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Modification of the sector theorem of Kondo-Tanaka

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Abstract: Kondo-Tanaka proved that if a rotationally symmetric plane M_m is von Mangoldt or Cartan-Hadamard outside a compact set and has finite total curvature, then it has a sector with no pair of cut points. We show that the condition of finite total curvature can be removed.

Key words: Radial curvature, critical point, surface of revolution, finite topological type, finite total curvature, cut point, conjugate point

1. Introduction

Let (M,p) denote a complete, noncompact Riemannian manifold M with arbitrarily chosen basepoint p. Let (M_m, o) denote a rotationally symmetric plane M_m together with its origin o, where M_m equals $\mathbb{R}^2$ equipped with a smooth, complete Riemannian metric $g_m := dr^2 + m^2(r)d\theta^2$ with $m(0) = 0$ and $m'(0) = 1$. Given M_m, define a *sector of angular measure* δ, $V(\delta)$, as $V(\delta) := \{q \in M_m | 0 < \theta(q) < \delta\}$. It is assumed that always $0 < \delta \leq \pi$.

Global Riemannian geometry seeks to relate geometric data to topological data. It is often of particular interest if we can show that a certain set of traits imply that a noncompact manifold M is topologically finite, i.e. that it is homeomorphic to the interior of a compact set with boundary. The Toponogov comparison theorem was extended in [3] to open complete manifolds with radial sectional curvature bounded below by the curvature of a rotationally symmetric plane, M_m, with finite total curvature and a sector with no pair of cut points. Kondo-Tanaka used this extended version to prove the following important result:

Theorem 1.1 (Main Theorem of [3]) *Let (M,p) be a complete open Riemannian n-manifold whose radial curvature at basepoint p is bounded below by that of a noncompact rotationally symmetric plane M_m with finite total curvature and a sector with no pair of cut points. Then M is of finite topological type.*

It is natural to wonder what types of rotationally symmetric planes satisfy the condition of Theorem 1.1 requiring a sector with no pair of cut points. A von Mangoldt or Cartan-Hadamard plane contains $V(\pi)$ free of any pair of cut points; recall that in a *von Mangoldt plane*, G_m (the sectional curvature function) is nonincreasing in r; in a *Cartan-Hadamard plane*, $G_m \leq 0$ for all r. In [4], Kondo-Tanaka show that the cut-point-free sector requirement for M_m can be dropped if the total curvature is strictly less than 2π. But the requirement stands when the total curvature is equal to 2π.

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Apart from applications such as in Theorem 1.1, the behavior of geodesics and their cut points in rotationally symmetric planes is a substantive area of study in its own right; see for example [5], chapter 7. All things considered, the *Sector Theorem*, a result in [3], is of interest:

Theorem 1.2 (Sector Theorem) *Let M_m be a noncompact rotationally symmetric plane that is von Mangoldt or Cartan-Hadamard outside a compact set. If M_m admits finite total curvature, then there exists a sector that has no pair of cut points.*

In the above theorem, M_m being von Mangoldt or Cartan-Hadamard outside a compact set means respectively that G_m is nonincreasing or $G_m \leq 0$ on $[R, \infty)$ for some $R \in (0, \infty)$. Equivalently, we can say that M_m is von Mangoldt or Cartan-Hadamard outside $\overline{B_R(o)}$ to denote the above. In the **main result of this paper,** we remove the requirement of finite total curvature in Theorem 1.2:

Theorem 1.3 (Main result) *Let M_m be a noncompact rotationally symmetric plane that is von Mangoldt or Cartan-Hadamard outside a compact set. Then there exists a sector that has no pair of cut points.*

2. Notations, conventions, and definitions

Theorem 1.1 uses the notion of the radial sectional curvature of M being bounded below by that of M_m. We here define this notion:

