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On the eigenstructure of the q-Durrmeyer operators

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Abstract: The purpose of this paper is to establish the eigenvalues and the eigenfunctions of both the q-Durrmeyer operators $D_{n,q}$ and the limit q-Durrmeyer operators $D_{\infty,q}$ introduced by V. Gupta in the case $0 < q < 1$. All moments for $D_{n,q}$ and $D_{\infty,q}$ are provided. The coefficients for the eigenfunctions of the operators are explicitly derived and the eigenfunctions of these operators are illustrated by graphical examples.

Key words: q-integers, q-Stirling numbers, q-Bernstein-Durrmeyer operators, moments, eigenvalues, eigenfunctions

1. Introduction

After the Bernstein polynomials [2] had been constructed to uniformly approximate continuous functions on $[0,1]$, there were very few works on these polynomials until the 30th. In 1930, when Bernstein organized a mathematical congress in Kharkiv, breakthrough results were presented on these polynomials by a few speakers including such mathematicians as Kantorovich, Khlodovskii and Voronovskaya, see [22].

Kantorovich used polynomials of the form

$$\Phi_n(f;x) = \sum_{k=0}^{n} \varphi_{nk}(f) p_{nk}(x), \tag{1.1}$$

where $p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, 1, \ldots, n$ are the Bernstein basis polynomials and $\varphi_{nk}(f)$ are positive functionals which are not necessarily linear. Obviously, $\varphi_{nk}(f) = f(k/n)$ gives the Bernstein polynomials. Kantorovich selected

$$\varphi_{nk}(f) = (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt,$$

and obtained the polynomials—called, nowadays, the Kantorovich polynomials $K_n(f;x)$—which converge to f almost everywhere whatever measurable function f is, see [13].

Later, Lorentz [15] proved that the sequence $\{K_n(f;x)\}$ approximates $f \in L_p[0,1]$ in the L_p-norm. It turned out that the Kantorovich polynomials approximate functions $f \in C[0,1]$ with respect to the uniform norm, as well. After the congress, the situation has been changed drastically concerning the investigation and application of Bernstein polynomials and related linear positive operators. The research in this field is still going

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on due to its wide range of applications, both in mathematics and engineering like computer aided geometric design [9].

The idea of Kantorovich to replace the coefficients of Bernstein basis polynomials by integrals rather than the values at given points was further developed in 1967 by Durrmeyer [6] who proposed to take $\varphi_{nk}(f)$ in (1.1) as

$$\varphi_{nk}(f) = (n+1)\int_0^1 f(t)p_{nk}(t)\,dt,$$

which leads to the new positive linear operators given below.

Definition 1.1 *Let $f \in C[0,1]$. The Bernstein-Durrmeyer operators are defined by*

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_0^1 f(t)p_{nk}(t)\,dt, \quad n = 1,2,\ldots, \tag{1.2}$$

where $p_{nk}(x)$, $k = 0,\ldots,n$ are the Bernstein basis polynomials.

Along with the development of q-calculus, a variety of q-analogues of classical linear positive operators occured. Starting from the well-known papers by Lupaş [16] and Phillips [20], the researches on q-analogues have been going extensively revealing new applications [3, 17].

As for the Durrmeyer operator, the first q-analogue was introduced by Derriennic [5] in 2005. Derriennic proved that, when considered in $C[0,1]$, the polynomials $D_n(f;x)$ provide uniform approximation and possess degree-reducing properties on polynomials. Moreover, the operators $D_n(f;x)$ are self-adjoint in $L_2[0,1]$ and converge almost everywhere to f when f is integrable on $[0,1]$. Other q-analogues were defined by Gupta [10] and Gupta and Wang [11].

In the present paper, the q-Durrmeyer operators given in [10] are studied. For the convenience of the reader, let us recall the needed notations and definitions associated with q-calculus, see [1, Chapter 10].

Let $q > 0$. For any nonnegative integer n, the q-integers $[n]_q$ are defined by

$$[0]_q := 0, \quad [n]_q := 1 + q + \cdots + q^{n-1}, \quad n = 1,2,\ldots.$$

The following two expressions are q-analogues of the factorials and binomial coefficients which are called q-factorials and q-binomial coefficients, respectively, given by

$$[0]_q! := 1, \quad [n]_q! := [1]_q[2]_q\cdots[n]_q, \quad n = 1,2,\ldots,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For integers $0 \leqslant k \leqslant n$, the q-binomial coefficient can be expressed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1.3}$$

where $(x;q)_n$ stands for the q-analogue of the Pochhammer symbol defined by, for each $x \in \mathbb{C}$,

$$(x;q)_0 := 1, \quad (x;q)_n := \prod_{j=0}^{n-1}(1 - xq^j), \quad (x;q)_\infty := \prod_{j=0}^{\infty}(1 - xq^j).$$

The Gauss q-binomial formula is valid, see [1, Chapter 10, Corollary 10.2.2]:

$$(-x;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k. \tag{1.4}$$

Two binomial formulae, which are the generalization of Taylor's expansion in q-calculus (see [1, Chapter 10, Corollary 10.2.2]), are presented. The first one is the Euler identity

$$\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}, \quad |q| < 1, |x| < 1, \tag{1.5}$$

the other one is the Rothe identity

$$\frac{1}{(x;q)_n} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k, \quad |x| < 1. \tag{1.6}$$

