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On positive periodic solutions to third-order integro-differential equations with distributed delays

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Abstract: In this paper, we investigate the existence of positive periodic solutions of a third-order nonlinear integro-differential equation with distributed delays, by using the Green function and the Krasnosel'skii fixed point theorem in cones of Banach spaces, providing new results on this field. Three examples are analyzed to illustrate the effectiveness of the abstract results.

Key words: Fixed point theorem, positive periodic solutions, third-order delay differential equations, distributed delays

1. Introduction

Delay differential equations form an important branch of functional differential equations that take into account the dependence on their history, which makes it possible to predict the future evolution of the system in a more reliable and efficient way. Such processes take place in the theories of optimal control, population dynamics, economics, mathematical ecological models, and other biotechnological systems, but they are characterized by specific properties which make their study difficult in both concepts and techniques. Note that many results concerning the theory of delay functional differential equations can be found in the monographs by Hale and Meyer [16], Hale and Lunel [17], Kuang [19]amongst others.

Third-order differential equations with and without delay have been studied by many authors, since they describe several models derived from natural phenomena, such as wave propagation in thermally relaxing viscous fluids or flexible space structures with internal damping, for example, a thin uniform rectangular panel and a spaceship with flexible attachments, and many others (see, e.g., [5]- [7], [12], [14], [15], [26], [27]). This means that it is interesting and meaningful to study the properties of the solutions of the differential equations of the third order with and without delay.

Recently, intensive scientific work has been carried out in various dynamical aspects of third-order delay differential equations, functional delay differential equations, and many results have been reported in the literature. For instance, uniqueness of periodic solutions and some other fundamental properties of solutions

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of certain delay differential equations have been discussed in ([1]- [4], [10], [11], [18], [20]- [25], [29], [32], [33]). Some classical tools have been used to study delay differential equation in the literature, including the fixed point index theorem [25], Krasnoselskii's fixed point theorem ([2], [3]), the fixed point theorem in cones ([23], [24], [29], [33]), fixed point theorem of Leray-Schauder type [11], Mawhin's continuous theorem [4], and the continuation theorem of coincidence degree theory [1].

As far as we know, the existence of positive periodic solutions for third-order integro-differential equations with distributed delay that are considered in the present paper have not been investigated yet. For this reason, in this paper, we make a first attempt to fill this gap and obtain new sufficient conditions for the existence of one positive periodic solution thanks to the use of a fixed point theorem on cones. We are particularly motivated and inspired by the papers [2], [22], [23], [24], [25], [33] and the references therein.

In [28], the following third-order nonlinear delay differential equation with periodic coefficients is considered:

$$y'''(t)+p(t)y''(t)+q(t)y'(t)+r(t)y(t)=f(t,y(t)). \tag{1.1}$$

By applying a fixed point theorem index, the authors derived some verifiable sufficient conditions for the existence of a positive periodic solution to (1.1).

Very recently, Ardjouni and Djoudi [2] considered the following third-order nonlinear delay differential equation with periodic coefficients:

$$y'''(t)+p(t)y''(t)+q(t)y'(t)+r(t)y(t)=f(t,y(t-\tau(t)))+\frac{d}{dt}g(t,y(t-\tau(t))). \tag{1.2}$$

Thanks to the Green function and the Krasnoselskii fixed point theorem, they obtained a set of easily verifiable sufficient conditions for the existence of positive periodic solutions to equation (1.2).

In 2006, Li and Wang [22] studied the existence of positive periodic solutions of the following two kinds of nonlinear neutral differential equations

$$\frac{d}{dt}\left(y(t)-cy(t-\tau(t))\right)=-a(t)y(t)+g(t,y(t-\tau(t)), \tag{1.3}$$

and

$$\frac{d}{dt}\left(y(t)-c\int_{-\infty}^{0}K(r)\,y(t+r)\,dr\right)=-a(t)y(t)+b(t)\int_{-\infty}^{0}K(r)\,g(t,y(t+r))\,dr. \tag{1.4}$$

By using the theory of fixed point index in cones, sufficient conditions are presented for the existence of positive periodic solutions of two kinds of neutral differential equations with periodic coefficients.

In [33], the authors analyzed the following impulsive equations

$$\begin{array}{c} y'(t)=-a(t)y(t)+\int_{-\infty}^{0}K(r)\,g(t,y(t+r))\,dr, t\neq t_k,\\ y(t_k^+)=y(t_k^-)+I_k(y(t_k)),\ t=t_k, k\in\mathbb{Z}, \end{array} \tag{1.5}$$

$$\begin{array}{c} y'(t)=a(t)y(t)-\int_{-\infty}^{0}K(r)\,g(t,y(t+r))\,dr, t\neq t_k,\\ y(t_k^+)=y(t_k^-)+I_k(y(t_k)),\ t=t_k, k\in\mathbb{Z}, \end{array} \tag{1.6}$$

where $y(t_k^+)$ and $y(t_k^-)$ represent the right and the left limit of $y(t_k)$, $a\in C(\mathbb{R},\mathbb{R}^+)$, $g\in C(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+)$, $I_k\in C(\mathbb{R}^+,\mathbb{R}^+)$, a and $g(t,y)$ are $\omega-$periodic functions, where ω is a positive constant. Moreover, $K\in C(\mathbb{R}^-,\mathbb{R}^+)$

with $\int_{-\infty}^{0} K(r)\,dr = 1$. There exists an integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $I_{(k+q)}(y(t_{k+q})) = I_k(y(t_k))$, $k \in \mathbb{Z}$, where $0 < t_1 < t_2 < ... < t_q < \omega$.

By using the theory of fixed point index in cones, existence theory for single and multiple positive periodic solutions to a kind of nonautonomous Volterra integro-differential equations with impulse effects $(1.5), (1.6)$ has been investigated.

Motivated by the above statements, in this paper we study a third-order nonlinear integro-differential equations with distributed delays of the following form, by using a fixed point on a cone:

$$\begin{aligned} y'''(t) + p(t)y''(t) + q(t)y'(t) + r(t)y(t) &\\ = \frac{d}{dt}\int_{-\infty}^{0} K(r)\,c(t)\,y(t+r)\,dr &\\ +d(t)\int_{-\infty}^{0} K(r)\,f(t, y(t+r))\,dr, & \end{aligned} \tag{1.7}$$

where $d \in C(\mathbb{R}, \mathbb{R}^+ \setminus \{0\}), c, p, q, r \in C(\mathbb{R}, \mathbb{R}^+)$. The function $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Moreover, K is a continuous and integrable function on $]-\infty, 0]$ with $\int_{-\infty}^{0} K(r)\,dr = 1$.

In order to prove our main results in the next section, we need the following definition and theorem.

Definition 1.1. (see [30]) *Let X be a Banach space and let K be a closed, nonempty subset of X. K is said to be a cone if*

i) $\alpha x + \beta y \in K$ for all $x, y \in K$ and all $\alpha, \beta \geq 0$;

ii) $y, -y \in K$ imply $y = 0$.

We now state the Krasnosel'skii fixed point theorem.

Theorem 1.1. (see [14], Theorem 2.3.3 on p. 93). *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\Phi : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to K,$$

be a completely continuous operator such that either

a) $\|\Phi y\| \leq \|y\|$ for $y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \geq \|y\|$ for $y \in K \cap \partial\Omega_2$; or

b) $\|\Phi y\| \geq \|y\|$ for $y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \leq \|y\|$ for $y \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

This paper is devoted to studying the existence of positive periodic solutions of Equation (1.7) by using the Krasnosel'skii fixed point theorem and some mathematical analysis techniques. Equation (1.7) is a nonneutral third-order nonlinear differential equation, including the existing classical third-order differential equations, as well as the equations considered in ([27, 28]). Therefore, the results of this paper are more general and better applicable. The research method of this paper is different from the existing research methods (see, e.g., [2, 25, 28]). For more results about third-order differential equation with and without delays, see, e.g, ([1, 4, 8, 10, 25–27, 31]) and cited references.

The organization of this paper is as follows. In Section 2, we present the inversion of equation and state some preliminary results needed in later sections. We also describe the Green function of Eq. (1.7), which plays an important role in this paper. In Section 3, we establish our main results ensuring the existence of positive periodic solutions by applying the fixed point theorem in cones established previously. In Section 4, three examples are exhibited to illustrate that our results are feasible and more general than previous ones in the literature.

