The application of Brzdek's fixed point theorem in the stability problem of the Drygas functional equation

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Abstract: Using the Brzdek fixed point theorem, we establish the Hyers–Ulam stability problem of Drygas functional equations

$$\delta(x+y-z) + \delta(x-y) + \delta(-y-z) + \delta(y) = \delta(x-y-z) + \delta(y-z) + \delta(x+y) + \delta(-y)$$

for all $x, y, z \in A$.

Key words: Brzdek fixed point theorem, Drygas functional equation, stability

1. Introduction

The stability theory of functional equations started with the talk of Ulam held at the Wisconsin University in 1940 (see [21]).

Hyers [12] was the first to introduce the result concerning the stability of functional equations. Indeed, he obtained a famous theorem on the stability of the additive functional equation while he was trying to answer the question of Ulam.

The method provided by Hyers [12] which produces the additive function will be called a direct method. This method is the most important and powerful tool concerning the stability of different functional equations. That is, the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution [2, 8, 20].

Isac and Rassias [13] were the first to provide applications of stability problem of functional equations for the proof of new fixed point theorems with applications in 1996. Using fixed point methods, the stability problems of several functional equations have been discussed (among others) in the following papers: (see [15, 18]).

During the last years, a number of results concerning the stability have been obtained by different ways [3, 5–7, 17, 19] and applied to a number of functional equations, functional inequalities, and differential equation.

Also, many mathematicians studied the stabilities Pexiderized additive-Drygas equation, additive-quadratic equation, and the Drygas' equation (see [11], [10], and [14]).

Throughout the paper, we suppose that A is a normed space and B is a Banach space. We recall some basic definitions.

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A mapping $\delta : A \to B$ is said to be additive if it satisfies

$$\delta(x + y) = \delta(x) + \delta(y)$$

for all $x, y \in A$. A mapping $\delta : A \to B$ is called quadratic if δ satisfies the functional equation

$$\delta(x + y) + \delta(x - y) = 2\delta(x) + 2\delta(y)$$

for all $x, y \in A$.

Various studies on stability of the quadratic functional equation can be found in [4, 16].

We say a mapping $\delta : A \to B$ satisfies the Drygas functional equation if

$$\delta(x + y) + \delta(x - y) = 2\delta(x) + \delta(y) + \delta(-y)$$

for all $x, y \in A$. This functional equation was considered by Drygas [1] to characterize quasiinner product spaces. The general solution of this functional equation was given by Ebanks et al. [9] as

$$\delta(x) = Q(x) + K(x)$$

for all $x \in A$, where $Q : A \to B$ is a quadratic mapping and $K : A \to B$ is an additive mapping.

In this note, using the Brzdek fixed point method, we find the stability results of the functional equation

$$\delta(x + y - z) + \delta(x - y) + \delta(-y - z) + \delta(y) = \delta(x - y - z) + \delta(y - z) + \delta(x + y) + \delta(-y) \tag{1}$$

for all $x, y, z \in A$.

In 2011, Brzdek et al. [1] gave a simple fixed point theorem. Before stating Brzdek fixed point theorem, let us introduce some hypothesis, which we will use in the sequel.

(H1) A is a nonempty set and B is a Banach space.

(H2) $\delta_1, \ldots, \delta_k : A \to A$ and $\lambda_1, \ldots, \lambda_k : A \to \mathbb{R}_+$ are given maps.

(H3) $\mathcal{J} : B^A \to B^A$ is an operator satisfying the inequality

$$\|\mathcal{J}g(x) - \mathcal{J}h(x)\| \leq \sum_{i=1}^{k} \lambda_i(x) \|g(\delta_i(x)) - h(\delta_i(x))\|$$

for all $g, h : A \to B$ and $x \in A$.

(H4) $\Lambda : \mathbb{R}_+^A \to \mathbb{R}_+^A$ is a linear operator defined by

$$\Lambda \mathcal{F}(x) := \sum_{i=1}^{k} \lambda_i(x) \mathcal{F}(\delta_i(x))$$

for $\mathcal{F} : A \to \mathbb{R}_+$ and $x \in A$.

Theorem 1.1 *[1] Suppose that hypotheses (H1)–(H4) are satisfied. Assume that there are functions $\eta : A \to \mathbb{R}_+$ and $\varphi : A \to B$ such that, for all $x \in A$,*

$$\|\mathcal{J}\varphi(x) - \varphi(x)\| \leq \eta(x)$$

and

$$\eta^*(x) := \sum_{n=0}^{\infty} \Lambda^n \eta(x) < \infty$$

hold. Then, for all $x \in A$ the limit

$$T(x) := \lim_{n \to \infty} \mathcal{J}^n \varphi(x)$$

exists and the function $T : A \to B$ so defined is a unique fixed point of $\mathcal{J}$ with

$$\|\varphi(x) - T(x)\| \leq \eta^*(x)$$

for all $x \in A$.