The q-integral in the interval $[0,a]$, first introduced by Thomae [21] and later by Jackson [12], is defined as

$$\int_0^a f(t)\, d_q t := (1-q)a \sum_{j=0}^{\infty} q^j f(aq^j). \tag{1.7}$$

Definition 1.2 *[10] Let $0 < q < 1$, $f \in C[0,1]$. The q-Durrmeyer operator $D_{n,q} : C[0,1] \to C[0,1]$ is given by*

$$D_{n,q}(f;x) = \sum_{k=0}^{n} A_{nk}(q;f) p_{nk}(q;x), \tag{1.8}$$

where

$$A_{nk}(q;f) := [n+1]_q q^{-k} \int_0^1 f(t) p_{nk}(q;qt)\, d_q t, \tag{1.9}$$

and

$$p_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k = 0, 1, \ldots, n.$$

Observe that, for $q = 1$, (1.8) gives the classical Bernstein-Durrmeyer operators (1.2). In this paper, new results on the operator (1.8) are presented.

The present paper aims to highlight the eigenvalues and the eigenfunctions of the q-Durrmeyer operators $D_{n,q}$ and their limit operators $D_{\infty,q}$. For the Bernstein type operators, the intensive research on the eigenstructure was initiated in [4] and, afterwards, has been carried out in a number of papers, see, for example, [5, 7, 14, 19]. This paper is organized as follows. In Section 1, some preliminary results, which will be used in the next sections, are presented. In Section 2, all moments of the q-Durrmeyer operators $D_{n,q}$ are obtained by means of the recurrence relation which is derived in the same section. Section 3 deals with the eigenvalues and the corresponding eigenfunctions of $D_{n,q}$. To derive the explicit formula for the coefficients of the eigenfunctions, a similar technique is used with that of S. Cooper and S. Waldron found in [4]. Section 4 is devoted to some results concerning the limit q-Durrmeyer operators such as moments, eigenvalues and eigenfunctions. It should be noted that the moments of $D_{\infty,q}$ are proved with a different point of view from [10]. The last section contains some illustrations of the eigenfunctions of $D_{n,q}$ and $D_{\infty,q}$.

2. Moments of $D_{n,q}$

In [10, Theorem 1], Gupta obtained $D_{n,q}(t^i;x)$, $i = 0,1,2$ by using the integral representation of q-Beta function. It is noted that, unlike the q-Bernstein operators, the operators $D_{n,q}(f;x)$ leave invariant only the constant functions, not all of the linear ones. In this section, all moments will be evaluated, explicitly.

Lemma 2.1 *For $f(t) = t^m$, $m = 0,1,\ldots,$ the coefficients $A_{nk}(q;f)$ in (1.8) have the form*

$$A_{nk}(q;t^m) = \frac{(q^{k+1};q)_m}{(q^{n+2};q)_m}, \quad k = 0,1,\ldots,n.$$

Proof By the definition (1.9) of $A_{nk}(q;f)$, one has

$$A_{nk}(q;t^m) = [n+1]_q q^{-k}\int_0^1 t^m p_{nk}(q;qt)\,d_qt = [n+1]_q \begin{bmatrix} n \\ k \end{bmatrix}_q \int_0^1 t^{m+k}(qt;q)_{n-k}\,d_qt.$$

Taking (1.3) and (1.7) into account, one obtains

$$A_{nk}(q;t^m) = [n+1]_q \frac{(q;q)_n(1-q)}{(q;q)_k(q;q)_{n-k}} \sum_{j=0}^{\infty} (q^{m+k+1})^j (q^{j+1};q)_{n-k}.$$

As $(q^{j+1};q)_{n-k} = (q;q)_{n-k+j}/(q;q)_j$, one gets

$$\begin{aligned} A_{nk}(q;t^m) &= [n+1]_q \frac{(q;q)_n(1-q)}{(q;q)_k(q;q)_{n-k}} \sum_{j=0}^{\infty} (q^{m+k+1})^j \frac{(q;q)_{n-k+j}}{(q;q)_j} \\ &= [n+1]_q \frac{(q;q)_n(1-q)}{(q;q)_k} \sum_{j=0}^{\infty} \begin{bmatrix} n-k+j \\ j \end{bmatrix}_q (q^{m+k+1})^j. \end{aligned}$$

Applying the Rothe identity (1.6) to the last sum, one obtains

$$A_{nk}(q;t^m) = \frac{(q;q)_{n+1}}{(q;q)_k (q^{m+k+1};q)_{n-k+1}} = \frac{(q^{k+1};q)_m}{(q^{n+2};q)_m},$$

which completes the proof. □

At this stage, one has

$$D_{n,q}(t^m;x) = \sum_{k=0}^{n} A_{nk}(q;t^m)p_{nk}(q;x) = \sum_{k=0}^{n} \frac{(q^{k+1};q)_m}{(q^{n+2};q)_m} p_{nk}(q;x).$$

Expanding the product $(q^{k+1};q)_m$ in powers of $[k]_q$, one can express the moments of q-Durrmeyer operators in terms of the moments of q-Bernstein polynomials as follows:

$$D_{n,q}(t^m;x) = \frac{1}{(q^{n+2};q)_m}\sum_{r=0}^{m}\beta_q(r,m)[n]_q^r B_{n,q}(t^r;x), \tag{2.1}$$

in which $\beta_q(r,m) > 0$ are coefficients defined by

$$(q^{k+1};q)_m = \sum_{r=0}^{m}\beta_q(r,m)[k]_q^r. \tag{2.2}$$

In the next lemma, we propose the recurrence relation for the coefficients $\beta_q(r,m)$.