Definition 2.1 *Let G be the sectional curvature function for M, and for any meridian $\mu(t)$ emanating from $o = \mu(0)$, define the* **radial sectional curvature function** *of M_m as $G_m \circ \mu : [0, \infty) \to \mathbb{R}$. We say that (M, p) has* **radial sectional curvature bounded below** *by that of (M_m, o) if, along every unit-speed minimal geodesic $\gamma : [0, a) \to M$ emanating from $p = \gamma(0)$, we have $G(\sigma_t) \geq G_m(\mu(t))$ for all $t \in [0, a)$ and all 2-dimensional subspaces σ_t spanned by $\gamma'(t)$ and an element of $T_{\gamma(t)}M$.*

Two theorems foundational to Theorem 1.1 are the Isotopy Lemma and the extended Toponogov comparison theorem. The Isotopy Lemma, given as Theorem 2.1 below, is a part of the critical point theory of distance functions by Grove-Shiohama [2]; recall that given (M, p), a point $q \in M$ is a *critical point of $d(\cdot, p)$* (the distance function to p) if, given any $v \in T_qM$, there exists a minimal geodesic γ emanating from q to p such that $\measuredangle(\dot{\gamma}(0), v) \leq \frac{\pi}{2}$:

Theorem 2.1 (Isotopy Lemma) *Given (M, p), suppose that for R_1, R_2 with $0 < R_1 < R_2 \leq \infty$, $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ has no critical point of $d(\cdot, p)$. Then $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ is homeomorphic to $\partial B_{R_1}(p) \times [R_1, R_2]$.*

The extended Toponogov comparison theorem is developed in [3]. It enables us to compare triangles in M and M_m and draw conclusions on corresponding angle inequalities. Its application requires the existence of a cut-point-free sector in M_m, and hence the requirement in Theorem 1.1. Theorem 1.1 works by showing that if its conditions are satisfied, then the critical points of $d(\cdot, p)$ are confined to $B_R(p)$, $R < \infty$.

All geodesics are parametrized by arclength. The term *segments* refers to minimizing geodesics. Let ∂_r, ∂_θ denote the vector fields dual to dr, $d\theta$ on $\mathbb{R}^2$. Given $q \neq o$, denote its polar coordinates by θ_q, r_q. Let μ_q, η_q denote the geodesics defined on $[0, \infty)$ that start at q in the direction of ∂_r, $-\partial_r$, respectively. Set

$\kappa_{\gamma(t)} := \angle(\dot{\gamma}(t), \partial_r)$. For all geodesic segments $\gamma : [0, \ell] \to M_m$, assume $r_{\gamma(\ell)} \geq r_{\gamma(0)}$. We write $\dot{r}$, $\dot{\theta}$, and $\dot{\gamma}$ for the derivatives of $r_{\gamma(t)}$, $\theta_{\gamma(t)}$, and $\gamma(t)$ by t, while m' denotes $\frac{dm}{dr}$, and proceed similarly for higher derivatives. The *total curvature* of M_m, $c(M_m)$, takes on the usual definition:

$$c(M_m) := \int_{M_m} G_m dM \tag{2.2}$$

We state below an important result (by Alexis Clairaut), which gives rise to the terms Clairaut constant and Clairaut relation, used in this paper:

Theorem 2.3 *Let γ be a geodesic in a rotationally symmetric plane M_m such that γ does not intersect the origin. Then there exists a constant c such that $m(r) \sin \kappa_{\gamma(s)} = c$ for all s.*

A useful consequence of the above result is that since $0 \leq \sin \kappa_{\gamma(s)} \leq 1$ for all s, $0 \leq c \leq m(r_{\gamma(s)})$, where $c = m(r_{\gamma(s)})$ only at points where γ is tangent to a parallel and $c = 0$ when γ is tangent to a meridian.

Definition 2.2 *In the setting of Theorem 2.3, the constant c is called the* **Clairaut constant** *of geodesic γ; the equation $m(r) \sin \kappa_{\gamma(s)} = c$ is called the* **Clairaut relation.**

3. Proof of Theorem 1.3

Remark 3.1 We acknowledge that our proof is a modification of the proof in [3]; much of the relevant original content in [3] remains the same, so we hereby give due credit and respect to Kondo-Tanaka.