Throughout the paper, $\mathbb{N}_0$ denotes the set of all nonnegative integers.

2. Stability of functional equation (1): via fixed point method

In this section, using the Brzdek fixed point method, we find the stability results of the functional equation (1).

Lemma 2.1 *Suppose that a function $\delta : A \to B$ satisfying $\delta(0) = 0$ and*

$$\delta(x + y - z) + \delta(x - y) + \delta(-y - z) + \delta(y) = \delta(x - y - z) + \delta(y - z) + \delta(x + y) + \delta(-y) \tag{2}$$

for all $x, y, z \in A$. Then the function δ satisfies the Drygas equation on A.

Proof Letting $z = y$ in (2) yields

$$\delta(x) + \delta(x - y) + \delta(-2y) + \delta(y) = \delta(x - 2y) + \delta(x + y) + \delta(-y) \tag{3}$$

for all $x, y \in A$. Replace y by $-y$ in (3) to get

$$\delta(x) + \delta(x + y) + \delta(2y) + \delta(-y) = \delta(x + 2y) + \delta(x - y) + \delta(y) \tag{4}$$

for all $x, y \in A$.

Adding (3) and (4) side by side, we obtain

$$2\delta(x) + \delta(2y) + \delta(-2y) = \delta(x + 2y) + \delta(x - 2y)$$

i.e.

$$\delta(x + y) + \delta(x - y) = 2\delta(x) + \delta(y) + \delta(-y)$$

for all $x, y \in A$. $\qquad\square$

Theorem 2.2 *Let A be a normed space and B be a Banach space. Suppose that a function $\delta : A \to B$ satisfying and*

$$\|\delta(x + y - z) + \delta(x - y) + \delta(-y - z) + \delta(y) - \delta(x - y - z) - \delta(x + y) - \delta(y - z) - \delta(-y)\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \tag{5}$$

for some $\theta \geq 0$, $p, q, r > 0$ and all $x, y, z \in A$. Then there exists a unique function $\mathfrak{D} : A \to B$ satisfying the Drygas equation on A such that

$$\|\delta(x) - \mathfrak{D}(x) - \delta(0)\| \leq \begin{cases} \frac{2\theta}{2^{1+q+r} - 2^{p+2q+2r}} \|x\|^{p+q+r} & 0 < p+q+r < 1 \\\\ \left(\frac{2\theta}{2^{p+2q+2r} - 2^{1+q+r}} + \frac{2\theta}{2^{2+q+r} - 2^{p+2q+2r}} \right) \|x\|^{p+q+r} & 1 < p+q+r < 2 \\\\ \frac{2\theta}{2^{p+2q+2r} - 2^{q+r+2}} \|x\|^{p+q+r} & p+q+r > 2 \end{cases}$$

for all $x \in A$.

Proof Define a new function $\widehat{\delta} : A \to B$ by $\widehat{\delta}(x) = \delta(x) - \delta(0)$ for all $x \in A$. So, $\widehat{\delta}(0) = 0$ and by (5) the function $\widehat{\delta}$ satisfying

$$\|\widehat{\delta}(x+y-z) + \widehat{\delta}(x-y) + \widehat{\delta}(-y-z) + \widehat{\delta}(y) - \widehat{\delta}(x-y-z) - \widehat{\delta}(x+y) - \widehat{\delta}(y-z) - \widehat{\delta}(-y)\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \quad (6)$$

for all $x, y, z \in A$.

First, we consider the case $p + q + r > 2$.

Setting $z = y$ in (6) yields

$$\|\widehat{\delta}(x) + \widehat{\delta}(x-y) + \widehat{\delta}(-2y) + \widehat{\delta}(y) - \widehat{\delta}(x-2y) - \widehat{\delta}(x+y) - \widehat{\delta}(-y)\| \leq \theta \|x\|^p \|y\|^{q+r} \quad (7)$$

for all $x, y \in A$. Replace y by $-y$ in (7) to gain

$$\|\widehat{\delta}(x) + \widehat{\delta}(x+y) + \widehat{\delta}(2y) + \widehat{\delta}(-y) - \widehat{\delta}(x+2y) - \widehat{\delta}(x-y) - \widehat{\delta}(y)\| \leq \theta \|x\|^p \|y\|^{q+r} \quad (8)$$

for all $x, y \in A$.

It follows from (7) and (8) that

$$\|\widehat{\delta}(x+2y) + \widehat{\delta}(x-2y) - 2\widehat{\delta}(x) - \widehat{\delta}(2y) - \widehat{\delta}(-2y)\| \leq 2\theta \|x\|^p \|y\|^{q+r} \quad (9)$$

for all $x, y \in A$.