Lemma 2.2 *For $m = 0,1,\ldots,$ the coefficients $\beta_q(r,m)$ in (2.2) obey the conditions: $\beta_q(0,m) = (q;q)_m$, $\beta_q(m,m) = (1-q)^m q^{m(m+1)/2}$, $\beta_q(r,m) = 0$ if $r > m$ or $r < 0$ and*

$$\beta_q(r,m) = (1-q^m)\beta_q(r,m-1) + q^m(1-q)\beta_q(r-1,m-1), \quad r = 1,2,\ldots,m-1. \tag{2.3}$$

Proof Obviously, for $m = 0$, equality (2.2) implies that $\beta_q(0,0) = 1$. Also, it is evident that $\beta_q(r,m) = 0$ if $r > m$ or $r < 0$. Further, assuming such an expression as in (2.2), one has

$$\begin{aligned}\sum_{r=0}^{m}\beta_q(r,m)[k]_q^r &= (q^{k+1};q)_m = (q^{k+1};q)_{m-1}(1-q^{k+m})\\ &= (1-q^m+q^m(1-q)[k]_q)\sum_{r=0}^{m-1}\beta_q(r,m-1)[k]_q^r\\ &= \sum_{r=0}^{m-1}(1-q^m)\beta_q(r,m-1)[k]_q^r + \sum_{r=1}^{m}q^m(1-q)\beta_q(r-1,m-1)[k]_q^r.\end{aligned}$$

As $\beta_q(m,m-1) = \beta_q(-1,m-1) = 0$, the sums on the right side can be written as a single sum:

$$\sum_{r=0}^{m}\beta_q(r,m)[k]_q^r = \sum_{r=0}^{m}\left((1-q^m)\beta_q(r,m-1) + q^m(1-q)\beta_q(r-1,m-1)\right)[k]_q^r.$$

Comparing the coefficients of $[k]_q^r$ on both sides, for $r = 0$, one gets

$$\beta_q(0,m) = (1-q^m)\beta_q(0,m-1)$$

which gives

$$\beta_q(0,m) = (1-q^m)(1-q^{m-1})\cdots(1-q)\beta_q(0,0) = (q;q)_m.$$

Also, for $r = m$, one finds

$$\beta_q(m,m) = q^m(1-q)\beta_q(m-1,m-1),$$

which gives

$$\beta_q(m,m) = (1-q)^m q^m q^{m-1}\cdots q\beta_q(0,0) = (1-q)^m q^{m(m+1)/2}.$$

Finally, for $r = 1, 2, \ldots, m-1$, one receives (2.3). □

The next theorem gives the explicit formula for the moments $D_{n,q}(t^m; x)$. To present the result, one needs q-Stirling numbers of the second kind [8],

$$S_q(i,j) = \frac{1}{[j]_q!\, q^{j(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix}_q [j-r]_q^i, \tag{2.4}$$

with $S_q(0,0) = 1$, $S_q(i,0) = 0$ for $i > 0$, $S_q(i,j) = 0$ for $j > i$ and the eigenvalues $\lambda_{m,q}^{(n)}$ of the q-Bernstein operators [19],

$$\lambda_{0,q}^{(n)} = \lambda_{1,q}^{(n)} = 1, \quad \lambda_{m,q}^{(n)} = \left(1 - \frac{1}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right) \cdots \left(1 - \frac{[m-1]_q}{[n]_q}\right), \quad m = 2, 3, \ldots, n.$$

With the help of the numbers $\beta_q(r,m)$, one can evaluate $D_{n,q}(t^m; x)$.

Theorem 2.3 *For $m = 0, 1, \ldots,$ there holds*

$$D_{n,q}(t^m; x) = \sum_{r=0}^{m} a_{n,q}(r,m) x^r,$$

where

$$a_{n,q}(r,m) = \frac{[n]_q^r \, \lambda_{r,q}^{(n)}}{(q^{n+2}; q)_m} \sum_{i=r}^{m} \beta_q(i,m) S_q(i,r). \tag{2.5}$$

Proof It is known that $B_{n,q}(t^r, x)$ is a polynomial of degree $\min\{r, n\}$ (see [18]) and has the form [8],

$$B_{n,q}(t^r; x) = \sum_{i=0}^{r} \frac{S_q(r,i)}{[n]_q^{r-i}} \lambda_{i,q}^{(n)} x^i. \tag{2.6}$$

Putting (2.6) into (2.1), one obtains

$$D_{n,q}(t^m; x) = \frac{1}{(q^{n+2}; q)_m} \sum_{r=0}^{m} \sum_{i=0}^{r} \beta_q(r,m) [n]_q^i S_q(r,i) \lambda_{i,q}^{(n)} x^i.$$

Changing the order of sums and swapping the roles of r and i, one obtains

$$D_{n,q}(t^m; x) = \sum_{r=0}^{m} \left(\frac{[n]_q^r \, \lambda_{r,q}^{(n)}}{(q^{n+2}; q)_m} \sum_{i=r}^{m} \beta_q(i,m) S_q(i,r) \right) x^r,$$

as desired. □

3. On the eigenconfiguration of $D_{n,q}$

In this section, the eigenvalues and the corresponding eigenfunctions of $D_{n,q}$ are obtained. Notice that, in distinction to $B_{n,q}$, all eigenvalues of $D_{n,q}$ are simple.

