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Operators affiliated to Banach lattice properties and their enveloping norms

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Abstract: Several recent papers were devoted to various modifications of limited, Grothendieck, and Dunford–Pettis operators, etc., through involving the Banach lattice structure. In the present paper, it is shown that many of these operators appear as operators affiliated to well-known properties of Banach lattices, like the disjoint (dual) Schur property, the disjoint Grothendieck property, the property (d), the sequential w*-continuity of the lattice operations, etc. We also introduce new classes of operators such as the s-GPP-operators, s-BDP-operators, and bi-sP-operators. It is proved that the spaces consisting of regular versions of the above-mentioned operators are all the Banach spaces. The domination problem for these operators is investigated.

Key words: Banach lattice, affiliated operators, enveloping norm, domination problem

1. Introduction and preliminaries

Throughout the paper, vector spaces are real; operators are linear and bounded; letters X, Y stand for Banach spaces; and E, F for Banach lattices. We denote by B_X the closed unit ball of X; by $\mathrm{L}(X,Y)$ the space of all bounded operators from X to Y; and by E_+ the positive cone of E. An operator $T : E \to F$ is called *regular* if $T = T_1 - T_2$ for some $T_1, T_2 \in \mathrm{L}_+(E,F)$. We denote by $\mathrm{L}_r(E,F)$ ($\mathrm{L}_{ob}(E,F)$, $\mathrm{L}_{oc}(E,F)$) the space of all regular (o-bounded, o-continuous) operators from E to F.

1.1. Recall that a bounded $A \subseteq X$ is said to be a *limited* set (resp. a DP-*set*) if each w*-null (resp. w-null) sequence in X' is uniformly null on A. Similarly, a bounded $A \subseteq E$ is called an a-*limited* set (resp. an a-DP-*set*) if each disjoint w*-null (resp. disjoint w-null) sequence in E' is uniformly null on A (cf. [5, 6, 11, 15]). Each relatively compact set is limited, and each limited set is an a-limited DP-set, and each DP-set is an a-DP-set.

Proposition 1.1 (cf. [9]) Let $A \subseteq X$ be limited. Then:

(i) Every sequence in A has a w-Cauchy subsequence.

(ii) If X is either separable or else reflexive, then A is relatively compact.

(iii) If ℓ^1 does not embed in X, then A is relatively w-compact.

We include proof of the following useful technical fact.

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Lemma 1.2 Let $A \subseteq X$ and $B \subseteq X'$ be nonempty. Then:

(i) A sequence (f_n) in X' is uniformly null on A iff $f_n(a_n) \to 0$ for each sequence (a_n) in A.

(ii) A sequence (x_n) in X is uniformly null on B iff $b_n(x_n) \to 0$ for each sequence (b_n) in B.

Proof i) The necessity is obvious. Let $f_n(a_n) \to 0$ for each (a_n) in A. Suppose $\lim\limits_{n\to\infty}\sup(\sup\limits_{a\in A}|f_n(a)|) \geq 3\varepsilon > 0$. Choose an increasing sequence (n_k) satisfying $\sup\limits_{a\in A}|f_{n_k}(a)| \geq 2\varepsilon$ for all $k \in \mathbb{N}$, and pick $a_{n_k} \in A$ with $|f_{n_k}(a_{n_k})| \geq \varepsilon$ for each k. Letting $a_n := a_{n_1}$ for all $n \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$ gives $f_n(a_n) \not\to 0$. The obtained contradiction proves that (f_n) is uniformly null on A.
ii) The proof is similar. □

A bounded $B \subseteq X'$ (resp. $B \subseteq E'$) is called an L-*set* (resp. an a-L-*set*) if each w-null sequence in X (resp. each disjoint w-null sequence in E) is uniformly null on B (cf. [22]). The next fact follows from Lemma 1.2.

Proposition 1.3 A bounded subset A of X is

(i) limited iff $f_n(a_n) \to 0$ for all w*-null (f_n) in X' and all (a_n) in A;

(ii) a DP-set iff $f_n(a_n) \to 0$ for all w-null (f_n) in X' and all (a_n) in A.

A bounded subset B of X' is

(iii) an L-set iff $b_n(x_n) \to 0$ for all (b_n) in B and all w-null (x_n) in X.

A bounded subset A of E is

(iv) a-limited iff $f_n(a_n) \to 0$ for all disjoint w*-null (f_n) in E' and all (a_n) in A;

(v) an a-DP-set iff $f_n(a_n) \to 0$ for all disjoint w-null (f_n) in E' and all (a_n) in A.

A bounded subset B of E' is

(vi) an a-L-set iff $b_n(x_n) \to 0$ for all (b_n) in B and all disjoint w-null (x_n) in E.