Replacing x and y by $\frac{x}{2}$ and $\frac{x}{4}$ in (9), respectively, we gain

$$\left\| \widehat{\delta}(x) - 3\widehat{\delta}\left(\frac{x}{2}\right) - \widehat{\delta}\left(-\frac{x}{2}\right) \right\| \leq \frac{2\theta}{2^{p+2q+2r}} \|x\|^{p+q+r} \quad (10)$$

for all $x \in A$. Let $\mathcal{J} : B^A \to B^A$ and $\eta : A \to \mathbb{R}_+$ be defined by

$$\mathcal{J}g(x) = 3g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right), \qquad g \in B^A$$

and

$$\eta(x) = \frac{2\theta}{2^{p+2q+2r}} \|x\|^{p+q+r}$$

for all $x \in A$. So, inequality (10) becomes as follows

$$\|\mathcal{J}\widehat{\delta}(x) - \widehat{\delta}(x)\| \leq \eta(x)$$

for all $x \in A$. For all $g, h \in B^A$ and $x \in A$

$$\|\mathcal{J}g(x) - \mathcal{J}h(x)\| \le 3 \left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right)\right\|;$$

thus, $\mathcal{J} : B^A \to B^A$ satisfies the condition (H3) with $\lambda_1(x) = 3, \lambda_2(x) = 1, \delta_1(x) = \frac{x}{2}$ and $\delta_2(x) = -\frac{x}{2}$. By (H4), the operator $\Lambda : \mathbb{R}_+^A \to \mathbb{R}_+^A$ is defined by:

$$\Lambda\mathcal{F}(x) = 3\mathcal{F}\left(\frac{x}{2}\right) + \mathcal{F}\left(-\frac{x}{2}\right), \qquad \mathcal{F} \in \mathbb{R}_+^A$$

for all $x \in A$. Hence,

$$\Lambda\eta(x) = 3\eta\left(\frac{x}{2}\right) + \eta\left(-\frac{x}{2}\right) = 4\eta\left(\frac{x}{2}\right) = \frac{4}{2^{p+q+r}}\eta(x), \qquad \eta \in \mathbb{R}_+^A$$

for all $x \in A$. By linearity Λ and induction on n, we can prove

$$\Lambda^n\eta(x) = \left(\frac{4}{2^{p+q+r}}\right)^n \eta(x), \qquad n \in \mathbb{N}_0$$

for all $x \in A$.

Now, if $p + q + r > 2$, then $\frac{4}{2^{p+q+r}} < 1$. Therefore, the series $\sum_{n=0}^{\infty} \Lambda^n\eta(x)$ is convergent for all $x \in A$ and

$$\eta^*(x) = \sum_{n=0}^{\infty} \Lambda^n\eta(x) = \sum_{n=0}^{\infty} \left(\frac{4}{2^{p+q+r}}\right)^n \eta(x) = \frac{2^{p+q+r}}{2^{p+q+r} - 4}\eta(x) = \frac{2\theta}{2^{p+2q+2r} - 2^{q+r+2}}\|x\|^{p+q+r}$$

for all $x \in A$. By Theorem 1.1, there exists a function $\mathfrak{D} : A \to B$ such

$$\mathfrak{D}(x) = \lim_{n \to \infty} \mathcal{J}^n\widehat{\delta}(x), \qquad \mathfrak{D}(x) = 3\mathfrak{D}\left(\frac{x}{2}\right) + \mathfrak{D}\left(-\frac{x}{2}\right)$$

and

$$\|\widehat{\delta}(x) - \mathfrak{D}(x)\| \le \frac{2\theta}{2^{p+2q+2r} - 2^{q+r+2}}\|x\|^{p+q+r}$$

i.e.

$$\|\delta(x) - \mathfrak{D}(x) - \delta(0)\| \le \frac{2\theta}{2^{p+2q+2r} - 2^{q+r+2}}\|x\|^{p+q+r}$$

for all $x \in A$.

Next, we show that $\mathfrak{D}$ satisfies the Drygas equation. It follows from (6) that

$$\|\mathcal{J}\widehat{\delta}(x+y-z) + \mathcal{J}\widehat{\delta}(x-y) + \mathcal{J}\widehat{\delta}(-y-z) + \mathcal{J}\widehat{\delta}(y) - \mathcal{J}\widehat{\delta}(x-y-z) - \mathcal{J}\widehat{\delta}(x+y) - \mathcal{J}\widehat{\delta}(y-z) - \mathcal{J}\widehat{\delta}(-y)\|$$

$$\leq 3\left\|\widehat{\delta}\left(\frac{x+y-z}{2}\right) + \widehat{\delta}\left(\frac{x-y}{2}\right) + \widehat{\delta}\left(\frac{-y-z}{2}\right) + \widehat{\delta}\left(\frac{y}{2}\right)\right.$$