Theorem 3.1 *For all* $0 < q < 1$, *the operator* $D_{n,q}$ *has* $(n+1)$ *simple eigenvalues* $\eta_{m,q}^{(n)}$ *given by*

$$\eta_{m,q}^{(n)} := a_{n,q}(m,m) = \frac{[n]_q^m \, \lambda_{m,q}^{(n)}}{(q^{n+2};q)_m}(1-q)^m q^{m(m+1)/2}, \quad m = 0,1,\dots,n. \tag{3.1}$$

Proof Notice that $D_{n,q}(1;x) = 1$ and, for $m = 1,\dots,n$, the polynomial $D_{n,q}(t^m;x)$ can be written as

$$D_{n,q}(t^m;x) = \eta_{m,q}^{(n)} x^m + P_{m-1}(x), \tag{3.2}$$

where P_{m-1} is a polynomial of degree at most $m-1$ and $\eta_{m,q}^{(n)}$ are given by (3.1). For $j < m \leqslant n$, the fraction $\eta_{m,q}^{(n)}/\eta_{j,q}^{(n)}$ equals

$$\frac{\eta_{m,q}^{(n)}}{\eta_{j,q}^{(n)}} = \frac{[n]_q^{m-j}(1-q)^{m-j} q^{[m(m+1)-j(j+1)]/2} \lambda_{m,q}^{(n)}}{(q^{n+j+2};q)_{m-j} \lambda_{j,q}^{(n)}}.$$

Obviously,

$$\frac{[n]_q^{m-j}(1-q)^{m-j}}{(q^{n+j+2};q)_{m-j}} = \prod_{k=j+2}^{m+1} \frac{1-q^n}{1-q^{n+k}} < 1$$

and

$$\frac{\lambda_{m,q}^{(n)}}{\lambda_{j,q}^{(n)}} = \prod_{k=j}^{m-1} \left(1 - \frac{[k]_q}{[n]_q}\right) < 1$$

Therefore, $\eta_{m,q}^{(n)}/\eta_{j,q}^{(n)} < 1$, which gives $\eta_{m,q}^{(n)} < \eta_{j,q}^{(n)}$ for $j < m$, meaning that $\eta_{m,q}^{(n)}$ are distinct.

By (3.2), the matrix representation of $D_{n,q}$ in the standard basis $\{1, x, x^2, \dots, x^n\}$ has the form

$$\begin{bmatrix} 1 & * & * & * & \dots & * \\ 0 & \eta_{1,q}^{(n)} & * & * & \dots & * \\ 0 & 0 & \eta_{2,q}^{(n)} & * & \dots & * \\ 0 & 0 & 0 & \eta_{3,q}^{(n)} & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & * \\ 0 & 0 & 0 & 0 & \dots & \eta_{n,q}^{(n)} \end{bmatrix}.$$

Therefore, the numbers $\eta_{m,q}^{(n)}$, $m = 0,\dots,n$ are the eigenvalues of $D_{n,q}$. □

Remark 3.2 *It is worth pointing out that* $\eta_{m,1}^{(n)}$ *are the eigenvalues of the classical Bernstein-Durrmeyer operators.*

Theorem 3.3 *For* $n \in \mathbb{N}$ *and* $m = 0,1,\dots,n$, *the monic polynomials* $\varphi_m^{(n)}(q;x)$, *which are the eigenfunctions of* $D_{n,q}(f;x)$ *corresponding to the eigenvalue of* $\eta_{m,q}^{(n)}$, *have the form*

$$\varphi_m^{(n)}(q;x) = \sum_{r=0}^{m} c_{n,q}(r,m) x^r, \tag{3.3}$$

where $c_{n,q}(m,m) = 1$ *and*

$$c_{n,q}(m-j,m) = \frac{1}{\eta_{m,q}^{(n)} - \eta_{m-j,q}^{(n)}} \sum_{i=0}^{j-1} c_{n,q}(m-i,m) a_{n,q}(m-j,m-i), \quad j = 1,2,\ldots,m-1. \tag{3.4}$$

Proof Let the eigenfunctions of $D_{n,q}(f;x)$ be represented by

$$\varphi_m^{(n)}(q;x) = \sum_{r=0}^{m} c_{n,q}(r,m) x^r, \quad c_{n,q}(m,m) := 1. \tag{3.5}$$

Since $\varphi_m^{(n)}(q;x)$ are the eigenfunctions of $D_{n,q}(f;x)$, one can write

$$D_{n,q}(\varphi_m^{(n)}(q;x);x) = \eta_{m,q}^{(n)} \varphi_m^{(n)}(q;x) \tag{3.6}$$