2. Green's function and periodicity

In this section, we state and define the Green function for periodic solutions of third-order nonlinear integro-differential equations with distributed delays (1.7).

Let

$$C_\omega = \left\{ y \in C\left(\mathbb{R}, \mathbb{R}\right),\ y(t+\omega) = y(t) \text{ for } t \in \mathbb{R} \right\},$$

with the norm

$$\|y\| = \max_{t\in[0,\omega]} |y(t)|.$$

It is easy to verify that $(C_\omega, \|.\|)$ is a Banach space.

In this paper, we give the assumptions as follows that will be used in the main results.

(A1) There exist differentiable ω–periodic functions $a_1, a_2 \in C\left(\mathbb{R}, \mathbb{R}^+\right)$ and a positive real constant ρ such that

$$\begin{cases} a_1\left(t\right) + \rho = p\left(t\right), \\ a_1'\left(t\right) + a_2\left(t\right) + \rho a_1\left(t\right) = q\left(t\right), \\ a_2'\left(t\right) + \rho a_2\left(t\right) = r\left(t\right). \end{cases}$$

(A2) $p, q, r \in C\left(\mathbb{R}, \mathbb{R}^+\right)$ are ω–periodic functions and

$$\int_0^\omega q\left(u\right) du > 0, \int_0^\omega p\left(u\right) du > \rho\omega.$$

(A3) The function $f \in C\left(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+\right)$ is a $\omega-$periodic in t, and $c \in C\left(\mathbb{R}, \mathbb{R}^+\right)$, $d \in C\left(\mathbb{R}, \mathbb{R}^+ \setminus \{0\}\right)$ are $\omega-$periodic. For all $y \in \mathbb{R}^+$,

$$f\left(t, y\right) \geq \rho \frac{c\left(t\right)}{d\left(t\right)} y, \forall t \in \left[0, \omega\right].$$

We consider

$$y'''(t) + p(t)y''\left(t\right) + q(t)y'\left(t\right) + r(t)y\left(t\right) = h\left(t\right), \tag{2.1}$$

where $h \in C_\omega$. Obviously, by condition (A1), Eq. (2.1) can be transformed into

$$\begin{cases} z'\left(t\right) + \rho z\left(t\right) = h\left(t\right), \\ y''\left(t\right) + a_1(t)y'\left(t\right) + a_2(t)y\left(t\right) = z\left(t\right). \end{cases}$$

Lemma 2.1 (see [3]). If $z, h \in C_\omega$, *then* z *is a solution of equation*

$$z'\left(t\right) + \rho z\left(t\right) = h\left(t\right),$$

if and only if

$$z(t) = \int_t^{t+\omega} G_1(t,s)\, h(s)\, ds, \tag{2.2}$$

where

$$G_1(t,s) = \frac{\exp(\rho(s-t))}{\exp(\rho\omega) - 1}. \tag{2.3}$$

Corollary 2.1 (see [21]). *Green function G_1 satisfies the following properties*

$$\begin{aligned}
G_1(t+\omega, s+\omega) &= G_1(t,s), G_1(t,t+\omega) = G_1(t,t)\exp(\rho\omega), \\
G_1(t+\omega, s) &= G_1(t,s)\exp(-\rho\omega),\ G_1(t,s+\omega) = G_1(t,s)\exp(\rho\omega), \\
\frac{\partial}{\partial t} G_1(t,s) &= -\rho G_1(t,s), \\
\frac{\partial}{\partial s} G_1(t,s) &= \rho G_1(t,s),
\end{aligned}$$

and

$$m_1 \le G_1(t,s) \le M_1, \tag{2.4}$$

where

$$M_1 = \frac{1}{\exp(\rho\omega) - 1},\ m_1 = \frac{\exp(\rho\omega)}{\exp(\rho\omega) - 1}. \tag{2.5}$$

Lemma 2.2 (see [32]). *Suppose that $(A1)$ and $(A2)$ hold and*

$$\frac{\overline{R}_1 \left[\exp\left(\int_0^\omega a_1(u)\, du\right) - 1\right]}{Q_1 \omega} \ge 1, \tag{2.6}$$

where

$$\overline{R}_1 = \max_{t \in [0,\omega]} \left| \int_t^{t+\omega} \frac{\exp\left(\int_0^\omega a_1(u)\, du\right)}{\exp\left(\int_0^\omega a_1(u)\, du\right) - 1} a_2(s)\, ds \right|,$$

$$Q_1 = \left(1 + \exp\left(\int_0^\omega a_1(u)\, du\right)\right)^2 \overline{R}_1^2.$$

Then there are continuous $\omega-$ periodic functions a and b such that

$$b(t) > 0, \qquad \int_0^\omega a(u)\, du > 0,$$

and

$$a(t) + b(t) = a_1(t),\ b'(t) + a(t)\, b(t) = a_2(t),\ \textit{for } t \in \mathbb{R}.$$

Lemma 2.3 (see [21]). *Suppose the conditions of Lemma 2.2 hold and $y \in C_\omega$. Then, the equation*

$$y''(t) + a_1(t)\, y'(t) + a_1(t)\, y(t) = h(t),$$

possesses an $\omega-$ periodic solution. Moreover, the periodic solution can be expressed by

$$y(t)=\int_t^{t+T} G_2(t,s)\,h(t)\,ds, \tag{2.7}$$

where

$$\begin{aligned} G_2(t,s) &= \frac{\int_t^s \exp\left[\int_t^u b(v)\,dv+\int_u^s a(v)\,dv\right]du}{\left[\exp\left(\int_0^\omega a(u)\,du\right)-1\right]\left[\exp\left(\int_0^\omega b(u)\,du\right)-1\right]} \\ &\quad +\frac{\int_s^{t+\omega} \exp\left[\int_t^u b(v)\,dv+\int_u^{s+\omega} a(v)\,dv\right]du}{\left[\exp\left(\int_0^\omega a(u)\,du\right)-1\right]\left[\exp\left(\int_0^\omega b(u)\,du\right)-1\right]}. \end{aligned} \tag{2.8}$$

Corollary 2.2 (see [32]). *The Green function G_2 satisfies the following properties:*

$$\begin{aligned} G_2(t+\omega,s+\omega) &= G_2(t,s),\ G_2(t,t+\omega)=G_2(t,t), \\ G_2(t+\omega,s) &= \exp\left(-\int_0^\omega b(v)\,dv\right) \\ &\quad \times\left[G_2(t,s)+\int_t^{t+\omega} E(t,u)\,F(u,s)\,du\right], \\ \frac{\partial}{\partial s}G_2(t,s) &= a(s)\,G_2(t,s)-\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^\omega b(v)\,dv\right)-1}, \\ \frac{\partial}{\partial t}G_2(t,s) &= -b(t)\,G_2(t,s)+\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^\omega a(v)\,dv\right)-1}, \end{aligned}$$

where

$$E(t,s)=\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^\omega b(v)\,dv\right)-1},F(t,s)=\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^\omega a(v)\,dv\right)-1}. \tag{2.9}$$

Lemma 2.4 (see [21]). *Let $A=\int_0^\omega a_1(u)du$ and $B=\omega^2\exp\left(\frac{1}{\omega}\int_0^\omega \ln\left(a_2(u)\right)du\right)$. If*

$$A^2\geq 4B, \tag{2.10}$$

then

$$\begin{aligned} \min\left\{\int_0^\omega a(u)du,\int_0^\omega b(u)du\right\} &\geq l, \\ \max\left\{\int_0^\omega a(u)du,\int_0^\omega b(u)du\right\} &\leq L, \end{aligned}$$

where

$$l=\frac{1}{2}\left(A-\sqrt{A^2-4B}\right),\ L=\frac{1}{2}\left(A+\sqrt{A^2-4B}\right). \tag{2.11}$$