$$\left. -\widehat{\delta}\left(\frac{x-y-z}{2}\right) - \widehat{\delta}\left(\frac{x+y}{2}\right) - \widehat{\delta}\left(\frac{y-z}{2}\right) - \widehat{\delta}\left(\frac{-y}{2}\right)\right\|$$

$$+\left\|\widehat{\delta}\left(-\frac{x+y-z}{2}\right) + \widehat{\delta}\left(-\frac{x-y}{2}\right) + \widehat{\delta}\left(-\frac{-y-z}{2}\right) + \widehat{\delta}\left(-\frac{y}{2}\right)\right.$$

$$\left. -\widehat{\delta}\left(-\frac{x-y-z}{2}\right) - \widehat{\delta}\left(-\frac{x+y}{2}\right) - \widehat{\delta}\left(-\frac{y-z}{2}\right) - \widehat{\delta}\left(\frac{y}{2}\right)\right\|$$

$$\leq 3\theta\left\|\frac{x}{2}\right\|^p\left\|\frac{y}{2}\right\|^q\left\|\frac{z}{2}\right\|^r + \theta\left\|\frac{x}{2}\right\|^p\left\|\frac{y}{2}\right\|^q\left\|\frac{z}{2}\right\|^r = \left(\frac{4\theta}{2^{p+q+r}}\right)\|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in A$. By induction on n, we gain

$$\|\mathcal{J}^n\widehat{\delta}(x+y-z) + \mathcal{J}^n\widehat{\delta}(x-y) + \mathcal{J}^n\widehat{\delta}(-y-z) + \mathcal{J}^n\widehat{\delta}(y)$$

$$-\mathcal{J}^n\widehat{\delta}(x-y-z) - \mathcal{J}^n\widehat{\delta}(x+y) - \mathcal{J}^n\widehat{\delta}(y-z) - \mathcal{J}^n\widehat{\delta}(-y)\|$$

$$\leq \theta\left(\frac{4}{2^{p+q+r}}\right)^n\|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in A$ and $n \in \mathbb{N}_0$. By letting $n \to \infty$, we arrive at

$$\mathfrak{D}(x+y-z) + \mathfrak{D}(x-y) + \mathfrak{D}(-y-z) + \mathfrak{D}(y) = \mathfrak{D}(x-y-z) + \mathfrak{D}(x+y) + \mathfrak{D}(y-z) + \mathfrak{D}(-y)$$

for all $x, y, z \in A$. Thus, by Lemma 2.1, the function $\mathfrak{D}$ satisfies the Drygas equation.

Let us now consider the case $0 < p + q + r < 1$. The proof runs in the similar way as before; thus, we only give a sketch.

Taking $y = \frac{x}{2}$ in (9), we get

$$\|\frac{1}{3}\widehat{\delta}(2x) - \frac{1}{3}\widehat{\delta}(-x) - \widehat{\delta}(x)\| \leq \frac{1}{3} \cdot \frac{2\theta}{2^{q+r}}\|x\|^{p+q+r} \tag{11}$$

for all $x \in A$. Define the functions $\mathcal{J} : B^A \to B^A$ and $\eta : A \to \mathbb{R}_+$ by

$$\mathcal{J}g(x) = \frac{1}{3}g(2x) - \frac{1}{3}g(-x), \qquad g \in B^A$$

and

$$\eta(x) = \frac{1}{3} \cdot \frac{2\theta}{2^{q+r}}\|x\|^{p+q+r}$$

for all $x \in A$. Hence, the Inequality (11) is as follows:

$$\|\mathcal{J}\widehat{\delta}(x) - \widehat{\delta}(x)\| \leq \eta(x)$$

for all $x \in A$. It is clear that $\mathcal{J}$ satisfies (H3) with $\lambda_1(x) = \lambda_2(x) = \frac{1}{3}, \delta_1(x) = 2x$ and $\delta_2(x) = -x$ for all $x \in A$. Now, we define a function $\Lambda : \mathbb{R}_+^A \to \mathbb{R}_+^A$ by

$$\Lambda \mathcal{F}(x) = \frac{1}{3}\mathcal{F}(2x) + \frac{1}{3}\mathcal{F}(-x), \qquad \mathcal{F} \in \mathbb{R}_+^A$$

for all $x \in A$. Hence,

$$\Lambda \eta(x) = \frac{1}{3}\eta(2x) + \frac{1}{3}\eta(-x) = \frac{2^{p+q+r} + 1}{3}\eta(x)$$

for all $x \in A$. By induction on $n \in \mathbb{N}_0$, we arrive at

$$\Lambda^n \eta(x) = \left(\frac{2^{p+q+r} + 1}{3}\right)^n \eta(x)$$

for all $x \in A$. If $p + q + r < 1$, then $\frac{2^{p+q+r}+1}{3} < 1$; thus, the serie $\sum_{n=0}^{\infty} \Lambda^n \eta(x)$ is convergent for all $x \in A$. Also,