1.2. We recall the following properties of Banach spaces and describe operators affiliated to these properties.

Definition 1.4 A Banach space X is said to possess:

a) the *Schur property* (briefly, $X \in$ (SP)) if each w-null sequence in X is norm null;

b) the *Grothendieck property* (briefly, $X \in$ (GP)) if each w*-null sequence in X' is w-null;

c) the *Dunford–Pettis property* (briefly, $X \in$ (DPP)) if $f_n(x_n) \to 0$ for each w-null (f_n) in X' and each w-null (x_n) in X;

d) the *Gelfand–Phillips property* (briefly, $X \in$ (GPP)) if each limited subset of X is relatively compact (cf. [22, p.424]).

e) the *Bourgain–Diestel property* (briefly, $X \in$ (BDP)) if each limited subset of X is relatively w-compact [19].

Dedekind complete AM-spaces with a strong order unit belong to (GP), for a comprehensive recent source on the Grothendieck property see [23]. All separable and all reflexive Banach spaces belong to (GPP) [9]. A Dedekind σ-complete Banach lattice E belongs (GPP) iff E has o-continuous norm [10]. In particular, $c_0, \ell^1 \in$ (GPP), yet $\ell^\infty \notin$ (GPP). Clearly, (GPP) $\Rightarrow$ (BDP). By [9], $X \in$ (BDP) whenever X contains no copy of ℓ^1. Redistributing properties (cf. [2, 24]) between the domain and range in Definition 1.4, we obtain the following list of affiliated operators.

Definition 1.5 An operator $T : X \to Y$ is called:

a) an [SP]-*operator* if (Tx_n) is norm null for each w-null (x_n) in X;

b) a [GP]-*operator* if $(T'f_n)$ is w-null in X' for each w*-null (f_n) in Y';

c) a [DPP]-*operator* if $f_n(Tx_n) \to 0$ for each w-null (f_n) in Y' and each w-null (x_n) in X;

d) a [GPP]-*operator* if T carries limited sets onto relatively compact sets;

e) a [BDP]-*operator* if T carries limited sets onto relatively w-compact sets.

Note that the [SP]-operators coincide with Dunford–Pettis operators, the [GP]-operators coincide with Grothendieck operators, whereas the [DPP]-operators agree with weak Dunford–Pettis operators of [1, p.349]. Although, this terminology may overlap the existing classical one, it looks more natural as it is based on properties of spaces rather than on sometimes artificial names for operators.

Definition 1.6 Let $\mathcal{P}$ be a class of operators between Banach spaces. A Banach space X is said to be *affiliated with* $\mathcal{P}$ if $I_X \in \mathcal{P}$. In this case we write $X \in (\mathcal{P})$.

It should be clear that if P is one of the five properties mentioned in Definition 1.4, then $X \in (P)$ iff X affiliated with $[P]$-operators; symbolically $([P]) = P$. It is worth noticing that the reflexivity/finite dimensionality of Banach spaces is affiliated with w-compact/compact operators and vice versa.

1.3. We recall the following classes of operators.

Definition 1.7 An operator

a) $T : X \to F$ is *almost Grothendieck* (shortly, T is a-G) if T' takes disjoint w*-null sequences of F' to w-null sequences of X' [21, Def.3.1].

b) $T : X \to F$ is *almost limited* (shortly, T is Lm) if $T(B_X)$ is a-limited; i.e. T' takes disjoint w*-null sequences of F' to norm null sequences of X' [16].

c) $T : E \to Y$ is *almost Dunford–Pettis* (shortly, T is a-DP) if T takes disjoint w-null sequences to norm null ones [30].

d) $T : E \to Y$ is *almost weak Dunford–Pettis* (shortly, T is a-wDP) if $f_n(Tx_n) \to 0$ whenever (f_n) is w-null in Y' and (x_n) is disjoint w-null in E.

e) $T : E \to Y$ is o-*limited* (shortly, T is o-Lm) if $T[0,x]$ is limited for all $x \in E_+$; i.e. $(T'f_n)$ is uniformly null on all order intervals $[0,x] \subseteq E_+$ for each w*-null (f_n) of Y' [26].

f) $T : E \to F$ is *almost* o-*limited* (shortly, T is a-o-Lm) if $T[0,x]$ is a-limited for all $x \in E_+$; i.e. $(T'f_n)$ is uniformly null on all order intervals $[0,x] \subseteq E_+$ for each disjoint w*-null (f_n) of F' [27, Def.3.1].

Clearly: a-Lm$(X,F) \subseteq$ a-G(X,F); a-DP$(E,Y) \subseteq$ a-wDP(E,Y); Lm$(E,Y) \subseteq$ o-Lm(E,Y); and o-Lm$(E,F) \subseteq$ a-o-Lm(E,F).