In view of the expression (3.5), (3.6) becomes

$$\begin{aligned}\eta_{m,q}^{(n)} \sum_{s=0}^{m} c_{n,q}(s,m) x^s &= \sum_{r=0}^{m} c_{n,q}(r,m) D_{n,q}(t^r;x) \\ &= \sum_{r=0}^{m} c_{n,q}(r,m) \sum_{s=0}^{r} a_{n,q}(s,r) x^s = \sum_{s=0}^{m} \sum_{r=s}^{m} c_{n,q}(r,m) a_{n,q}(s,r) x^s.\end{aligned}$$

Comparing the coefficient of x^s, one gets

$$\eta_{m,q}^{(n)} c_{n,q}(s,m) = \sum_{r=s}^{m} c_{n,q}(r,m) a_{n,q}(s,r).$$

Replacing s by $m-j$ and r by $m-i$ yields

$$\eta_{m,q}^{(n)} c_{n,q}(m-j,m) = \sum_{i=0}^{j} c_{n,q}(m-i,m) a_{n,q}(m-j,m-i)$$

which gives

$$c_{n,q}(m-j,m) = \frac{1}{\eta_{m,q}^{(n)} - \eta_{m-j,q}^{(n)}} \sum_{i=0}^{j-1} c_{n,q}(m-i,m) a_{n,q}(m-j,m-i),$$

as desired. □

4. On the limit operators

The focus is on the limit q-Durrmeyer operator introduced in [10]. This operator emerged as a limit of $D_{n,q}(f;x)$. More precisely, it was shown that $\{D_{n,q}\} \to D_{\infty,q}$ as $n \to \infty$ in strong operator topology.

Writing

$$p_{\infty k}(q;x) = \lim_{n\to\infty} p_{nk}(q;x) = \frac{x^k (x;q)_\infty}{(q;q)_k}, \quad k = 0,1,\ldots,$$

the operator $D_{\infty,q}$ can be expressed as follows:

Definition 4.1 *[10] Let $q \in (0,1)$, $f \in C[0,1]$. The limit q-Durrmeyer operator is defined by*

$$D_{\infty,q}(f;x) = \begin{cases} \sum_{k=0}^{\infty} A_{\infty k}(q;f)p_{\infty k}(q;x), & x \in [0,1), \\ f(1), & x = 1 \end{cases}$$

where

$$A_{\infty k}(q;f) := \frac{q^{-k}}{1-q}\int_0^1 f(t)p_{\infty k}(q;qt)\,d_q t, \tag{4.1}$$

and

$$p_{\infty k}(q;x) = \frac{(x,q)_\infty\, x^k}{(q,q)_k}, \quad k = 0,1,\ldots$$

Lemma 4.2 *For all $m = 0,1,\ldots,$ one has*

$$A_{\infty k}(q;t^m) = (q^{k+1};q)_m, \quad k = 0,1,\ldots \tag{4.2}$$

Proof Using (4.1) with $f(t) = t^m$, one can write

$$A_{\infty k}(q;t^m) = \frac{q^{-k}}{1-q}\int_0^1 t^m p_{\infty k}(q;qt)\,d_q t = \frac{1}{(1-q)(q;q)_k}\int_0^1 t^{m+k}(qt;q)_\infty\,d_q t.$$

Applying (1.7) and the relation $(q;q)_\infty = (q;q)_j(q^{j+1};q)_\infty$, one obtains

$$A_{\infty k}(q;t^m) = \frac{1}{(q;q)_k}\sum_{j=0}^{\infty}(q^{m+k+1})^j(q^{j+1};q)_\infty = \frac{(q;q)_\infty}{(q;q)_k}\sum_{j=0}^{\infty}\frac{(q^{m+k+1})^j}{(q;q)_j}.$$

By virtue of the Euler identity (1.5), one arrives at

$$A_{\infty k}(q;t^m) = \frac{(q;q)_\infty}{(q;q)_k(q^{m+k+1};q)_\infty} = (q^{k+1};q)_m,$$

as stated. □

The next theorem gives the explicit formula for the moments of $D_{\infty,q}$.

Theorem 4.3 *For all $m = 0,1,\ldots,$ one has*

$$D_{\infty,q}(t^m;x) = \sum_{k=0}^{m} a_q^*(k,m)x^k, \tag{4.3}$$

where

$$a_q^*(k,m) = \sum_{s=k}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} s \\ k \end{bmatrix}_q (-1)^{k+s} q^{(s(s+1)+k(k-1))/2}. \tag{4.4}$$

Proof Using (4.1) and (4.2) results in

$$D_{\infty,q}(t^m;x) = \sum_{k=0}^{\infty} A_{\infty k}(q;t^m)p_{\infty k}(q;x) = (x;q)_\infty\sum_{k=0}^{\infty}\frac{(q^{k+1};q)_m x^k}{(q;q)_k}.$$

Employing Gauss q-binomial formula (1.4) leads to

$$(q^{k+1};q)_m = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q (-1)^s q^{s(s-1)/2} (q^{k+1})^s,$$

which allows to write

$$\begin{aligned} D_{\infty,q}(t^m;x) &= (x;q)_\infty \sum_{k=0}^{\infty} \left(\sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q (-1)^s q^{s(s-1)/2} (q^{k+1})^s \right) \frac{x^k}{(q;q)_k} \\ &= (x;q)_\infty \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q (-1)^s q^{s(s-1)/2} q^s \left(\sum_{k=0}^{\infty} \frac{(q^s x)^k}{(q;q)_k} \right). \end{aligned}$$