Corollary 2.3 (see [32]). *Functions* G_2, E, *and* F *satisfy*

$$\begin{aligned} m_2 &\leq G_2(t,s) \leq M_2, && (2.12) \\ E(t,s) &\leq \frac{e^L}{e^l - 1}, && (2.13) \\ F(t,s) &\leq e^L, && (2.14) \end{aligned}$$

where

$$m_2 = \frac{\omega}{(\exp(L) - 1)^2},\ M_2 = \frac{\omega \left(\exp \int_0^\omega a_1(u)du\right)}{(\exp(l) - 1)^2}, \tag{2.15}$$

and

$$\exp(L) \neq 1,\ \ \exp(l) \neq 1.$$

Lemma 2.4 (see [8]). *Suppose the conditions of Lemma 2.2 hold and* $h \in C_\omega$. *Then, the equation*

$$y'''(t) + p(t)y''(t) + q(t)y'(t) + r(t)y(t) = h(t),$$

possesses an ω*-periodic solution. Moreover, the periodic solution can be expressed by*

$$y(t) = \int_t^{t+\omega} G(t,s)\, h(s)\, ds, \tag{2.16}$$

where

$$G(t,s) = \int_t^{t+\omega} G_2(t,\sigma)\, G_1(\sigma,s)\, d\sigma. \tag{2.17}$$

Corollary 2.4 (see [25]). *The Green function* G *satisfies the following properties*

$$G(t+\omega, s+\omega) = G(t,s), G(t, t+\omega) = G(t,t)\exp(\rho\omega),$$

$$\begin{aligned} \frac{\partial}{\partial t} G(t,s) &= (\exp(-\rho\omega) - 1)\, G_1(t,s)\, G_2(t,s) \\ &\quad - b(t)\, G(t,s) + \int_t^{t+\omega} F(t,\sigma)\, G_1(\sigma,s)\, d\sigma, \\ \frac{\partial}{\partial s} G(t,s) &= \rho G(t,s), \end{aligned}$$

and

$$m \leq G(t,s) \leq M, \tag{2.18}$$

where

$$m = \frac{\omega^2}{(\exp(l) - 1)^2 (\exp(\rho\omega) - 1)},\ M = \frac{\omega^2 \exp\left(\rho\omega + \int_0^\omega a(u)du\right)}{(\exp(l) - 1)^2 (\exp(\rho\omega) - 1)}, \tag{2.19}$$

and

$$\exp(\rho\omega) \neq 1,\ \ \exp(l) \neq 1.$$

Before stating the main result of this paper, we establish the equivalent integral formulation for the solution of equation (1.7).

Lemma 2.5. *Assume that (A1)-(A3) and* (2.6) *hold. The function* $y(\cdot)$ *is an* $\omega-$*periodic solution of equation* (1.7) *if and only if* $y(\cdot)$ *is an* $\omega-$*periodic solution of the following equation*

$$\begin{aligned} y(t) &= \int_t^{t+\omega} G(t,s)\left[d(s)\int_{-\infty}^0 K(r)\, f(s,y(s+r))\,dr \right. \\ &\quad \left. -\rho\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,dr\right] ds \\ &\quad +(\exp(\rho\omega)-1)\,G(t,t)\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,dr. \end{aligned} \tag{2.20}$$

Proof Let $y\in C_\omega$ be a solution of (1.7). From Lemma 2.4, we have

$$\begin{aligned} y(t) &= \int_t^{t+\omega} G(t,s)\left[d(s)\int_{-\infty}^0 K(r)\, f(s,y(s+r))\,dr \right. \\ &\quad \left. +\frac{\partial}{\partial s}\left(\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,dr\right)\right] ds \\ &= \int_t^{t+\omega} G(t,s)\left(d(s)\int_{-\infty}^0 K(r)\, f(s,y(s+r))\,dr\right) ds \\ &\quad +\int_t^{t+\omega} G(t,s)\,\frac{\partial}{\partial s}\left(\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,dr\right) ds. \end{aligned} \tag{2.21}$$

Performing an integration by parts, we obtain

$$\begin{aligned} &\int_t^{t+\omega} G(t,s)\,\frac{\partial}{\partial s}\left(\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,dr\right) \\ =\;& G(t,s)\left(\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,dr\right)\Bigg|_t^{t+\omega} \\ &-\int_t^{t+\omega}\left[\frac{\partial}{\partial s}G(t,s)\right]\left(\int_{-\infty}^0 K(r)\,c(s)\,y(s+r))dr\right) ds \\ =\;& G(t,t+\omega)\int_{-\infty}^0 K(r)\,c(t+\omega)\,y(t+\omega+r))dr \\ &-G(t,t)\int_{-\infty}^0 K(r)\,c(t)\,y(t+r))dr-\int_t^{t+\omega}\left[\frac{\partial}{\partial s}G(t,s)\right]\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,drds \\ =\;& (G(t,t)\exp(\rho\omega))\int_{-\infty}^0 K(r)\,c(t)\,y(t+r)\,dr - G(t,t)\int_{-\infty}^0 K(r)\,c(t)\,y(t+r)\,dr \\ &-\rho\int_t^{t+\omega} G(t,s)\int_{-\infty}^0 K(r)\,c(s)\,y(s+r)\,drds \end{aligned}$$

$$
\begin{aligned}
&= (\exp(\rho\omega)-1)\, G(t,t)\int_{-\infty}^{0} K(r)\, g(t, y(t+r))dr \\
&\quad -\rho\int_{t}^{t+\omega} G(t,s)\int_{-\infty}^{0} K(r)\, c(s)\, y(s+r)\, drds.
\end{aligned} \tag{2.22}
$$

We obtain (2.20) by replacing (2.21) into (2.22). □

3. Existence of positive periodic solutions

This section is devoted to establish three theorems and two corollaries, which are the main results of our paper. In the analysis we will carry out, we use the idea and concept of Green's function for third-order periodic solutions and transform equation (1.7) into an equivalent integral one. Then, by means of the Krasnosel'skii fixed theorem in cones of Banach spaces, we show the existence of at least one positive periodic solution of Eq. (1.7) in three different theorems. Some new and interesting sufficient conditions are obtained to guarantee the existence of such positive periodic solutions to (1.7).

Since Eq. (2.20) is equivalent to Eq. (1.7), we just have to study the existence of positive periodic solutions to (2.20). To this end, we will use Theorem 1.1 where we consider $(X, \|.\|) = (C_\omega, \|.\|)$.

Now, let

$$
\delta = \frac{m}{M} = \frac{1}{\exp\left(\rho\omega + \int_0^\omega a(u)du\right)} \in (0,1),
$$

and define K as a cone in C_ω by

$$
K = \{y(\cdot) \in C_\omega : y(t) \geq 0, \text{ and } y(t) \geq \delta \|y\|, t \in [0,\omega]\},
$$

where m, M are given by (2.19). It is not difficult to check that K is a cone in C_ω.

Thanks to (2.20), we define operator $\Phi : C_\omega \to C_\omega$ by:

$$
\begin{aligned}
(\Phi y)(t) &= \int_{t}^{t+\omega} G(t,s)\left[\left(d(s)\int_{-\infty}^{0} K(r)\, f(s, y(s+r))\, dr\right)\right. \\
&\qquad \left. -\rho\left(\int_{-\infty}^{0} K(r)\, c(s)\, y(s+r)\, dr\right)\right] ds \\
&\quad + (\exp(\rho\omega)-1)\, G(t,t)\left(\int_{-\infty}^{0} K(r)\, c(t)\, y(t+r)\, dr\right).
\end{aligned} \tag{3.1}
$$

To simplify our description, we introduce the following constants

$$
\begin{aligned}
\pi^* &= \exp(L), d^* = \max_{t\in[0,\omega]} |d(t)|, c^* = \max_{t\in[0,\omega]} |c(t)|, b^* = \max_{t\in[0,\omega]} |b(t)|, \\
\widehat{d} &= \min_{t\in[0,\omega]} |d(t)|, \widehat{c} = \min_{t\in[0,\omega]} |c(t)|, \widetilde{\theta} = \exp(\rho\omega) - 1.
\end{aligned} \tag{3.2}
$$

Thus, the existence of a positive periodic solution of equation (1.7) is equivalent to finding a fixed point of operator Φ.