Broadly speaking, Lemmas 3.1, 3.2, 3.4, 3.5, and 3.6 are technical lemmas used to prove Lemma 3.8. Lemmas 3.7, 3.8, and 3.9 are used to prove Lemma 3.10. The culminating proof at the end of this section uses Lemmas 3.10, 3.11, and 3.12.

Lemma 3.1 below states a useful relationship between Clairaut constants and their corresponding geodesic segments in an important limiting process that will be used later.

Lemma 3.1 (Lemma 3.1, [3]) *Given M_m, let $V_i := V(\frac{1}{i})$ for each $i = 1, 2, \ldots$ Assume that there exist a constant $r_0 > 0$ and a sequence $\{\sigma_i : [0, \ell_i] \to V_i\}$ of geodesic segments such that $\sigma_i([0, \ell_i]) \bigcap \overline{B_{r_0}(o)} \neq \emptyset$ for each i and that $\liminf_{i\to\infty} r(\sigma_i(\ell_i)) > r_0$. Then, $\lim_{i\to\infty} c_i = 0$ holds, where c_i denotes the Clairaut constant of σ_i.*

Lemmas 3.2 and 3.4 give an important condition needed for a geodesic in M_m to have a conjugate point.

Lemma 3.2 (Propositions 7.2.1, 7.2.2, [5]) *Given $q \in M_m$, let $\gamma : [0, s] \to M_m$, $\gamma(0) = q$ be a geodesic not tangent to the parallel or meridian through q. If $\dot{r}_\gamma$ is nonzero on $[0, s)$, then there exists a Jacobi field $X(t)$ along γ that can be expressed as*

$$X(t) = \operatorname{sign}\left(\frac{\pi}{2} - \kappa_\gamma\right) \dot{r}(t) \int_{d(o,q)}^{r(t)} \frac{m(r)}{\sqrt{m^2(r) - c^2}^3} dr \left\{ -c \frac{\partial}{\partial r_{\gamma(t)}} + \dot{r}(t) \frac{\partial}{\partial \theta_{\gamma(t)}} \right\} \tag{3.3}$$

on $[0, s)$, where c is the Clairaut constant of γ.

Lemma 3.4 *Given $q \in M_m$, let $\gamma : [0,s] \to M_m$, $\gamma(0) = q$ be a geodesic that is not tangent to the parallel or meridian through q. If $\dot{r}_\gamma$ is nonzero on $[0,s)$, then there exists no conjugate point of q along $\gamma|_{[0,s)}$.*

Proof Each additive term in the expression for $X(t)$ in Lemma 3.2 carries $\dot{r}(t)$. Hence, $\dot{r}(t)$ nonzero on $[0,s)$ implies that the Jacobi field $X(t)$ is nonzero on $[0,s)$. □

Lemma 3.5 states conditions in which a geodesic segment cannot be tangent to a parallel in M_m from below.

Lemma 3.5 *Let M_m be such that $\liminf_{r\to\infty} m(r) > 0$. Let $\{\sigma_i : [0,\ell_i] \to M_m\}$ be a sequence of minimal geodesics such that $\ell_i \to \infty$, $c_i \neq 0$, and $c_i \to 0$. Then there exists $L > 0$ such that for all $i \geq L$, there does not exist any value t at which both $\dot{r}_{\sigma_i(t)} = 0$ and $\ddot{r}_{\sigma(t)} < 0$ hold.*

Proof By contradiction; suppose that for any $L > 0$, there exists $i \geq L$ such that $\dot{r}_{\sigma_i(t_i)} = 0$ and $\ddot{r}_{\sigma_i(t_i)} < 0$ for some t_i. Choose such a subsequence and denote it $\{\sigma_i\}$. By reflectional symmetry and uniqueness of geodesics, r_{σ_i} attains its absolute maximum at t_i. Since $c_i = m(r_{\sigma_i(t_i)})$, $c_i \to 0$, we have $m(r_{\sigma_i(t_i)}) \to 0$. Since $\liminf_{r\to\infty} m(r) > 0$, $m(r_{\sigma(t_i)}) \to 0$ implies $r_{\sigma_i(t_i)} \to 0$. But this is impossible, since $\ell_i \to \infty$ and σ_i is a minimal geodesic.