$$\eta^*(x) = \sum_{n=0}^{\infty} \Lambda^n \eta(x) = \sum_{n=0}^{\infty} \left(\frac{2^{p+q+r}+1}{3}\right)^n \eta(x) = \frac{3}{2 - 2^{p+q+r}}\eta(x) = \frac{2\theta}{2^{1+q+r} - 2^{p+2q+2r}}\|x\|^{p+q+r}$$

for all $x \in A$. By Theorem 1.1, there exists a function $\mathfrak{D} : A \to B$ that

$$\mathfrak{D}(x) = \lim_{n \to \infty} \mathcal{J}^n \widehat{\delta}(x), \qquad \mathfrak{D}(x) = \frac{1}{3}\mathfrak{D}(2x) - \frac{1}{3}\mathfrak{D}(-x)$$

and

$$\|\widehat{\delta}(x) - \mathfrak{D}(x)\| \leq \frac{2\theta}{2^{1+q+r} - 2^{p+2q+2r}}\|x\|^{p+q+r}$$

which implies

$$\|\delta(x) - \mathfrak{D}(x) - \delta(0)\| \leq \frac{2\theta}{2^{1+q+r} - 2^{p+2q+2r}}\|x\|^{p+q+r}$$

for all $x \in A$.

From the inequality (6), we gain

$$\|\mathcal{J}\widehat{\delta}(x + y - z) + \mathcal{J}\widehat{\delta}(x - y) + \mathcal{J}\widehat{\delta}(-y - z) + \mathcal{J}\widehat{\delta}(y) - \mathcal{J}\widehat{\delta}(x - y - z) - \mathcal{J}\widehat{\delta}(x + y) - \mathcal{J}\widehat{\delta}(y - z) - \mathcal{J}\widehat{\delta}(-y)\|$$
$$\leq \frac{2^{p+q+r} + 1}{3}\theta\|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in A$. By induction on $n \in \mathbb{N}_0$, we obtain

$$\|\mathcal{J}^n\widehat{\delta}(x + y - z) + \mathcal{J}^n\widehat{\delta}(x - y) + \mathcal{J}^n\widehat{\delta}(-y - z) + \mathcal{J}^n\widehat{\delta}(y)$$
$$- \mathcal{J}^n\widehat{\delta}(x - y - z) - \mathcal{J}^n\widehat{\delta}(x + y) - \mathcal{J}^n\widehat{\delta}(y - z) - \mathcal{J}^n\widehat{\delta}(-y)\|$$
$$\leq \theta\left(\frac{2^{p+q+r} + 1}{3}\right)^n \|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in A$. By letting $n \to \infty$, we have

$$\mathfrak{D}(x + y - z) + \mathfrak{D}(x - y) + \mathfrak{D}(-y - z) + \mathfrak{D}(y) = \mathfrak{D}(x - y - z) + \mathfrak{D}(x + y) + \mathfrak{D}(y - z) + \mathfrak{D}(-y)$$

for all $x, y, z \in A$. By using Lemma 2.1, the function $\mathfrak{D}$ satisfies the Drygas equation.

The only remaining case is when $1 < p + q + r < 2$.

Let $\widehat{\delta}_o$ and $\widehat{\delta}_e$ be the odd and even parts of $\widehat{\delta}$, respectively, i.e. $\widehat{\delta}_o = \frac{\widehat{\delta}(x) - \widehat{\delta}(-x)}{2}$, $\widehat{\delta}_e = \frac{\widehat{\delta}(x) + \widehat{\delta}(-x)}{2}$ for all $x \in A$ and $\widehat{\delta} = \widehat{\delta}_o + \widehat{\delta}_e$. It is evident that $\widehat{\delta}(0) = \widehat{\delta}_o(0) = \widehat{\delta}_e(0) = 0$. Thus, by (6), we get

$$\|\widehat{\delta}_o(x + y - z) + \widehat{\delta}_o(x - y) + \widehat{\delta}_o(-y - z) + \widehat{\delta}_o(y) - \widehat{\delta}_o(x - y - z) - \widehat{\delta}_o(x + y) - \widehat{\delta}_o(y - z) - \widehat{\delta}_o(-y)\|$$

$$\leq \frac{1}{2}(\|\widehat{\delta}(x + y - z) + \widehat{\delta}(x - y) + \widehat{\delta}(-y - z) + \widehat{\delta}(y) - \widehat{\delta}(x - y - z) - \widehat{\delta}(x + y) - \widehat{\delta}(y - z) - \widehat{\delta}(-y)\|$$

$$+ \|\widehat{\delta}(-x - y + z) + \widehat{\delta}(-x + y) + \widehat{\delta}(y + z) + \widehat{\delta}(-y) - \widehat{\delta}(-x + y + z) - \widehat{\delta}(-x - y) - \widehat{\delta}(-y + z) - \widehat{\delta}(y)\|)$$