By Euler identity (1.5), one gets

$$D_{\infty,q}(t^m;x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q (-1)^s q^{s(s+1)/2} (x;q)_s.$$

Again, taking into account Gauss q-binomial formula (1.4), one comes up with

$$\begin{aligned} D_{\infty,q}(t^m;x) &= \sum_{s=0}^{m} \sum_{k=0}^{s} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} s \\ k \end{bmatrix}_q (-1)^{k+s} q^{(s(s+1)+k(k-1))/2} x^k \\ &= \sum_{k=0}^{m} \sum_{s=k}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} s \\ k \end{bmatrix}_q (-1)^{k+s} q^{(s(s+1)+k(k-1))/2} x^k, \end{aligned}$$

which completes the proof. □

The previous theorem will be used to describe the polynomial eigenfunctions of $D_{\infty,q}$.

Theorem 4.4 *For each $m = 0, 1, \ldots,$ the operator $D_{\infty,q}$ has a polynomial eigenfunction of degree m, corresponding to the eigenvalue $\eta^*_{m,q} = q^{m^2}$. The monic polynomial $\varphi^*_m(q;x)$ has the explicit representation*

$$\varphi^*_m(q;x) = \sum_{r=0}^{m} c^*_q(r,m) x^r, \tag{4.5}$$

where $c^*_q(m,m) = 1$ and for $j = 1, 2, \ldots, m$,

$$c^*_q(m-j,m) = \frac{1}{q^{m^2} - q^{(m-j)^2}} \sum_{i=0}^{j-1} c^*_q(m-i,m) a^*_q(m-j,m-i).$$

Proof Using (4.3), one writes

$$D_{\infty,q}(t^m;x) = a^*_q(m,m)x^m + \text{l.o.t.},$$

where, by (4.4), $a^*_q(m,m) = q^{m^2}$ and l.o.t. stands for "lower order terms". Also, (4.3) implies that the vector space $\mathcal{P}_n$ of polynomials of degree at most n is invariant under $D_{\infty,q}$ and that the matrix representation of $D_{\infty,q}$ in the standart basis $\{1, x, x^2, \ldots, x^n\}$ of $\mathcal{P}_n$ is upper triangular matrix with diagonal entries being

$\eta^*_{m,q} = q^{m^2}$, $m = 0,1,\ldots,n$. Therefore, $\eta^*_{m,q} = q^{m^2}$ are the eigenvalues of $D_{\infty,q}$. As the eigenvalues are distinct, there exists a one-dimensional eigenspace associated with q^{m^2} which is spanned by a polynomial of degree exactly m.

Let $\varphi^*_m(q;x)$ be a monic eigenpolynomial of degree m given by (4.5) with $c^*_q(m,m) = 1$. Using the fact that $D_{\infty,q}(\varphi^*_m;x) = q^{m^2}\varphi^*_m(q;x)$ results in

$$q^{m^2}\sum_{s=0}^{m} c^*_q(s,m)x^s = \sum_{r=0}^{m} c^*_q(r,m)D_{\infty,q}(t^r;x) = \sum_{r=0}^{m} c^*_q(r,m)\sum_{s=0}^{r} a^*_q(s,r)x^s = \sum_{s=0}^{m}\sum_{r=s}^{m} c^*_q(r,m)a^*_q(s,r)x^s.$$

Comparing the coefficient of x^s in both sides brings about

$$q^{m^2}c^*_q(s,m) = \sum_{r=s}^{m} c^*_q(r,m)a^*_q(s,r) = c^*_q(s,m)a^*_q(s,s) + \sum_{r=s+1}^{m} c^*_q(r,m)a^*_q(s,r).$$

Hence

$$c^*_q(s,m) = \frac{1}{q^{m^2} - q^{s^2}}\sum_{r=s+1}^{m} c^*_q(r,m)a^*_q(s,r).$$

Replacing s by $m-j$ and r by $m-i$, one obtains the desired result. □

In the remaining part of the paper, the asymptotic behaviour, when $n\to\infty$, of the eigenfunctions (3.3) is investigated. To this aim, one needs the following auxiliary results.

Lemma 4.5 *Let $\beta_q(r,m)$ be as in (2.2) and $S_q(i,r)$ be the q-Stirling numbers of the second kind as in (2.4). Then, for $0 \leqslant r \leqslant m$, one has*

$$\sum_{i=r}^{m}\beta_q(i,m)S_q(i,r) = (1-q)^r\sum_{s=r}^{m}\begin{bmatrix} m \\ s\end{bmatrix}_q\begin{bmatrix} s \\ r\end{bmatrix}_q(-1)^{r+s}q^{s(s+1)/2}.$$