□

Remark 3.2 Given any $q \in M$, M a complete Riemannian manifold, we define the *segment domain* of q as

$$\{v \in T_qM \;\mid \exp_q tv : [0,1] \to M \text{ is a minimal geodesic}\}$$

It is well known that the segment domain of any $q \in M$ is star-shaped and closed. The *interior* of the segment domain of q, denoted $I(q)$, is likewise defined as

$$\{v \in T_qM \;\mid \exp_q tv : [0,1) \to M \text{ is a minimal geodesic}\}$$

Note that $\exp_q$ is one-to-one on $I(q)$, so if x is in the image of $I(q)$, denoted $I(q)^*$, there exists a unique minimizing geodesic γ connecting q to x, and there exists $\epsilon > 0$ such that γ minimizes on $(0, d(q,x) + \epsilon)$. Hence, if x is conjugate to q, x cannot be in $I(q)^*$.

Lemma 3.6 *Let $\{\sigma_i : [0,\ell_i] \to M_m\}$ be a sequence of minimal geodesics converging to $\sigma : [0,\ell] \to M_m$, where σ is a subarc of a meridian. For all i large enough, $\sigma_i(\ell_i)$ is in $I(\sigma_i(0))^*$ and $\sigma_i(0)$ is in $I(\sigma_i(\ell_i))^*$.*

Proof Since any subarc of a meridian is distance-minimizing, $\sigma(\ell)$ is in $I(\sigma(0))^*$. Hence for i large enough, $\sigma_i(\ell_i)$ is also in $I(\sigma(0))^*$. It follows that $\sigma(0)$ is in $I(\sigma_i(\ell_i))^*$, since the above implies that $\sigma(0)$ is joined to $\sigma_i(\ell_i)$ by a unique minimal geodesic and $\sigma(0)$ cannot be conjugate to $\sigma_i(\ell_i)$. So for i large enough, $\sigma_i(0)$ is in $I(\sigma_i(\ell_i))^*$. It must also follow that $\sigma_i(\ell_i)$ is in $I(\sigma_i(0))^*$. □

Below we give the original version of Lemma 3.3 in [3], followed by our modified version. Both versions are key to our proof of Theorem 1.3.

Lemma 3.7 (Lemma 3.3, [3]) *Let M_m have finite total curvature. For each $r > 0$, there exists a number $\delta(r) \in (0,\pi)$ such that $\sigma([0,\ell]) \cap \overline{B_r(o)} = \emptyset$ holds for any minimal geodesic segment $\sigma : [0,\ell] \to V(\delta(r)) \subset M$, along which $\sigma(0)$ is conjugate to $\sigma(\ell)$.*

Lemma 3.8 *Let M_m be such that $\liminf_{r\to\infty} m(r) > 0$. For each $r > 0$, there exists a number $\delta(r) \in (0, \pi)$ such that $\sigma([0,\ell]) \cap \overline{B_r(o)} = \emptyset$ holds for any minimal geodesic segment $\sigma : [0,\ell] \to V(\delta(r)) \subset M$, along which $\sigma(0)$ is conjugate to $\sigma(\ell)$.*

Proof By contradiction. To establish the existence of $\delta(r) \in (0, \pi)$, all we need to do is show that there exists $\delta(r) > 0$, since we have $|\theta(\sigma(0)) - \theta(\sigma(\ell))| < \pi$ for any minimal geodesic segment $\sigma : [0,\ell] \to M \setminus \{o\}$. Put $V_i := V(\frac{1}{i})$ for each i. Assume that there exists a constant $r_0 > 0$ and a sequence of minimal geodesic segments $\{\sigma_i : [0,\ell_i] \to V_i\}$, with $\sigma_i(0)$ conjugate to $\sigma_i(\ell_i)$ along σ_i, such that $\sigma_i([0,\ell_i]) \cap \overline{B_{r_0}(o)} \neq \emptyset$ for each i.

We want to establish that the sequence of Clairaut constants, $\{c_i\}$, converges to 0 as $i \to \infty$. We do this by showing that $\lim_{i\to\infty} \ell_i = \infty$; indeed, this implies $\liminf_{i\to\infty} r_{\sigma_i(l_i)} > r_0$, whereupon by Lemma 3.1 $\{c_i\} \to 0$.