$$\leq \theta \|x\|^p \|y\|^q \|z\|^r$$

and similarly

$$\|\widehat{\delta}_e(x + y - z) + \widehat{\delta}_e(x - y) + \widehat{\delta}_e(-y - z) + \widehat{\delta}_e(y) - \widehat{\delta}_e(x - y - z) - \widehat{\delta}_e(x + y) - \widehat{\delta}_e(y - z) - \widehat{\delta}_e(-y)\|$$

$$\leq \theta \|x\|^p \|y\|^q \|z\|^r$$

for all $x, y, z \in A$. So, $\widehat{\delta}_o$ and $\widehat{\delta}_e$ satisfy the inequality (6). It follows that

$$\|\widehat{\delta}_o(x + 2y) + \widehat{\delta}_o(x - 2y) - 2\widehat{\delta}_o(x) - \widehat{\delta}_o(2y) - \widehat{\delta}_o(-2y)\| \leq 2\theta \|x\|^p \|y\|^{q+r} \tag{12}$$

and

$$\|\widehat{\delta}_e(x + 2y) + \widehat{\delta}_e(x - 2y) - 2\widehat{\delta}_e(x) - \widehat{\delta}_e(2y) - \widehat{\delta}_e(-2y)\| \leq 2\theta \|x\|^p \|y\|^{q+r} \tag{13}$$

for all $x, y \in A$.

Replacing y by $\frac{x}{2}$ in (12), and applying the oddness of a function $\widehat{\delta}_o$ gives

$$\|\widehat{\delta}_o(2x) - 2\widehat{\delta}_o(x)\| \leq \frac{2\theta}{2^{q+r}} \|x\|^{p+q+r}$$

for all $x \in A$. Substitute $\frac{x}{2}$ in the place of x in the above inequality, we gain

$$\|\widehat{\delta}_o(x) - 2\widehat{\delta}_o\left(\frac{x}{2}\right)\| \leq \frac{2\theta}{2^{p+2q+2r}} \|x\|^{p+q+r}$$

for all $x \in A$. Define

$$\mathcal{J}g(x) = 2g\left(\frac{x}{2}\right), \qquad g \in B^A,$$

$$\Lambda \mathcal{F}(x) = 2\mathcal{F}\left(\frac{x}{2}\right), \qquad \mathcal{F} \in \mathbb{R}_+^A$$

and $\eta(x) = \frac{2\theta}{2^{p+2q+2r}} \|x\|^{p+q+r}$ for all $x \in A$. Also,

$$\Lambda \eta(x) = \frac{2}{2^{p+q+r}} \eta(x)$$

for all $x \in A$. Since $p + q + r > 1$, the series $\sum_{n=0}^{\infty} \Lambda^n \eta(x)$ is convergent for all $x \in A$ and

$$\eta^*(x) = \sum_{n=0}^{\infty} \Lambda^n \eta(x) = \frac{2\theta}{2^{p+2q+2r} - 2^{1+q+r}} \|x\|^{p+q+r}$$

for all $x \in A$. So, by Theorem 1.1, there exists $\mathfrak{D}_o : A \to B$ so that

$$\mathfrak{D}_o(x) = \lim_{n \to \infty} \mathcal{J}^n \widehat{\delta}_o(x), \qquad \mathfrak{D}_o(x) = 2\mathfrak{D}_o\left(\frac{x}{2}\right)$$

and

$$\|\widehat{\delta}_o(x) - \mathfrak{D}_o(x)\| \leq \frac{2\theta}{2^{p+2q+2r} - 2^{1+q+r}} \|x\|^{p+q+r}$$

for all $x \in A$. Further,

$$\|\mathcal{J}^n \widehat{\delta}_o(x+y-z) + \mathcal{J}^n \widehat{\delta}_o(x-y) + \mathcal{J}^n \widehat{\delta}_o(-y-z) + \mathcal{J}^n \widehat{\delta}_o(y)$$

$$-\mathcal{J}^n \widehat{\delta}_o(x-y-z) - \mathcal{J}^n \widehat{\delta}_o(x+y) - \mathcal{J}^n \widehat{\delta}_o(y-z) - \mathcal{J}^n \widehat{\delta}_o(-y)\|$$

$$\leq \theta \left(\frac{2}{2^{p+q+r}}\right)^n \|x\|^p \|y\|^q \|z\|^r$$

for all $n \in \mathbb{N}_0$ and $x, y, z \in A$. Therefore, $\mathfrak{D}_o$ satisfies the Drygas equation.