Lemma 3.1. *Assume (A1)-(A3) hold. Then* Φ: $K \to K$ *is well defined.*

Proof From (3.1), it is easy to verify that $(\Phi y)(t)$ is continuous in t. Moreover, for any $y \in K$,

$$\begin{aligned}
&(\Phi y)(t+\omega)\\
&= \int_{t+\omega}^{t+2\omega} G(t+\omega, s)\left[d(s)\left(\int_{-\infty}^{0} K(r) f(s, y(s+r))\, dr\right)\right.\\
&\left. -\rho\left(\int_{-\infty}^{0} K(r) c(s) y(s+r)\, dr\right)\right] ds\\
&+ (\exp(\rho\omega)-1)\, G(t+\omega, t+\omega) \times\\
&\qquad \times \left(\int_{-\infty}^{0} K(r) c(t+\omega) y(t+\omega+r)\, dr\right)\\
&= \int_{t}^{t+\omega} G(t+\omega, s+\omega)\left[d(s+\omega)\left(\int_{-\infty}^{0} K(r) f(s+\omega, y(s+\omega+r))\, dr\right)\right.\\
&\left. -\rho\left(\int_{-\infty}^{0} K(r) c(s+\omega) y(s+\omega+r)) dr\right) ds\right]\\
&+ (\exp(\rho\omega)-1)\, G(t+\omega, t+\omega)\left(\int_{-\infty}^{0} K(r) c(t+\omega) y(t+\omega+r)) dr\right)
\end{aligned}$$

$$\begin{aligned}
&= \int_{t}^{t+\omega} G(t, s)\left[d(s)\left(\int_{-\infty}^{0} K(r) f(s, y(s+r))\, dr\right)\right.\\
&\left. -\rho\left(\int_{-\infty}^{0} K(r) c(s) y(s+r)\, dr\right)\right] ds\\
&+ (\exp(\rho\omega)-1)\, G(t, t)\left(\int_{-\infty}^{0} K(r) c(t) y(t+r)) dr\right)\\
&= (\Phi y)(t).
\end{aligned}$$

Therefore, $(\Phi y) \in C_\omega$. By condition (A3), we have

$$(\Phi y)(t) \geq 0, \text{ for all } y \in K.$$

Also, for $y \in K$, by using (2.19) and (3.1), we deduce

$$\begin{aligned}
|(\Phi y)| \leq\ & M \int_{0}^{\omega}\left[\left(d(s) \int_{-\infty}^{0} K(r) f(s, y(s+r))\, dr\right)\right.\\
& -\rho\left(\int_{-\infty}^{0} K(r) c(s) y(s+r)\, dr\right)\Big] ds\\
& \left. + (\exp(\rho\omega)-1)\left(\int_{-\infty}^{0} K(r) c(t) y(t+r)) dr\right)\right].
\end{aligned}$$

Noticing that

$$G(t,s)\left[d(t)\int_{-\infty}^{0}K(r)\,f(t,y(t+r))\,dr-\rho\int_{-\infty}^{0}K(r)\,c(t)y(t+r))dr\right]\geq 0,$$

we obtain

$$\begin{aligned}(\Phi y)(t) &\geq m\int_{0}^{\omega}\left[\left(d(s)\int_{-\infty}^{0}K(r)\,f(s,y(s+r))\,dr\right)\right.\\ &\quad -\rho\left(\int_{-\infty}^{0}K(r)\,c(s)\,y(s+r)\,dr\right)\Big]\,ds\\ &\qquad \left.+(\exp(\rho\omega)-1)\left(\int_{-\infty}^{0}K(r)\,c(t)y(t+r))dr\right)\right]\\ &\geq \frac{m}{M}\|\Phi y\|\geq\delta\|\Phi y\|.\end{aligned}$$

That is, ΦK is contained in K. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Assume that (A1)-(A3),* $(2.6),(2.10)$ *hold. Then* $\Phi:K\to K$ *is completely continuous.*

Proof Now we have to show that Φ is continuous. Let $y_n,(n=1,2,...)$ be a sequence in K such that

$$\lim_{n\to\infty}\|y_n-y_0\|=0.$$

Since K is closed, we have $y_0\in K$. Since $y,f(t,y)$ are ω-periodic and continuous functions, $f(s,y_n(s+r))\to f(s,y_0(s+r))$ uniformly, for $s\in[0,\omega]$.

$$\begin{aligned}&|(\Phi y_n)(t)-(\Phi y_0)(t)|\\ \leq\ &\int_{t}^{t+\omega}|G(t,s)|\,|d^*|\left|\int_{-\infty}^{0}K(r)\,|f(t,y_n(t+r))-f(t,y_0(t+r))|\,dr\right|ds\\ &+\rho\int_{t}^{t+\omega}|G(t,s)|\int_{-\infty}^{0}K(r)\,|f(t,y_n(t+r))\,dr-f(t,y_0(t+r))|\,dr\\ &+\widetilde{\theta}c^*M\int_{-\infty}^{0}K(r)\,|y_n(t+r)-y_0(t+r)|)dr.\end{aligned}$$

By Lebesgue dominated convergence theorem and (3.1), we obtain

$$\|(\Phi y_n)-(\Phi y_0)\|\to 0.$$

This shows that Φ is a continuous on K.

Next, we show that Φ is completely continuous. Let λ be any positive constant and

$$S_\lambda=\{y\in K:\|y\|\leq\lambda\},$$

be a bounded set in K. Then for $y \in K$,

$$\begin{aligned}
\left|\left(\Phi y\right)(t)\right| &\leq M\left[\int_0^{\omega}\left|d(s)\right|\left|\int_{-\infty}^{0} K(r) f(s, y(s+r))\, dr\right|\right. \\
&+\rho\left(\int_{-\infty}^{0} K(r)\left|c(s)\right|\left|y(s+r)\right| dr\right) ds \\
&\left.+\left(\exp(\rho\omega)-1\right)\int_{-\infty}^{0} K(r)\left|c(t)\right|\left|y(t+r))\right| dr\right] \\
&\leq Md^{*}\int_0^{\omega}\int_{-\infty}^{0} K(r)\, drds \max_{s\in[0,\omega],\|y\|\leq\lambda}\left|f(s,y)\right| \\
&+\rho Mc^{*}\int_0^{\omega}\int_{-\infty}^{0} K(r)\, drds \max_{s\in[0,\omega]}\left|y(s)\right| \\
&+M\widetilde{\theta}c^{*}\int_{-\infty}^{0} K(r)\, dr \max_{s\in[0,\omega]}\left|y(s)\right| \\
&\leq M\omega d^{*}\max_{s\in[0,\omega],\|y\|\leq\lambda}\left|f(s,y)\right| + \rho\omega Mc^{*}\lambda + M\lambda\widetilde{\theta}c^{*} \\
&= U.
\end{aligned}$$

Therefore, for any $y \in S_\lambda$,

$$\|\Phi u\| \leq U,$$

which implies that $\Phi(S_\lambda)$ is a uniformly bounded set. By taking the derivative in (3.1),

$$\begin{aligned}
&\frac{d(\Phi y)(t)}{dt} \\
= &\int_t^{t+\omega}\left[\left(\exp(-\rho\omega)-1\right)G_1(t,t)\,G_2(t,s) - b(t)\,G(t,s)\right. \\
&\left.+\int_t^{t+\omega} F(t,\mu)\,G_1(t,\mu)\,d\mu\right]\times \\
&\times\left[d(s)\int_{-\infty}^{s} K(r) f(s, y(s+r))\, dr - \rho\int_{-\infty}^{0} K(r)\, c(s) y(s+r))dr\right] ds.
\end{aligned}$$

Consequently, by invoking $(2.5), (2.14), (2.15), (2.19), (3.2),$ we have

$$\begin{aligned}
\left|\frac{d(\Phi y)(t)}{dt}\right| &= \int_t^{t+\omega}\left[\left|(\exp(-\rho\omega)-1)G_1(t,t)G_2(t,s)-b(t)G(t,s)\right|\right. \\
&\quad \left. + \int_t^{t+\omega}\left|F(t,\mu)G_1(t,\mu)\right|d\mu\right]\times \\
&\quad \times\left[\left|d(s)\int_{-\infty}^{0}K(r)f(s,y(s+r))dr\right|\right. \\
&\qquad \left. +\rho\left|\int_{-\infty}^{0}K(r)c(s)y(s+r))dr\right|\right]ds \\
&\leq \left[\omega(\exp(-\rho\omega)-1)M_1M_2+\omega b^{\star}M+\omega M_1\pi^{*}\right] \\
&\quad \times\left[d^{*}\max_{s\in[0,\omega],\|y\|\leq\lambda}|f(s,y)|+\rho\lambda\right] \\
&= Q,
\end{aligned}$$

which implies that $\dfrac{d(\Phi y)(t)}{dt}$ is also uniformly bounded, for any $y\in S_\lambda$. Hence, $\{\Phi y : y\in K, \|y\|\leq\lambda\}$ is a family of uniformly bounded and equicontinuous functions on $[0,\omega]$. In view of the Arzelà-Ascoli Theorem (see, Royden [30]), operator Φ is completely continuous. The proof of Lemma 3.2 is complete. □

We can now state and prove our main results.