Suppose $\lim_{i\to\infty} \ell_i < \infty$ or does not exist. Then there exists $M < \infty$ such that given any N, there exists $i \geq N$ such that $\ell_i \leq M$. Then we have a subsequence of $\{\sigma_i\}$ such that the endpoints $\{\sigma_i(0)\}$, $\{\sigma_i(\ell_i)\}$ are confined to a compact set. Let $\{\sigma_i\}$ denote this subsequence. Since each σ_i is a minimal geodesic, $\{\sigma_i\}$ must lie in a bounded set. By the Arzela-Ascoli theorem, there exists a geodesic σ to which some subsequence $\{\sigma_{i_j}\}$ converges, and by construction σ must be a subarc of a meridian. Let $\sigma(0)$ be the point to which $\{\sigma_{i_j}(0)\}$ converges and let $\sigma(\ell)$ be the point to which $\{\sigma_{i_j}(\ell_{i_j})\}$ converges. For j large enough, $\sigma_{i_j}(0)$ is in $I(\sigma_{i_j}(\ell_{i_j}))^*$ and $\sigma_{i_j}(\ell_{i_j})$ is in $I(\sigma_{i_j}(0))^*$ by Lemma 3.6. Remark 3.2 implies that $\sigma_{i_j}(0)$ cannot be conjugate to $\sigma_{i_j}(\ell_{i_j})$, a contradiction. Hence we establish that $\liminf_{i\to\infty} r_{\sigma_i(\ell_i)} > r_0$.

Since $\sigma_i(0)$ and $\sigma_i(\ell_i)$ are conjugate, there exists a positive parameter value a_i at which $\dot{r}_{\sigma_i} = 0$ by Lemma 3.4. From our work above, we have $c_i \to 0$ and $\ell_i \to \infty$, and by assumption $\liminf_{r\to\infty} m(r) > 0$, so by Lemma 3.5, there exists J such that for all $i > J$, we cannot have $\ddot{r}_{\sigma_i}(a_i) < 0$. From this point on, assume $i > J$ always. Since σ_i is tangent to a parallel from above, $r_{\sigma_i(a_i)}$ is the absolute minimum of r_{σ_i}, implying $r_{\sigma_i(a_i)} \in B_{r_0}(o)$.

Let $u_i \in [a_i, \ell_i]$ be a parameter value of σ_i such that $r_{\sigma_i(u_i)} = r_0$. Set $\triangle_i :=$ the triangle $o\sigma_i(a_i)\sigma_i(u_i)$. This triangle lies in $\overline{B_{r_0}(o)} \bigcap V_i$. The angle at $\sigma_i(a_i)$ equals $\frac{\pi}{2}$ by construction. The angle at $o < \frac{1}{i}$, so it tends to 0 as $i \to \infty$. This implies that the area of $\triangle_i$ tends to 0 as $i \to \infty$.

Now consider the angle at $\sigma(u_i)$. On the one hand, since $c_i \to 0$, the angle at $\sigma(u_i)$ must go to 0. On the other hand, the curvature function $G_m(r)$ attains its maximum and minimum on $[0, r_0]$, so $\int_{\triangle_i} G_m \to 0$ as $i \to \infty$. The Gauss-Bonnet theorem gives { sum of the interior angles } $= \pi + \int_{\triangle_i} G_m$, so we have { sum of the interior angles } $\to \pi$ as $i \to \infty$. This means that the angle at $\sigma_i(u_i)$ must approach $\frac{\pi}{2}$ as $i \to \infty$, a contradiction.