Proof Using $S_q(i,r) = 0$ for $r > i$ and (2.4), one has

$$\begin{aligned}\sum_{i=r}^{m}\beta_q(i,m)S_q(i,r) &= \frac{1}{[r]_q!\,q^{r(r-1)/2}}\sum_{i=0}^{m}\beta_q(i,m)\sum_{j=0}^{r}\begin{bmatrix} r \\ j\end{bmatrix}_q(-1)^j q^{j(j-1)/2}[r-j]_q^i \\ &= \frac{1}{[r]_q!\,q^{r(r-1)/2}}\sum_{j=0}^{r}\begin{bmatrix} r \\ j\end{bmatrix}_q(-1)^j q^{j(j-1)/2}\sum_{i=0}^{m}\beta_q(i,m)[r-j]_q^i.\end{aligned}$$

By virtue of (2.2),

$$\sum_{i=r}^{m}\beta_q(i,m)S_q(i,r) = \frac{1}{[r]_q!\,q^{r(r-1)/2}}\sum_{j=0}^{r}\begin{bmatrix} r \\ j\end{bmatrix}_q(-1)^j q^{j(j-1)/2}(q^{r-j+1};q)_m.$$

Taking into account the Gauss q-binomial formula (1.4), one obtains

$$\begin{aligned}\sum_{i=r}^{m}\beta_q(i,m)S_q(i,r) &= \frac{1}{[r]_q!\,q^{r(r-1)/2}}\sum_{j=0}^{r}\begin{bmatrix} r \\ j\end{bmatrix}_q(-1)^j q^{j(j-1)/2}\sum_{s=0}^{m}\begin{bmatrix} m \\ s\end{bmatrix}_q(-1)^s q^{s(s-1)/2}(q^{r-j+1})^s \\ &= \frac{1}{[r]_q!\,q^{r(r-1)/2}}\sum_{s=0}^{m}\begin{bmatrix} m \\ s\end{bmatrix}_q(-1)^s q^{s(s-1)/2}(q^{r+1})^s\sum_{j=0}^{r}\begin{bmatrix} r \\ j\end{bmatrix}_q(-1)^j q^{j(j-1)/2}(q^{-s})^j.\end{aligned}$$

Using (1.4) again, one ends up with

$$\sum_{i=r}^{m} \beta_q(i,m) S_q(i,r) = \frac{1}{[r]_q!\, q^{r(r-1)/2}} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q (-1)^s q^{s(s-1)/2} (q^{r+1})^s (q^{-s};q)_r.$$

Since $(q^{-s};q)_r = 0$ for $s = 0,1,\ldots,r-1$ and

$$(q^{-s};q)_r = (-1)^r q^{-sr+\frac{r(r-1)}{2}} \frac{(q;q)_s}{(q;q)_{s-r}},$$

one arrives at

$$\begin{aligned}\sum_{i=r}^{m} \beta_q(i,m) S_q(i,r) &= \frac{1}{[r]_q!} \sum_{s=r}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q (-1)^{r+s} q^{s(s+1)/2} \frac{(q;q)_s}{(q;q)_{s-r}} \\ &= (1-q)^r \sum_{s=r}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} s \\ r \end{bmatrix}_q (-1)^{r+s} q^{s(s+1)/2},\end{aligned}$$

which completes the proof. □

The next result is on the limits of the coefficients of the moments (2.5) and (4.4).

Lemma 4.6 *For* $0 \leqslant r \leqslant m,$ *one has*

$$\lim_{n\to\infty} a_{n,q}(r,m) = a_q^*(r,m).$$

Proof Indeed, $\lim_{n\to\infty} [n]_q^r = (1-q)^{-r}$ and $\lim_{n\to\infty} \lambda_{r,q}^{(n)} = q^{r(r-1)/2}$. Hence, using Lemma 4.5 and the fact that $(q^{n+2};q)_m \to 1$ as $n \to \infty$, one receives the result on taking the limit on both sides of (2.5). □

Corollary 4.7 *For* $r = m$*, the following result is valid:*

$$\lim_{n\to\infty} \eta_{m,q}^{(n)} = q^{m^2}, \quad m = 0,1,\ldots$$

The assertion below demonstrates the uniform convergence of the eigenfunctions of $D_{n,q}$ to those of $D_{\infty,q}$ on any compact set.

Theorem 4.8 *For* $0 \leqslant r \leqslant m,$ *one has*

$$\lim_{n\to\infty} c_{n,q}(r,m) = c_q^*(r,m).$$

Proof Obviously, the statement is true for $r = m$. Suppose that $\lim_{n\to\infty} c_{n,q}(m-i,m) = c_q^*(m-i,m)$ exists for $i = 0,1,\ldots,r-1$ where $1 \leqslant r \leqslant m$. Notice that

$$\lim_{n\to\infty} \eta_{m,q}^{(n)} = \eta_{m,q}^*.$$

Now, taking the limits of both sides of (3.4) and using the induction hypothesis together with Lemma 4.6, one gets,

$$\lim_{n\to\infty} c_{n,q}(m-r,m) = \frac{1}{q^{m^2} - q^{(m-j)^2}} \sum_{i=0}^{r-1} c_q^*(m-i,m) a_q^*(m-r,m-i) = c_q^*(m-r,m),$$