Theorem 3.1. *In addition to conditions (A1)-(A3),* $(2.6), (2.10)$*, let us further assume that the following assumptions hold:*

(A4)

$$\liminf_{y\to 0}\frac{f(t,y)}{y}=\alpha(t), \text{ and } \limsup_{y\to\infty}\frac{f(t,y)}{y}=\beta(t), \tag{3.3}$$

where α,β *are continuous* $\omega-$*periodic functions on* $\mathbb{R}$.

(A5)

$$\int_0^{\omega}d(s)\alpha(s)ds\geq\rho\int_0^{\omega}c(s)ds+\frac{1}{m\delta}-\widetilde{\theta}\widehat{c}, \tag{3.4}$$

and

$$\int_0^{\omega}d(s)\beta(s)ds\leq\rho\int_0^{\omega}c(s)ds+\frac{1}{M}\left(1-\widetilde{\theta}c^{*}\right). \tag{3.5}$$

Then, equation (1.7) possesses at least one positive $\omega-$ *periodic solution.*

Proof We first construct two sets Ω_1 and Ω_2 in order to apply Theorem 1.1. Since $\liminf_{y\to 0}\frac{f(t,y)}{y}=\alpha(t)$, there exists $R_1>0$ such that

$$f(t,y)\geq\alpha(t)y, \text{ for } 0<y\leq R_1.$$

Define now the open subset

$$\Omega_1=\{y(\cdot)\in C_\omega : \|y\|<R_1\}.$$

Then, $\partial\Omega_1 = \{y(\cdot) \in C_\omega : \|y\| = R_1\}$, while $y \in K \cap \partial\Omega_1$, that is $\delta \|y\| \le y \le \|y\| = R_1$. Noticing that

$$G(t,s)\left(d(s)\int_{-\infty}^{0} K(r) f(s, y(s+r))\, dr - \rho\left(\int_{-\infty}^{0} K(r) c(s) y(s+r) dr\right)\right) \ge 0,$$

by noticing inequality (3.4) and the definition of Φ, we obtain

$$\begin{aligned}
(\Phi y)(t) \quad &\ge \quad m\int_t^{t+\omega}\left[d(s)\int_{-\infty}^{0} K(r) f(s, y(s+r))\, dr \,-\, \rho\left(\int_{-\infty}^{0} K(r)\, c(s)\, y(s+r)\, dr\right)\right] ds \\
&\quad +m\widetilde{\theta}\int_{-\infty}^{0} K(r)\, c(t) y(t+r)) dr \qquad (3.6)\\
&\ge \quad m\int_0^{\omega}[d(s)\,\alpha(s) - \rho c(s)]\, y(s)\, ds + m\widetilde{\theta} c(t)\, y(t) \\
&\ge \quad m\delta R_1 \int_0^{\omega}[d(s)\,\alpha(s) - \rho c(s)]\, ds + m\delta R_1 \widetilde{\theta}\widehat{c} \\
&\ge \quad \frac{m\delta R_1}{m\delta} - m\delta R_1\widetilde{\theta}\widehat{c} + m\delta R_1\widetilde{\theta}\widehat{c} \\
&> \quad R_1 = \|y\|.
\end{aligned}$$

Therefore, we conclude that

$$\|\Phi y\| > \|y\| \text{ for } y \in K \cap \partial\Omega_1. \qquad (3.7)$$

Next we construct the set Ω_2. Since $\limsup\limits_{y\to\infty}\dfrac{f(t,y)}{y} = \beta(t)$, there exists N such that

$$g(t,y) \le \beta(t)\, y, \text{ for } y \ge N.$$

Let

$$R_2 > \max\left\{\frac{M\theta}{1 - c_2^*\widetilde{\theta} - M\int_0^{\omega}(d(s)\,\beta(s) - \rho c_1(s))\, ds}, N\right\} > R_1,$$

where

$$\theta = \omega\left(d^*\widetilde{m}(f) + \rho c_2^* N\right),\ \widetilde{m}(f) = \max_{(t,u)\in[0,\omega]\times[0,N]} f(t,y).$$

Define now the open set $\Omega_2 = \{y(\cdot) \in C_\omega : \|y\| < R_2\}$, $E_1 = \{y(\cdot) \in C_\omega : \|y\| < N\}$, $E_2 = \{y(\cdot) \in C_\omega : \|y\| > N\}$, obviously $\overline{\Omega}_1 \subset \Omega_2$. If $y \in K \cap \partial\Omega_2$, by $(3.1), (3.5), (2.19)$, and $(A3)$, we have

$$\begin{aligned}
(\Phi y)(t) &\leq M \int_0^\omega \left[d(s) \left(\int_{-\infty}^0 K(r) f(s, y(s+r))\, dr \right) \right. \\
&\quad \left. - \rho \left(\int_{-\infty}^0 K(r) c(s) y(s+r)\, dr \right) \right] ds \\
&\quad + M(\exp(\rho\omega) - 1) \int_{-\infty}^0 K(r) c(s) y(s+r)\, dr \\
&\leq M \int_{E_1} \left[d(s) \left(\int_{-\infty}^0 K(r) f(s, y(s+r))\, dr \right) \right. \\
&\quad \left. - \rho \left(\int_{-\infty}^0 K(r) c(s) y(s+r)\, dr \right) \right] ds \\
&\quad + M \int_{E_2} \left[d(s) \left(\int_{-\infty}^0 K(r) f(s, y(s+r))\, dr \right) \right. \\
&\quad \left. - \rho \left(\int_{-\infty}^0 K(r) c(s) y(s+r)\, dr \right) \right] ds + M\widetilde{\theta} \int_{-\infty}^0 K(r) c(s) y(s+r)\, dr \\
&\leq M\omega d^* \max_{(s,u)\in[0,\omega]\times[0,N]} f(s, y(s+r)) + M\omega\rho c^* N \\
&\quad + M \int_{E_2} \left[d(s) \int_{-\infty}^0 K(r) f(s, y(s+r))\, dr - \rho \int_{-\infty}^0 K(r) c(s) y(s+r)\, dr \right] ds \\
&\quad + M\widetilde{\theta} c^* R_2 \\
&\leq M\omega \left(d^* \widetilde{m}(f) + \rho c^* N \right) + M R_2 \int_0^\omega [d(s)\beta(s) - \rho c(s)]\, ds + M\widetilde{\theta} c^* R_2 \\
&\leq M\theta + M R_2 \int_0^\omega [d(s)\beta(s) - \rho c(s)]\, ds + M\widetilde{\theta} c^* R_2 \\
&< R_2,
\end{aligned} \tag{3.8}$$

and therefore

$$\|\Phi y\| > \|y\|, \forall y \in K \cap \partial\Omega_2. \tag{3.9}$$

By Lemma 3.2, Φ is a completely continuous operator; from (3.7) and (3.9), condition (i) of Theorem 1.1 is fulfilled. Thus, Φ has at least one fixed point $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. By Theorem 1.1, we can conclude that Φ has a fixed point $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and with $R_1 \leq \|y\| \leq R_2$, and $y \geq \delta \|y\| \geq \delta R_1 > 0$. In other words, equation (1.7) has at least one positive periodic solution. □

Theorem 3.2. *Suppose that (A1)-(A3),* $(2.6), (2.10)$ *hold, and the following conditions are fulfilled,*

$$A_0 = \lim_{\|y\|\to 0} \max_{t\in[0,\omega]} \frac{f(t,y) - \rho \frac{c(t)}{d(t)} y}{\|y\|} = 0, \tag{3.10}$$

$$A_\infty = \lim_{y\in K, \|y\|\to\infty} \min_{t\in[0,\omega]} \frac{f(t,y) - \rho \frac{c(t)}{d(t)} y}{\|y\|} = \infty. \tag{3.11}$$