In the similar way, taking $y = \frac{x}{2}$ in (13), and applying the evenness of a function $\widehat{\delta}_e$ gives

$$\left\|\widehat{\delta}_e(x) - \frac{1}{4}\widehat{\delta}_e(2x)\right\| \leq \frac{\theta}{2^{1+q+r}} \|x\|^{p+q+r}$$

for all $x \in A$. We consider

$$\mathcal{J}g(x) = \frac{1}{4}g(2x), \qquad g \in B^A,$$

$$\Lambda \mathcal{F}(x) = \frac{1}{4}\mathcal{F}(2x), \qquad \mathcal{F} \in \mathbb{R}_+^A$$

and $\eta(x) = \frac{\theta}{2^{1+q+r}} \|x\|^{p+q+r}$ for all $x \in A$. Also,

$$\Lambda \eta(x) = \frac{2^{p+q+r}}{4} \eta(x)$$

for all $x \in A$. Since $p + q + r < 2$, the serie $\sum_{n=0}^{\infty} \Lambda^n \eta(x)$ is convergent for all $x \in A$ and

$$\eta^*(x) = \sum_{n=0}^{\infty} \Lambda^n \eta(x) = \frac{2\theta}{2^{2+q+r} - 2^{p+2q+2r}} \|x\|^{p+q+r}$$

for all $x \in A$. So, by Theorem 1.1, there exists $\mathfrak{D}_e : A \to B$ so that

$$\mathfrak{D}_e(x) = \lim_{n \to \infty} \mathcal{J}^n \widehat{\delta}_o(x), \qquad \mathfrak{D}_e(x) = \frac{1}{4}\mathfrak{D}_e(2x)$$

and

$$\|\widehat{\delta}_e(x) - \mathfrak{D}_e(x)\| \le \frac{2\theta}{2^{2+q+r} - 2^{p+2q+2r}} \|x\|^{p+q+r}$$

for all $x \in A$. It is easy to check that

$$\|\mathcal{J}^n \widehat{\delta}_e(x + y - z) + \mathcal{J}^n \widehat{\delta}_e(x - y) + \mathcal{J}^n \widehat{\delta}_e(-y - z) + \mathcal{J}^n \widehat{\delta}_e(y)$$

$$-\mathcal{J}^n \widehat{\delta}_e(x - y - z) - \mathcal{J}^n \widehat{\delta}_e(x + y) - \mathcal{J}^n \widehat{\delta}_e(y - z) - \mathcal{J}^n \widehat{\delta}_e(-y)\|$$

$$\le \theta \left(\frac{2^{p+q+r}}{4}\right)^n \|x\|^p \|y\|^q \|z\|^r$$

for all $n \in \mathbb{N}_0$ and $x, y, z \in A$. Therefore, $\mathfrak{D}_e$ satisfies the Drygas equation.

It follows that $\mathfrak{D} = \mathfrak{D}_o + \mathfrak{D}_o$ satisfies (2) and

$$\|\widehat{\delta}(x) - \mathfrak{D}(x)\| \le \left(\frac{2\theta}{2^{p+2q+2r} - 2^{1+q+r}} + \frac{2\theta}{2^{2+q+r} - 2^{p+2q+2r}}\right) \|x\|^{p+q+r}$$

or

$$\|\delta(x) - \mathfrak{D}(x) - \delta(0)\| \le \left(\frac{2\theta}{2^{p+2q+2r} - 2^{1+q+r}} + \frac{2\theta}{2^{2+q+r} - 2^{p+2q+2r}}\right) \|x\|^{p+q+r}$$

for all $x \in A$.

Finally, we show the uniqueness of $\mathfrak{D}$. We prove the case $p + q + r > 2$. Suppose that $\mathfrak{D}_1, \mathfrak{D}_2 : A \to B$ satisfy the Drygas equation on A and

$$\|\delta(x) - \mathfrak{D}_1(x) - \delta(0)\| \le \theta_1 \|x\|^p, \qquad \|\delta(x) - \mathfrak{D}_2(x) - \delta(0)\| \le \theta_2 \|x\|^p$$

for some $\theta_1, \theta_2 \ge 0$ and for all $x \in A$. Thus,

$$\|\mathfrak{D}_1(x) - \mathfrak{D}_2(x)\| \le (\theta_1 + \theta_2)\|x\|^p$$

for all $x \in A$. Hence,

$$\mathfrak{D}_1(x) = 3\mathfrak{D}_1\left(\frac{x}{2}\right) + \mathfrak{D}_1\left(-\frac{x}{2}\right), \qquad \mathfrak{D}_2(x) = 3\mathfrak{D}_2\left(\frac{x}{2}\right) + \mathfrak{D}_2\left(-\frac{x}{2}\right)$$

for all $x \in A$. Therefore,

$$\|\mathfrak{D}_1(x) - \mathfrak{D}_2(x)\| \le 3\left\|\mathfrak{D}_1\left(\frac{x}{2}\right) - \mathfrak{D}_2\left(\frac{x}{2}\right)\right\| + \left\|\mathfrak{D}_1\left(-\frac{x}{2}\right) - \mathfrak{D}_2\left(-\frac{x}{2}\right)\right\| \le \frac{4}{2^{p+q+r}}(\theta_1 + \theta_2)\|x\|^p$$