$\square$

Lemma 3.9 *Suppose M_m is von Mangoldt or Cartan-Hadamard outside a compact set. If $\liminf_{r\to\infty} m(r) = 0$, then M_m has finite total curvature.*

Proof We prove our claim by showing that $\lim_{r\to\infty} m'(r)$ exists and is finite. Let $R > 0$ be such that M_m is von Mangoldt or Cartan-Hadamard on $M_m \setminus \overline{B_R(o)}$. There exists $r_0 > R$ at which $m' < 0$, for if $m'(r) \geq 0$ for all $r > R$, then $\liminf_{r\to\infty} m(r) > 0$. Because $m(r) > 0$ on $r > 0$, we cannot have $m'(r) \leq m'(r_0)$ on

$[r_0, \infty)$. Hence there exists $r_1 > r_0$ such that $m'(r_1) < 0$ and $m''(r_1) > 0$. Also $G_m(r_1) < 0$. Since M_m is von Mangoldt or Cartan-Hadamard on (R, ∞), $G_m(r) \leq 0$ on $[r_1, \infty)$, implying $m''(r) \geq 0$ on $[r_1, \infty)$. We claim $m' < 0$ on $[r_1, \infty)$. Indeed, if for some $r \geq r_1$ $m' \geq 0$, then $m''[r_1, \infty) \geq 0$ implies $m' \geq 0$ for all $r \geq r_1$, implying $\liminf_{r\to\infty} m(r) > 0$. Since m' is an increasing function on $[r_1, \infty)$ that is bounded above by 0, it must converge to a finite number.

□

Lemma 3.10 *Let M_m be von Mangoldt or Cartan-Hadamard outside a compact set. Then for each $r > 0$, there exists a constant number $\delta(r) \in (0, \pi)$ such that $\sigma([0, \ell]) \cap \overline{B_r(o)} = \emptyset$ holds for any minimal geodesic segment $\sigma : [0, \ell] \to V(\delta(r)) \subset M$, along which $\sigma(0)$ is conjugate to $\sigma(\ell)$.*

Proof Either $\liminf_{r\to\infty} m(r) > 0$ or $\liminf_{r\to\infty} m(r) = 0$. If $\liminf_{r\to\infty} m(r) > 0$, then the claim holds by Lemma 3.8. If $\liminf_{r\to\infty} m(r) = 0$, then Lemma 3.9 applies, so M_m has finite total curvature. Lemma 3.7 then implies the claim. □

Lemma 3.11 *If $q \in V(\delta)$ has a cut point in $V(\delta)$, then q must also have a conjugate point in $V(\delta)$.*

Proof Suppose $q \in V(\delta)$ has a cut point $x \in V(\delta)$. If x is conjugate to q, we are done, so suppose not. Then let α, β be minimal geodesics connecting q to x and bounding a region D; since α, β cannot be tangent to any meridian, $D \subset V(\delta)$. The boundary of D only meets C_q, the set of cut points of q, at x because α, β are minimal. Since C_q is a tree by Theorem 4.2.1 in [5], the interior of D contains an endpoint of C_q that is conjugate to q. □

Lemma 3.12 *Let M_m be von Mangoldt or Cartan-Hadamard outside $\overline{B_{R_1}(o)}$. Then for any $\delta \leq \pi$, there does not exist a geodesic $\sigma : [0, \ell] \to V(\delta) \setminus \overline{B_{R_1}(o)}$ containing conjugate points.*

Proof (This proof is modeled on the proofs of Theorem 3.4 in [3] and Theorem 7.3.1 in [5]). Let γ_x be the minimal geodesic joining q to x. Suppose $M_m \setminus \overline{B_{R_1}(o)}$ is Cartan-Hadamard. Then $G_m \leq 0$ along γ_x, implying that q cannot be conjugate to x along γ_x.

Suppose $M_m \setminus \overline{B_{R_1}(o)}$ is von Mangoldt. By Corollary 4.2.1 in [5] and Lemmas 2.5.19 and 2.5.22 in [1], we can find a normal cut point y in C_q (see Definition 2.5.12 in [1]) arbitrarily close to x such that $d(q, x) < d(q, y)$ and $\theta_x < \theta_y < \pi$. By Remark 2.5.21 in [1], there exists a minimal geodesic β_y connecting q to y such that

$$\measuredangle(\dot{\beta}_y(0), \dot{\eta}_q(0)) < \measuredangle(\dot{\gamma}_x(0), \dot{\eta}_q(0)), \tag{3.13}$$

and since y can be made arbitrarily close to x, we can ensure that β_y does not intersect $\overline{B_{R_1}(o)}$.

