in x\,\overline{\circ}\,y=x\,\overline{\circ}\,e=\{e\}$, $t\in z\,\overline{\circ}\,h=e\,\overline{\circ}\,e=\{e\}$. Then $t=e\in Q\cup\{e\}$.
(9) Let $x=e$, $y\in S$, $z\in S$, $h\in Q$. Then $t\in x\,\overline{\circ}\,y=e\,\overline{\circ}\,y=\{e\}$, $t\in z\,\overline{\circ}\,h=z\circ h=\{z\}*\{h\}\subseteq S*Q\subseteq S$. Then $S\ni t=e$, the case is impossible.
(10) Let $x=e$, $y\in S$, $z\in S$, $h=e$. Then $t\in x\,\overline{\circ}\,y=e\,\overline{\circ}\,y=\{e\}$, $t\in z\,\overline{\circ}\,h=z\,\overline{\circ}\,e=\{e\}$ and so $t=e\in Q\cup\{e\}$.
(11) Let $x=e$, $y\in S$, $z=e$, $h\in Q$. Then $t\in x\,\overline{\circ}\,y=e\,\overline{\circ}\,y=\{e\}$, $t\in z\,\overline{\circ}\,h=e\,\overline{\circ}\,h=\{e\}$ and so $t=e\in Q\cup\{e\}$.
(12) Let $x=e$, $y\in S$, $z=e$, $h=e$. Then $t\in x\,\overline{\circ}\,y=e\,\overline{\circ}\,y=\{e\}$, $t\in z\,\overline{\circ}\,h=e\,\overline{\circ}\,e=\{e\}$ and so

$t = e \in Q \cup \{e\}$.

(13) Let $x = e$, $y = e$, $z \in S$, $h \in Q$. Then $t \in x \,\bar{\circ}\, y = e \,\bar{\circ}\, e = \{e\}$, $t \in z \,\bar{\circ}\, h = z \circ h = \{z\} * \{h\} \subseteq S * Q \subseteq S$ and so $S \ni t = e$, the case is impossible.

(14) Let $x = e$, $y = e$, $z \in S$, $h = e$. Then $t \in x \,\bar{\circ}\, y = e \,\bar{\circ}\, e = \{e\}$, $t \in z \,\bar{\circ}\, h = z \,\bar{\circ}\, e = \{e\}$ and so $t = e \in Q \cup \{e\}$.

(15) Let $x = e$, $y = e$, $z = e$, $h \in Q$. Then $t \in x \,\bar{\circ}\, y = e \,\bar{\circ}\, e = \{e\}$, $t \in z \,\bar{\circ}\, h = e \,\bar{\circ}\, h = \{e\}$ and so $t = e \in Q \cup \{e\}$.

(16) Let $x = e$, $y = e$, $z = e$, $h = e$. Then $t \in x \,\bar{\circ}\, y = e \,\bar{\circ}\, e = \{e\}$, $t \in z \,\bar{\circ}\, h = e \,\bar{\circ}\, e = \{e\}$ and so $t = e \in Q \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a, b, f, g\}$ is a quasi-ideal of $\{a, b, c, d, e, f, g\}$ as

$$\begin{aligned} \Big(\{a,b,f,g\} * \{a,b,c,d,e,f,g\}\Big] \quad &\cap \quad \Big(\{a,b,c,d,e,f,g\} * \{a,b,f,g\}\Big] \\ &= \quad \Big(\{a,b,c,d,e,f\}\Big] \cap \Big(\{a,b,f\}\Big] \\ &= \quad \{a,b,c,d,f,e\} \cap \{a,b,f\} \\ &= \quad \{a,b,f\} \subseteq \{a,b,f,g\}; \end{aligned}$$

$x \in \{a, b, f, g\}$ and $\{a, b, c, d, e, f, g\} \ni y \le x$ implies $y \in \{a, b, f, g\}$, but $\{a, b, f, g, h\}$ is not a quasi-ideal of $\{a, b, c, d, e, f, g, h\}$ as $h \in \{a, b, f, g, h\}$ and $\{a, b, c, d, e, f, g\} \ni h \preceq h$, but $c \notin \{a, b, f, g, h\}$. □