which gives the desired result. □

5. Simulations

This section consists of some illustrative graphs for the eigenfunctions $\varphi_m^{(n)}(q;x)$ and $\varphi_m^*(q;x)$. Figure 1 shows the eigenfunctions $\varphi_m^{(6)}(q;x)$, for $m=0,1,\ldots,6$, normalized to have the uniform norm 1. In Figure 2, several members of the sequence $\{\varphi_4^{(n)}(q;x)\}$ together with their limit $\varphi_4^*(q;x)$ are drawn. As supported by graphics, the eigenfunctions of $D_{n,q}$ converge to those of $D_{\infty,q}$. Figure 3 indicates the eigenfunctions $\varphi_m^*(q;x)$, $m=0,1,\ldots,5$ of the limit q-Durrmeyer operator $D_{\infty,q}$. In Figures 4 and 5, unlike in the previous ones, the parameter q is varied and the eigenfunctions $\{\varphi_5^{(5)}(q;x)\}$ of the operators $D_{5,q}$ and $\{\varphi_5^*(q;x)\}$ of the operators $D_{\infty,q}$, respectively, for several values of q, are shown.

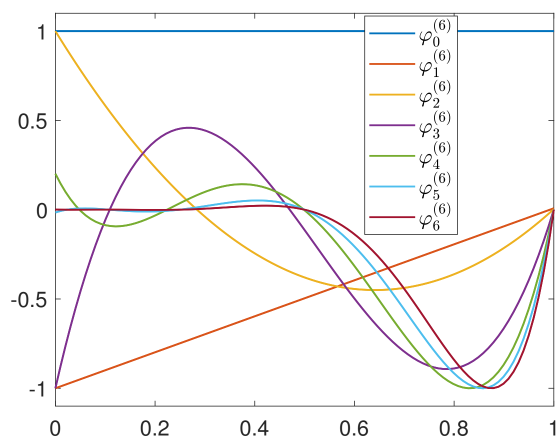

Figure 1. The normalized eigenfunctions of $D_{6,q}$ for $q=0.5$.

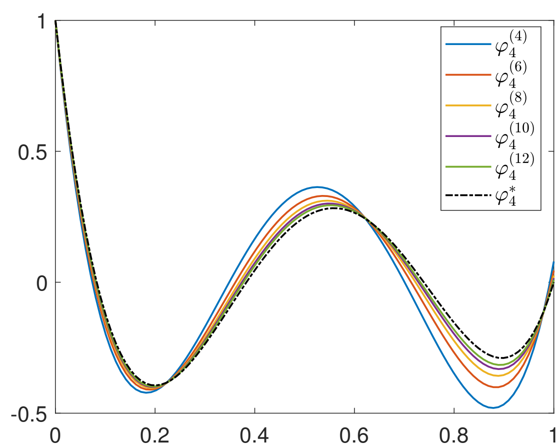

Figure 2. The eigenfunctions $\varphi_4^{(n)}(q;x)$ for different values of n and $\varphi_4^*(q;x)$ for $q=0.8$.

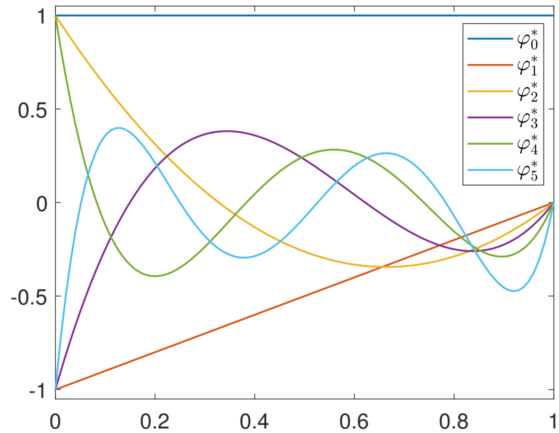

Figure 3. Several normalized eigenfunctions of $D_{\infty,q}$ for $q = 0.8$.

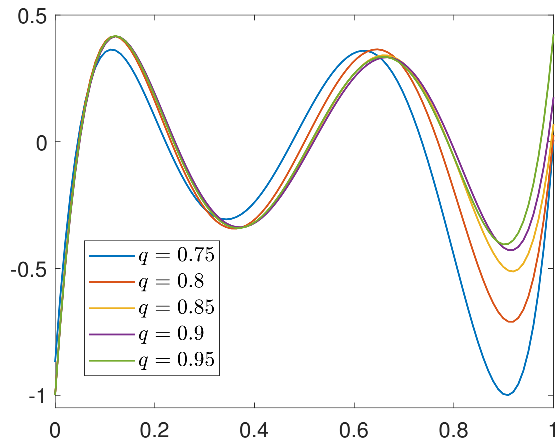

Figure 4. The eigenfunctions $\varphi_5^{(5)}(q;x)$ for a few values of q.

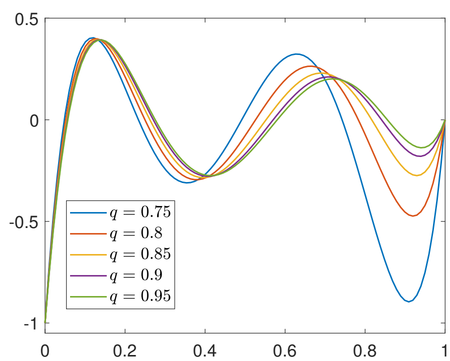

Figure 5. The eigenfunctions $\varphi_5^*(q;x)$ for a few values of q.

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