Then equation (1.7) *has at least one positive* $\omega-$*periodic solution.*

Proof i) In view of (3.10), there exist $\varepsilon > 0$ and corresponding $R_1 > 0$, such that

$$0 < \varepsilon < \frac{1-\widetilde{\theta}Mc^*}{Md^*},\ (0 \leq \widetilde{\theta}Mc^* < 1),$$

and

$$f(t,y) - \rho\frac{c(t)}{d(t)}y \leq \varepsilon\|y\|,\ \text{for}\ \|y\| \leq R_1, t \in [0,\omega].$$

Define the open subset $\Omega_1 = \{y(\cdot) \in C_\omega :\ \|y\| < R_1\}$, while $y \in K \cap \partial\Omega_1$, that is $\delta\|y\| \leq y \leq \|y\| = R_1$, we can have

$$\begin{aligned}
(\Phi y)(t) &= \int_t^{t+\omega} G(t,s)\left[\left(d(s)\int_{-\infty}^0 K(r)f(s,y(s+r))\,dr\right)\right.\\
&\quad \left.-\rho\left(\int_{-\infty}^0 K(r)c(s)y(s+r)\,dr\right)\right]ds\\
&\quad +(\exp(\rho\omega)-1)G(t,t)\left(\int_{-\infty}^0 K(r)c(t)y(t+r)\,dr\right)\\
&\leq M\int_t^{t+\omega} d(s)\int_{-\infty}^0 K(r)\left[f(s,y(s+r)) - \rho\frac{c(s)}{d(s)}\ y(s+r)\,dr\right]drds\\
&\quad +\widetilde{\theta}M\left(\int_{-\infty}^0 K(r)c(t)y(t+r)\,dr\right)\\
&\leq Md^*\varepsilon\|y\| + \widetilde{\theta}Mc^*\|y\|\\
&\leq \|y\|.
\end{aligned}$$

Hence,

$$\|\Phi y\| \leq \|y\|,$$

namely

$$\|\Phi y\| \leq \|y\|,\ \text{for each}\ y \in K \cap \partial\Omega_1. \tag{3.12}$$

ii) In view of (3.11), there exist D and corresponding $R_2 > R_1(\delta)^{-1}$ such that

$$\delta Dm\widehat{d} + \delta m\widehat{c}\widetilde{\theta} \geq 1, f(t,y) - \rho\frac{c(t)}{d(t)}y \geq D\|y\|,\ \text{for}\ y \geq \delta\|y\|,\ \text{and}\ \|y\| \geq \delta R_2.$$

Define $\Omega_2 = \{y(\cdot) \in C_\omega : \ \|y\| < R_2\}$, obviously, $\overline{\Omega}_1 \subset \Omega_2$. For $y \in K \cap \partial\Omega_2$, we can have

$$\begin{aligned}
(\Phi y)(t) &= \int_t^{t+\omega} G(t,s)\left[\left(d(s)\int_{-\infty}^{0} K(r) f(s, y(s+r))\, dr\right)\right. \\
&\quad - \rho\left.\left(\int_{-\infty}^{0} K(r) c(s) y(s+r)\, dr\right)\right] ds \\
&\quad + (\exp(\rho\omega) - 1) G(t,t)\left(\int_{-\infty}^{0} K(r) c(t) y(t+r)\, dr\right) \\
&\geq m\int_t^{t+\omega} d(s)\int_{-\infty}^{0} K(r)\left[f(s, y(s+r)) - \rho\frac{c(s)}{d(s)}\, y(s+r)\, dr\right] drds \\
&\quad + m\widetilde{\theta}\left(\int_{-\infty}^{0} K(r) c(t) y(t+r)\, dr\right) \\
&\geq m\widehat{d}\delta D R_2 + m\widehat{c}\widetilde{\theta}\delta R_2 \\
&\geq \left(m\widehat{d} D\delta + m\widehat{c}\widetilde{\theta}\delta\right) R_2 \\
&\geq R_2 = \|y\|,
\end{aligned}$$

which leads to

$$(\Phi y)(t) \geq \|y\|, \text{ for each } y \in K \cap \partial\Omega_2. \tag{3.13}$$

By Lemma 3.2, Φ is a completely continuous operator; from (3.12) and (3.13), condition (i) of Theorem 1.1 is fulfilled. Thus, Φ has at least one fixed point $y \in K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$, and $y \geq \delta\|y\| \geq \delta R_1 > 0$ which means $y(\cdot)$ is a ω–periodic positive solution of (1.7). This completes the proof. $\square$

For convenience and simplicity in the following discussion, we use the following notations:

$$F_{R_2}^{0} = \inf_{y \in K \cap \delta\Omega_2} \min_{t \in [0,\omega]} \left(\frac{f(t,y)}{y} - \rho\frac{c(t)}{d(t)}\right), \tag{3.14}$$

$$F_{R_1}^{\infty} = \sup_{y \in K \cap \delta\Omega_1} \max_{t \in [0,\omega]} \left(\frac{f(t,y)}{y} - \rho\frac{c(t)}{d(t)}\right). \tag{3.15}$$

Theorem 3.3. *Suppose that (A1)-(A3),* $(2.6), (2.10)$ *hold, and there are positive constants* R_1, R_2 *with* $R_1 < R_2$ *such that:*

$$M\left(d^* F_{R_1}^{\infty} + \widetilde{\theta} c^*\right) \leq 1, \tag{3.16}$$

and

$$m\left(\widehat{d} F_{R_2}^{0} + \widetilde{\theta}\widehat{c}\right) \geq 1. \tag{3.17}$$

Then, there exists an ω*–periodic solution which is a fixed point of* Φ *and satisfies* $R_1 \leq \|y\| \leq R_2$.

Proof Define two open sets Ω_1 and Ω_2 with $R_1 < R_2$. Let $\Omega_1 = \{y(\cdot) \in C_\omega : \ \|y\| < R_1\}$. Then for any $y \in K \cap \partial\Omega_1$, we have $\delta \|y\| \leq y \leq \|y\| = R_1$. From this, the definition of Φ and $F^{\infty}_{R_1}$, it follows that

$$\begin{aligned}
(\Phi y)(t) &= \int_t^{t+\omega} G(t,s)\left[\left(d(s)\int_{-\infty}^0 K(r) f(s, y(s+r))\, dr\right)\right. \\
&\quad \left. -\rho\left(\int_{-\infty}^0 K(r) c(s) y(s+r)\, dr\right)\right] ds \\
&\quad +(\exp(\rho\omega)-1) G(t,t)\left(\int_{-\infty}^0 K(r) c(t) y(t+r)\, dr\right) \\
&\leq M\int_t^{t+\omega} d(s)\int_{-\infty}^0 K(r)\left[f(s,y(s+r)) - \rho\frac{c(s)}{d(s)}\ y(s+r)\, dr\right] dr ds \\
&\quad +\widetilde{\theta} M\left(\int_{-\infty}^0 K(r) c(t) y(t+r)\, dr\right) \\
&\leq \left(Md^* F^{\infty}_{R_1} + \widetilde{\theta} M c^*\right)\|y\|.
\end{aligned} \tag{3.18}$$

Hence, in view of (3.16) and (3.18), we obtain

$$\|\Phi y\| \leq \|y\|, \text{ for each } y \in K \cap \partial\Omega_2. \tag{3.19}$$

On the other hand, let $\Omega_2 = \{y(\cdot) \in C_\omega\colon \ \|y\| < R_2\}$, obviously, $\overline{\Omega}_1 \subset \Omega_2$. For $y \in K \cap \partial\Omega_2$, then $\delta\|y\| \leq y \leq \|y\| = R_2$. From this, the definition of Φ and $F^0_{R_2}$, it follows that

$$\begin{aligned}
(\Phi y)(t) &= \int_t^{t+\omega} G(t,s)\left[\left(d(s)\int_{-\infty}^0 K(r) f(s, y(s+r))\, dr\right)\right. \\
&\quad \left. -\rho\left(\int_{-\infty}^0 K(r) c(s) y(s+r)\, dr\right)\right] ds \\
&\quad +(\exp(\rho\omega)-1) G(t,t)\left(\int_{-\infty}^0 K(r) c(t) y(t+r)\, dr\right) \\
&\geq m\int_t^{t+\omega} d(s)\int_{-\infty}^0 K(r)\left[f(s,y(s+r)) - \rho\frac{c(s)}{d(s)}\ y(s+r)\right] dr ds \\
&\quad +m\widetilde{\theta}\left(\int_{-\infty}^0 K(r) c(t) y(t+r)\, dr\right) \\
&\geq \left(m\widehat{d} F^0_{R_1} + m\widehat{c}\widetilde{\theta}\right)\|y\|.
\end{aligned} \tag{3.20}$$