Proposition 5.5 *Let $(S, \circ, \le)$ be an ordered hypersemigroup. If A is an interior ideal of $(S, \circ, \le)$, then $A \cup \{e\}$ is an interior ideal of $(S \cup \{e\}, \bar{\circ})$, but it is not an interior ideal of $(S \cup \{e\}, \bar{\circ}, \preceq)$.*

Proof We have $\Big(S \cup \{e\}\Big) \,\bar{*}\, \Big(A \cup \{e\}\Big) \,\bar{*}\, \Big(S \cup \{e\}\Big) \subseteq A \cup \{e\}$. Indeed:

Let $t \in \Big(S \cup \{e\}\Big) \,\bar{*}\, \Big(A \cup \{e\}\Big) \,\bar{*}\, \Big(S \cup \{e\}\Big)$. Then $t \in x \,\bar{\circ}\, y$ for some $x \in \Big(S \cup \{e\}\Big) \,\bar{*}\, \Big(A \cup \{e\}\Big)$, $y \in S \cup \{e\}$ and $x \in u \,\bar{\circ}\, v$ for some $u \in S \cup \{e\}$, $v \in A \cup \{e\}$. We have the cases:

(a) $y \in S$, $\big(u \in S$ or $u = e\big)$, $\big(v \in A$ or $v = e\big)$

(b) $y = e$, $\big(u \in S$ or $u = e\big)$, $\big(v \in A$ or $v = e\big)$.

So we have to check the following:

(1) $y \in S$, $u \in S$, $v \in A$

(2) $y \in S$, $u \in S$, $v = e$

(3) $y \in S$, $u = e$, $v \in A$

(4) $y \in S$, $u = e$, $v = e$

(5) $y = e$, $u \in S$, $v \in A$

(6) $y = e$, $u \in S$, $v = e$

(7) $y = e$, $u = e$, $v \in A$

(8) $y = e$, $u = e$, $v = e$.

(1) Let $y \in S$, $u \in S$, $v \in A$. Then we have

$$t \in x \,\overline{\circ}\, y = \{x\} \,\overline{*}\, \{y\} \subseteq (u \,\overline{\circ}\, v) \,\overline{*}\, \{y\} = \{u\} \,\overline{*}\, \{v\} \,\overline{*}\, \{y\} \subseteq S * A * S \subseteq A \subseteq A \cup \{e\}.$$

(2) Let $y \in S$, $u \in S$, $v = e$. Then $t \in u \,\overline{\circ}\, v = u \,\overline{\circ}\, e = \{e\} \subseteq A \cup \{e\}$.

(3) Let $y \in S$, $u = e$, $v \in A$. Then $t \in u \,\overline{\circ}\, v = e \,\overline{\circ}\, v = \{e\} \subseteq A \cup \{e\}$.

(4) Let $y \in S$, $u = e$, $v = e$. Then $t \in u \,\overline{\circ}\, v = e \,\overline{\circ}\, e = \{e\} \subseteq A \cup \{e\}$.

(5) Let $y = e$, $u \in S$, $v \in A$. Then $t \in x \,\overline{\circ}\, y = x \,\overline{\circ}\, e = \{e\} \subseteq A \cup \{e\}$.

(6) let $y = e$, $u \in S$, $v = e$. Then $t \in x \,\overline{\circ}\, y = x \,\overline{\circ}\, e = \{e\} \subseteq A \cup \{e\}$.