We now show that

$$\ell(\gamma_x) < \ell(\beta_y) \quad \text{and} \quad r_{\gamma_x(s)} > r_{\beta_y(s)} \tag{3.14}$$

for all $s \in (0, \ell(\gamma_x))$. For each $s \in (0, \ell(\gamma_x))$, since $\theta_y > \theta_x$, there exists a unique value $t(s)$ of β_y giving us

$$\theta_{\gamma(s)} = \theta_{\beta_y(t(s))}. \tag{3.15}$$

Since γ_x, β_y cannot intersect in their interiors we have $r_{\beta_y(t(s))} < r_{\gamma_x(s)}$. Hence for any given s, the set

$$S_s := \{t \in (0, \ell(\beta_y)) \ \mid \ r_{\beta_y(t)} < r_{\gamma_x(s)}\} \tag{3.16}$$

is nonempty. Now fix $s_0 \in (0, \ell(\gamma_x))$. Let (a,b) be the connected component of S_{s_0} containing $t(s_0)$. If we show that $s_0 \in (a,b)$, then we will have $r_{\gamma_x(s_0)} > r_{\beta_y(s_0)}$. If $(0,\ell(\gamma_x)) \subseteq (a,b)$ then $s_0 \in (a,b)$ and there is nothing to prove, so we can assume $a > 0$ or $b < \ell(\gamma_x)$. We have

$$r_{\gamma_x(s_0)} = r_{\beta_y(a)} = r_{\beta_y(b)}, \quad 0 \le \theta_{\beta_y(a)} < \theta_{\gamma_x(s_0)} = \theta_{\beta_y(t(s_0))} < \theta_{\beta_y(b)} < \pi \tag{3.17}$$

so the conditions for Lemma 7.3.2 in [5] are satisfied. It follows that

$$a = d(q, \beta_y(a)) < s_0 = d(q, \gamma_x(s_0)) < d(q, \beta_y(b)) = b, \tag{3.18}$$

implying $s_0 \in (a,b)$ and therefore $r_{\beta_y(s_0)} < r_{\gamma_x(s_0)}$. Since s_0 was arbitrary and $M_m \setminus \overline{B_{R_1}(o)}$ is von Mangoldt, we have $G_m(r_{\gamma_x(s)}) \le G_m(r_{\beta_y(s)})$ for all $s \in [0, \ell(\gamma_x)]$. Recalling that q is conjugate to x along γ_x and applying the Sturm Comparison Theorem, we have that q is conjugate to $\beta_y(t)$ along β_y for some $t \in (0, \ell(\gamma_x)]$. But this is impossible, since β_y minimizes the distance from q to y and $\ell(\beta_y) > \ell(\gamma_x)$. Hence q cannot be conjugate to x along γ_x, and this completes our proof.

$\square$

Proof [Proof of Theorem 1.3] Let M_m be von Mangoldt or Cartan-Hadamard outside $\overline{B_{R_0}(o)}$ for some $R_0 > 0$. Fix any $R_1 > R_0$, and in the setting of Lemma 3.10, let $\delta(R_1) \in (0, \pi)$ be the number such that if $\sigma : [0, \ell] \to V(\delta(R_1))$ is a minimal geodesic along which $\sigma(0)$ is conjugate to $\sigma(\ell)$, then

$$\sigma[0, \ell] \cap \overline{B_{R_1}(o)} = \emptyset. \tag{3.19}$$

Proceeding by contradiction, if $q \in V(\delta(R_1))$ has a cut point in $V(\delta(R_1))$, then by Lemma 3.11 there exists a point $x \in V(\delta(R_1))$ and a geodesic γ_x such that q is conjugate to x along γ_x. By Lemma 3.10, γ_x does not intersect $\overline{B_{R_1}(o)}$. But if γ_x lies entirely in $M_m \setminus \overline{B_{R_1}(o)}$, then by Lemma 3.12, q cannot be conjugate to any point along γ_x. $\square$

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