Hence, from (3.17) and (3.20), we obtain

$$(\Phi y)(t) \geq \|y\|, \text{ for each } y \in K \cap \partial\Omega_1. \tag{3.21}$$

By Lemma 3.2, Φ is a completely continuous operator, from (3.19) and (3.21), condition (i) of Theorem 1.1 is fulfilled. Thus, Φ has at least one fixed point $y \in K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$. It follows that $y(\cdot)$ is an $\omega-$periodic positive solution of (1.7) and $y \geq \delta\|y\| \geq \delta R_1 > 0$. This completes the proof. □

As consequence of Theorem 3.3, we state a corollary whose proof is similar to the proof of Theorem 3.3 and hence we omit it.

Corollary 3.1. *Suppose that (A1)-(A3),* (2.6), (2.10) *hold, and there are positive constants* R_1, R_2 *with* $R_1 < R_2$ *such that:*

$$M\left(d^* F^0_{R_2} + \widetilde{\theta} c^*\right) \le 1, \tag{3.22}$$

and

$$m\left(\widehat{d} F^{\infty}_{R_1} + \widetilde{\theta}\widetilde{c}\right) \ge 1. \tag{3.23}$$

Then, there exists an $\omega-$ *periodic solution which is a fixed point of* Φ *and satisfies* $R_1 \le \|y\| \le R_2$.

As a consequence of Theorem 3.3, we have the next Corollary.

Corollary 3.2. *Suppose that (A1)-(A3),* (2.6), (2.10) *hold, and there are positive constants* $R_1, R_2, ..., R_{n+1}$ *with* $R_1 < R_2 < ... < R_{n+1}$ *such that:*

$$\begin{aligned}
M\left(d^* F^{\infty}_{R_2} + \widetilde{\theta} c^*\right) &\le 1, \\
m\left(\widehat{d} F^0_{R_1} + \widehat{c}\widetilde{\theta}\right) &\ge 1, \\
M\left(d^* F^{\infty}_{R_4} + \widetilde{\theta} c^*\right) &\le 1, \\
m\left(\widehat{d} F^0_{R_3} + \widehat{c}\widetilde{\theta}\right) &\ge 1, \\
&\cdots\cdots\cdots \\
M\left(d^* F^{\infty}_{R_{n+1}} + \widetilde{\theta} c^*\right) &\le 1, \\
m\left(\widehat{d} F^0_{R_n} + \widehat{c}\widetilde{\theta}\right) &\ge 1.
\end{aligned}$$

Then, Eq. (1.7) has n *positive* $\omega-$*periodic solutions* $y_1, y_2, ..., y_n$ *with* $\|y_1\| \le \|y_2\| \le ... \le \|y_n\|$.

Remark 3.1. In our main results, condition (A3) is very important to guarantee that operator Φ maps K into itself.

4. Examples

Let us discuss three examples to illustrate our abstract theory.

Example 4.1. Let us consider the following third-order nonlinear integro-differential delay equation:

$$\begin{aligned}
&y'''(t) + p(t)y''(t) + q(t)y'(t) + r(t)y(t) \\
&= \frac{d}{dt}\int_{-\infty}^{0} K(r)\, c(t)\, y(t+r))dr \\
&+ d(t)\int_{-\infty}^{0} K(r)\, f(t, y(t+r))\, dr.
\end{aligned} \tag{4.1}$$

Corresponding to equation (1.7), we let

$$\begin{aligned} p(t) &= 31\times 10^{-2}, q(t) = 23\times 10^{-3}, \omega = 2\pi, \\ c(t) &= 1 + 0.1\left|\sin t\right|, r(t) = 2\times 10^{-4}. \end{aligned}$$

Also assume that

$$f\left(t,y\right) = \frac{0.4}{d(t)}\frac{y}{1+y} + \rho\frac{c(t)}{d(t)}y,$$

where $d\in C\left(\mathbb{R},\mathbb{R}^{+}\setminus\{0\}\right)$ is an arbitrary $2\pi-$periodic function.

Moreover, $K\in C\left(\mathbb{R}^{-},\mathbb{R}^{+}\right)$ is an arbitrary function satisfying

$$\int_{-\infty}^{0} K\left(r\right)dr = 1.$$

Thanks to direct computations, we obtain

$$\begin{aligned} a(t) &= 0.2, b(t) = 0.01, \delta = \frac{m}{M} \simeq 0.151, \\ a_1(t) &= 21\times 10^{-2}, a_2(t) = 2\times 10^{-3}, \rho = 0.1, \\ c^{*} &= 1.1, \widehat{c} = 1. \end{aligned}$$

It is clear that, $\forall y\in\mathbb{R}^{+}, t\in\left[0,2\pi\right],$ we have

$$f\left(t,y\right) - \rho\frac{c\left(t\right)}{d\left(t\right)}y = \frac{0.4}{d(t)}\frac{y}{1+y} \geq 0.$$

Again by straightforward computations, we have

$$\liminf_{y\to 0}\frac{f\left(t,y\right)}{y} = \frac{0.4}{d(t)} + \rho\frac{c(t)}{d(t)} = \alpha\left(t\right),$$

and

$$\limsup_{y\to\infty}\frac{f\left(t,y\right)}{y} = \rho\frac{c(t)}{d(t)} = \beta\left(t\right).$$

It is easy to check α,β are continuous 2π $-$periodic functions on $\mathbb{R}$.

From the above parameters, it follows that

$$\begin{aligned} &\int_0^{\omega} d\left(s\right)\alpha\left(s\right)ds - \int_0^{\omega}\rho c\left(s\right)ds - \frac{1}{m\delta} + \left(\exp\left(\rho\omega\right)-1\right)\widehat{c} \\ =\;& 0.4\omega - \frac{1}{m\delta} + \left(\exp\left(\rho\omega\right)-1\right)\widehat{c} \\ \simeq\;& 2.6 \geq 0. \end{aligned}$$

and

$$\begin{aligned} &\int_0^{\omega} d\left(s\right)\beta\left(s\right)ds - \rho\int_0^{\omega} c\left(s\right)ds - \frac{1}{M}\left(1-\left(\exp\left(\rho\omega\right)-1\right)c^{*}\right) \\ =\;& -\frac{1}{M}\left(1-\left(\exp\left(\rho\omega\right)-1\right)c^{*}\right) \\ \simeq\;& -54\times 10^{-8} \leq 0. \end{aligned}$$

Consequently, all conditions of Theorem 3.1 are fulfilled. Hence, we conclude that Equation (4.1) possesses at least one positive $2\pi-$periodic solution $y \in K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Example 4.2. Let us consider the following third-order nonlinear integro-differential delay equation:

$$\begin{aligned} & y'''(t) + p(t)y''(t) + q(t)y'(t) + r(t)y(t) \\ & = \frac{d}{dt}\int_{-\infty}^{0} K(r)\, c(t)\, y(t+r))dr \\ & +d(t)\int_{-\infty}^{0} K(r)\, f(t, y(t+r))\, dr. \end{aligned} \tag{4.2}$$

Corresponding to equation (1.7), we let

$$\begin{aligned} p(t) &= 31\times 10^{-2}, q(t) = 23\times 10^{-3}, \omega = 2\pi, \\ c(t) &= 4\left|\cos t\right| + 2, r(t) = 2\times 10^{-4}, d(t) = 0.2\left|\cos t\right| + 0.1. \end{aligned}$$

Also assume that

$$f(t, y) = 0.4y^2 + 2y.$$

Moreover, $K \in C(\mathbb{R}^-, \mathbb{R}^+)$ is an arbitrary function satisfying

$$\int_{-\infty}^{0} K(r)\, dr = 1.$$

Thanks to direct computations, we obtain

$$\begin{aligned} a(t) &= 0.2, b(t) = 0.01, \delta = \frac{m}{M} \simeq 0.151, \\ a_1(t) &= 21\times 10^{-2}, a_2(t) = 2\times 10^{-3}, \rho = 0.1, \\ c^* &= 6, \widehat{c} = 2. \end{aligned}$$