for all $x \in A$. By induction on $n \in \mathbb{N}_0$ we see that

$$\|\mathfrak{D}_1(x) - \mathfrak{D}_2(x)\| \le \left(\frac{4}{2^{p+q+r}}\right)^n (\theta_1 + \theta_2)\|x\|^p$$

which tends to 0 as $n \to \infty$ for all $x \in A$. This implies

$$\mathfrak{D}_1(x) = \mathfrak{D}_2(x)$$

for all $x \in A$. The proofs of the cases $0 < p + q + r < 1$ and $1 < p + q + r < 2$ runs as before. $\square$

Theorem 2.3 *Let A be a normed space and B be a Banach space. Suppose that a function $\delta : A \to B$ satisfying $\delta(0) = 0$ and*

$$\|\delta(x + y - z) + \delta(x - y) + \delta(-y - z) + \delta(y) - \delta(x - y - z) - \delta(x + y) - \delta(y - z) - \delta(-y)\| \leq \theta \qquad (14)$$

for some $\theta \geq 0$ and all $x, y, z \in A$. Then there exists a unique function $\mathfrak{D} : A \to B$ satisfying the Drygas equation on A such that

$$\|\delta(x) - \mathfrak{D}(x)\| \leq \frac{5}{3}\theta \leq 2\theta$$

for all $x \in A$.

Proof Putting $z = y$ in (14) yields

$$\|\delta(x) + \delta(x - y) + \delta(-2y) + \delta(y) - \delta(x - 2y) - \delta(x + y) - \delta(-y)\| \leq \theta \qquad (15)$$

for all $x, y \in A$. Replace y by $-y$ in (15) to have

$$\|\delta(x) + \delta(x + y) + \delta(2y) + \delta(-y) - \delta(x + 2y) - \delta(x - y) - \delta(y)\| \leq \theta \qquad (16)$$

for all $x, y \in A$.

It then follows from (15) and (16) that

$$\|\delta(x + 2y) + \delta(x - 2y) - 2\delta(x) - \delta(2y) - \delta(-2y)\| \leq 2\theta \qquad (17)$$

for all $x, y \in A$.

Replace (x, y) by $(2x, \frac{x}{2})$, next by (x, x), $(-x, -x)$ and $(x, \frac{x}{2})$ in (17), respectively, we obtain

$$\|\delta(3x) - 2\delta(2x) - \delta(-x)\| \leq 2\theta,$$
$$\|\delta(3x) + \delta(-x) - 2\delta(x) - \delta(2x) - \delta(-2x)\| \leq 2\theta,$$
$$\| - \delta(-3x) - \delta(x) + 2\delta(-x) + \delta(-2x) + \delta(2x)\| \leq 2\theta,$$
$$\|\delta(2x) - 3\delta(x) - \delta(-x)\| \leq 2\theta$$

for all $x \in A$. Hence, applying the triangle inequality and above the inequalities, we get

$$\|9\delta(x) - 2\delta(3x) + \delta(-3x)\| \leq 10\theta,$$

implying

$$\left\|\delta(x) - \frac{2}{9}\delta(3x) + \frac{1}{9}\delta(-3x)\right\| \leq \frac{10}{9}\theta,$$

for all $x \in A$. Let $\mathcal{J} : B^A \to B^A$ and $\eta : A \to \mathbb{R}_+$ be defined by

$$\mathcal{J}g(x) = \frac{2}{9}g(3x) - \frac{1}{9}g(-3x), \qquad g \in B^A$$

and

$$\eta(x) = \frac{10}{9}\theta$$

for all $x \in A$. So, it can be written

$$\|\mathcal{J}\delta(x) - \delta(x)\| \leq \eta(x)$$

for all $x \in A$. Consider the function $\Lambda : \mathbb{R}_+^A \to \mathbb{R}_+^A$ defined by

$$\Lambda\mathcal{F}(x) = \frac{2}{9}\mathcal{F}(3x) + \frac{1}{9}\mathcal{F}(-3x), \qquad \mathcal{F} \in \mathbb{R}_+^A$$

for all $x \in A$. Applying Theorem 1.1, we get a function $\mathfrak{D} : A \to B$ such that

$$\mathfrak{D}(x) = \lim_{n \to \infty} \mathcal{J}^n\delta(x), \qquad \mathfrak{D}(x) = \frac{2}{9}\mathfrak{D}(3x) - \frac{1}{9}\mathfrak{D}(-3x)$$

and

$$\|\delta(x) - \mathfrak{D}(x)\| \leq \frac{5}{3}\theta \leq 2\theta$$

for all $x \in A$. The remainder of the proof is in a similar way to that of Theorem 2.2. □

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