(7) Let $y = e$, $u = e$, $v \in A$. Then $t \in x \,\overline{\circ}\, y = x \,\overline{\circ}\, e = \{e\} \subseteq A \cup \{e\}$

(8) Let $y = e$, $u = e$, $v = e$. Then $t \in x \,\overline{\circ}\, y = e \,\overline{\circ}\, e = \{e\} \subseteq A \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a\}$ is an interior ideal element of $S = \{a, b, c, d, e, f, g\}$ as

$\{a, b, c, d, e, f, g\} * \{a\} = (a \circ a) \cup (b \circ a) \cdots (f \circ a) \cup (g \circ a) = \{a\}$,

$\{a, b, c, d, e, f, g\} * \{a\} * \{a, b, c, d, e, f, g\} = \{a\}$,

if $x \in \{a\}$ and $\{a, b, c, d, e, f, g\} \ni y \leq a$, then $y = a$.

However, $\{a\} \cup \{h\}$ is not an interior ideal of $\{a, b, c, d, e, f, g, h\}$. Indeed, $\{a, b, c, d, e, f, g, h\} \ni c \leq h$, but $c \notin \{a, h\}$. □

Note Concerning the ordered hypersemigroup $(S, \circ, \leq)$ given by Table 5 and Figure 3, it might be mentioned that

The ideals of $(S, \circ, \leq)$ are the sets: $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$, $\{a, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$, $\{a, b, e\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$, $\{a, b, c, d, e\}$, $\{a, b, c, d, e, f\}$, $\{a, b, g\}$ $\{a, b, c, g\}$, $\{a, b, d, g\}$, $\{a, b, c, d, g\}$, $\{a, b, e, g\}$, $\{a, b, c, e, g\}$, $\{a, b, d, e, g\}$, $\{a, b, c, d, e, g\}$ and S (total 22).

The quasi-ideals of $(S, \circ, \leq)$ are the ideals of S plus the sets $\{a, e\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{a, c, d, e\}$, $\{a, f\}$, $\{a, b, f\}$, $\{a, c, f\}$, $\{a, b, c, f\}$, $\{a, d, f\}$, $\{a, b, d, f\}$, $\{a, c, d, f\}$, $\{a, b, c, d, f\}$, $\{a, e, f\}$, $\{a, b, e, f\}$, $\{a, c, e, f\}$, $\{a, b, c, e, f\}$, $\{a, d, e, f\}$, $\{a, b, d, e, f\}$, $\{a, c, d, e, f\}$, $\{a, g\}$, $\{a, c, g\}$, $\{a, d, g\}$, $\{a, c, d, g\}$, $\{a, b, f, g\}$, $\{a, b, c, f, g\}$, $\{a, b, d, f, g\}$, $\{a, b, c, d, f, g\}$, $\{a, b, e, f, g\}$, $\{a, b, c, e, f, g\}$, $\{a, b, d, e, f, g\}$ (total 52).

The bi-ideals of $(S, \circ, \leq)$ are the quasi-ideals of S plus the sets $\{a, e, g\}$, $\{a, c, e, g\}$, $\{a, d, e, g\}$, $\{a, c, d, e, g\}$ (total 56).

References

[1] Fuchs L, Halperin I. On the imbedding of a regular ring in a regular ring with identity. Fundamenta Mathematicae 1964; 54: 285-290.

[2] Kehayopulu N. On adjoining identity to semigroups, greatest element of ordered sets. Mathematica Japonica 1991; 36 (4): 695-702.

[3] Kehayopulu N. On adjoining greatest element to ordered semigroups. Mathematica Japonica 1993; 38 (1): 61-66.

[4] Kehayopulu N. On hypersemigroups. Pure Mathematics and Applications. PU.M.A. 2015; 25 (2): 151-156. https://doi.org/10.1515/puma-2015-0015

[5] Kehayopulu N. On ordered hypersemigroups given by a table of multiplication and a figure. Turkish Journal of Mathematics 2018; 42 (4): 2045-060. https://doi.org/10.3906/mat-1711-53

[6] Kehayopulu N. From ordered semigroups to ordered hypersemigroups. Turkish Journal of Mathematics 2019; 43 (1): 21-35. https://doi.org/10.3906/mat-1806-104

[7] Kehayopulu N. Adjunction identity to hypersemigroups. Turkish Journal of Mathematics 2022; 46 (7): 2834-2853. https://doi.org/10.55730/1300-0098.3333