It is clear that, $\forall y \in \mathbb{R}^+, t \in [0, 2\pi]$, we have

$$f(t, y) - \rho\frac{c(t)}{d(t)}y = 0.4y^2 \geq 0.$$

It is easy to verify that

$$\begin{aligned} A_0 &= \lim_{\|y\|\to 0} \max_{t\in[0,\omega]} \frac{f(t, y) - \rho\frac{c(t)}{d(t)}y}{\|y\|} \\ &= \lim_{\|y\|\to 0} \max_{t\in[0,\omega]} \frac{0.4y^2}{\|y\|} = 0, \end{aligned}$$

and

$$\begin{aligned} A_\infty &= \lim_{y\in K, \|y\|\to\infty} \min_{t\in[0,\omega]} \frac{f(t, y) - \rho\frac{c(t)}{d(t)}y}{\|y\|} \\ &= \lim_{y\in K, \|y\|\to\infty} \min_{t\in[0,\omega]} \frac{0.4y^2}{\|y\|} = \infty. \end{aligned}$$

It is straightforward to show that all conditions of Theorem 3.2 are fulfilled. Hence, we conclude that equation (4.2) possesses at least one positive $2\pi-$periodic solution $y \in K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Example 4.3. Let us consider the following equation:

$$\begin{aligned} & y'''(t) + p(t)y''(t) + q(t)y'(t) + r(t)y(t) \\ &= \frac{d}{dt}\int_{-\infty}^{0} K(r)\, c(t)\, y(t+r))dr \\ & +d(t)\int_{-\infty}^{0} K(r)\, f(t, y(t+r))\, dr. \end{aligned} \tag{4.3}$$

Corresponding to equation (1.7), we let

$$\begin{aligned} p(t) &= 31\times 10^{-2}, q(t) = 23\times 10^{-3}, r(t) = 2\times 10^{-4}, \omega = 2\pi, \\ d(t) &= 10^{-4}(0.2 + 0.2\left|\sin t\right|), c(t) = 10^{-5}(1 + 0.5\left|\cos t\right|). \end{aligned}$$

Also assume that

$$f(t,y) = \frac{0.1\left(1 + 10^2\times 0.151^2\left|\sin t\right|\right)}{0.001 + y^2}y + \frac{1}{100}\times\frac{1 + 0.5\left|\cos t\right|}{0.2 + 0.2\left|\sin t\right|}y,$$

and $K \in C(\mathbb{R}^-, \mathbb{R}^+)$ satisfying

$$\int_{-\infty}^{0} K(r)\, dr = 1.$$

Then we can check that

$$\begin{aligned} a(t) &= 0.2,\ b(t) = 0.01, \rho = 0.1, \\ a_1(t) &= 21\times 10^{-2}, a_2(t) = 2\times 10^{-3}, \\ m &\simeq 10735.730,\ M \simeq 70706.185, \\ \delta &= \frac{m}{M} \simeq 0.151. \end{aligned}$$

It is clear that, $\forall y \in \mathbb{R}^+, t \in [0, 2\pi]$, we have

$$f(t,y) - \rho\frac{c(t)}{d(t)}y = \frac{0.1\left(1 + 10^2\times 0.151^2\left|\sin t\right|\right)y}{0.001 + y^2} \geq 0.$$

There are positive constants $R_1 = 0.02, R_2 = 10,$ such that

$$\begin{aligned} F_{R_1}^0 &= \inf_{y\in K\cap\delta\Omega_1}\min_{t\in[0,\omega]}\left(\frac{f(t,y)}{y} - \rho\frac{c(t)}{d(t)}\right) \\ &= \inf_{y\in K\cap\delta\Omega_1}\min_{t\in[0,\omega]}\left(\frac{0.1\left(1 + 10^2\delta^2\left|\sin t\right|\right)}{0.001 + y^2}\right) \\ &= \inf_{y\in K\cap\delta\Omega_1}\left(\frac{0.1}{0.001 + \left\|y\right\|^2}\right) \\ &= \frac{0.1}{0.001 + 0.0004} \simeq 71.428. \end{aligned}$$

$$\begin{aligned}
F_{R_2}^{\infty} &= \sup_{y\in K\cap\delta\Omega_2}\max_{t\in[0,\omega]}\left(\frac{f(t,y)}{y}-\rho\frac{c(t)}{d(t)}\right)\\
&= \sup_{y\in K\cap\delta\Omega_2}\max_{t\in[0,\omega]}\left(\frac{0.1\left(1+10^2\delta^2\left|\sin t\right|\right)}{0.001+y^2}\right)\\
&= \sup_{y\in K\cap\delta\Omega_2}\frac{0.1\left(1+10^2\delta^2\right)}{0.001+y^2}\\
&= \frac{0.1\left(1+10^2\delta^2\right)}{0.001+\delta^2R_2^2}\simeq 0.143.
\end{aligned}$$

Now, some simple calculations show that $\forall\,(t,s)\in\mathbb{R}\times\mathbb{R}$,

$$10735.730\simeq m\leq G(t,s)\leq M\simeq 70706.185.$$

For the above parameters, it is easy to verify that

$$M\left(d^*F_{R_2}^{\infty}+\widetilde{\theta}c^*\right)\simeq 0.713\leq 1,$$

and

$$m\left(\widehat{d}F_{R_1}^{0}+\widehat{c}\widetilde{\theta}\right)\simeq 15.430\geq 1,$$

where

$$\begin{aligned}
d^* &= \max_{t\in[0,\omega]}d(t)=0.4\times 10^{-4},\widehat{d}=\min_{t\in[0,\omega]}d(t)=0.2\times 10^{-4},\\
c^* &= \max_{t\in[0,\omega]}c(t)=0.5\times 10^{-5},\widehat{c}=\min_{t\in[0,\omega]}c(t)=10^{-5},\\
\widetilde{\theta} &= \exp(\rho\omega)-1\simeq 0.874.
\end{aligned}$$

All hypotheses of Theorem 3.3 are fulfilled and, therefore, Eq. (4.3) has at least one positive $2\pi-$periodic solution $y\in K\cap\left(\overline{\Omega}_2\setminus\Omega_1\right)$ satisfying

$$0.2=R_1\leq\|y\|\leq R_2=10.$$

Remark. 4.2. Observe that Example 4.1 and Example 4.2 cannot be analyzed by applying Theorem 3.1 in [25] (see also Theorem 3.1 in [2]). Indeed, in order to apply Theorem 3.1 in [2] or [25], we need to construct two mappings Φ_1 and Φ_2, Φ_1 is a contraction and Φ_2 is compact. Therefore, we express (4.1) or (4.2) as

$$(\Phi x)(t)=(\Phi_1x)(t)+(\Phi_2x)(t),$$

where $\Phi_1,\Phi_2:\ C_\omega\to C_\omega$ are given by

$$(\Phi_1x)(t)=(\exp(\rho\omega)-1)\,G(t,t)\int_{-\infty}^{0}K(r)\,c(s)\,x(s+r)\,dr,$$

$$\begin{aligned}(\Phi_2 x)(t) &= \int_t^{t+\omega} G(t,s)\left[d(s)\int_{-\infty}^{0} K(r)\, f(s, x(s+r))\, dr\right. \\ &\quad \left. - \rho \int_{-\infty}^{0} K(r)\, c(s)\, x(s+r)\, dr\right] ds.\end{aligned}$$

However, notice that

$$\exists \gamma > 1, \forall x, y \in C_\omega, \|\Phi_1 x - \Phi_1 y\| \leq \gamma \|x - y\|,$$

where

$$\gamma = M(\exp(\rho\omega) - 1)\, c^*.$$

This implies that Φ_1 is not contraction. Thus, Theorem 3.1 in [2] and [25] cannot be applied to equations (4.1) and (4.2). Therefore, the results in [2] and [25] are not applicable. However, the results obtained in our work are quite significant compared to the ones in the aforementioned papers [2, 25].

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Conflict of interests

The authors declare that they do not have any conflict of interest.

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