Let $\mathcal{P} \subseteq \mathrm{L}(E,F)$. We refer to elements of $\mathcal{P}$ as $\mathcal{P}$-operators and denote by $\mathcal{P}(E,F) := \mathcal{P}$ the set of all $\mathcal{P}$-operators in $\mathrm{L}(E,F)$. The $\mathcal{P}$-operators satisfy the *domination property* if $S \in \mathcal{P}$ whenever $0 \le S \le T \in \mathcal{P}$. An operator $T \in \mathrm{L}(E,F)$ is said to be $\mathcal{P}$-*dominated* if $\pm T \le U$ for some $U \in \mathcal{P}$.

1.4. Enveloping norms. Regularly $\mathcal{P}$-operators were introduced in [3, 18] and the enveloping norms on such operators in [4, 18]. Here we reproduce some of the basic results. By [29, Prop.1.3.6], $\mathrm{L}_r(E,F)$ is a Banach space under the *regular norm* $\|T\|_r := \inf\{\|S\| : \pm T \le S \in \mathrm{L}(E,F)\}$. Moreover, $\|T\|_r = \inf\{\|S\| : S \in \mathrm{L}(E,F), |Tx| \le S|x| \ \forall x \in E\} \ge \|T\|$ for every $T \in \mathrm{L}_r(E,F)$. If F is Dedekind complete, then $(\mathrm{L}_r(E,F), \|\cdot\|_r)$ is a Banach lattice and $\|T\|_r = \|\,|T|\,\|$ for every $T \in \mathrm{L}_r(E,F)$.

Definition 1.8 Let $\mathcal{P} \subseteq \mathrm{L}(E,F)$. An operator $T : E \to F$ is called a *regularly* $\mathcal{P}$-*operator* (shortly, an r-$\mathcal{P}$-*operator*), if $T = T_1 - T_2$ with $T_1, T_2 \in \mathcal{P} \cap \mathrm{L}_+(E,F)$. We denote by: $\mathcal{P}_r(E,F)$ the set of all regular operators in $\mathcal{P}(E,F)$; and by r-$\mathcal{P}(E,F)$ the set of all regularly $\mathcal{P}$-operators in $\mathrm{L}(E,F)$.

Proposition 1.9 ([3, Prop.1.6.2]) *Let* $\mathcal{P} \subseteq \mathrm{L}(E,F)$, $\mathcal{P} \pm \mathcal{P} \subseteq \mathcal{P} \ne \emptyset$, *and* $T \in \mathrm{L}(E,F)$. *Then the following hold*

(i) T *is an* r-$\mathcal{P}$-*operator iff* T *is a* $\mathcal{P}$-*dominated* $\mathcal{P}$-*operator.*

(ii) *Suppose* $\mathcal{P}$-*operators satisfy the domination property and the modulus* $|T|$ *exists in* $\mathrm{L}(E,F)$. *Then* T *is an* r-$\mathcal{P}$-*operator iff* $|T| \in \mathcal{P}$.

The replacement of the space $\mathrm{L}(E,F)$ in the definition

$$\|T\|_r := \inf\{\|S\| : \pm T \le S \in \mathrm{L}(E,F)\}$$

of the regular norm on $\mathrm{L}_r(E,F)$ by an arbitrary subspace $\mathcal{P}(E,F) \subseteq \mathrm{L}(E,F)$:

$$\|T\|_{\text{r-}\mathcal{P}} := \inf\{\|S\| : \pm T \le S \in \mathcal{P}(E,F)\} \qquad (T \in \text{r-}\mathcal{P}(E,F)) \tag{1.1}$$

gives the so-called *enveloping norm* on r-$\mathcal{P}(E,F)$ [4]. Furthermore

$$\|T\|_{\text{r-}\mathcal{P}} = \inf\{\|S\| : S \in \mathcal{P} \ \& \ (\forall x \in E)\ |Tx| \le S|x|\} \quad (T \in \text{r-}\mathcal{P}(E,F)) \tag{1.2}$$

by [4, Lm.4], and if $\mathcal{P}_1(E,F)$ is a subspace of $\mathcal{P}(E,F)$ then

$$\|T\|_{\text{r-}\mathcal{P}_1} \ge \|T\|_{\text{r-}\mathcal{P}} \ge \|T\|_r \ge \|T\| \qquad (\forall\, T \in \text{r-}\mathcal{P}_1(E,F)). \tag{1.3}$$

Proposition 1.10 ([4, Thm.6]) *Let $\mathcal{P} = \mathcal{P}(E, F)$ be a subspace of* $\mathrm{L}(E, F)$ *closed in the operator norm. Then* r-$\mathcal{P}(E, F)$ *is a Banach space under the enveloping norm.*

Let $\mathcal{P} = \mathcal{P}(E, F) \subseteq \mathrm{L}(E, F)$, and denote $\mathcal{P}' := \{T' : T \in \mathcal{P}\} \subseteq \mathrm{L}(F', E')$. Clearly, r-$\mathcal{P}'(F', E') = (\text{r-}\mathcal{P}(E, F))'$. Since $\|S'\| = \|S\|$, it follows from (1.1)

$$\|T'\|_{\text{r-}\mathcal{P}'} = \inf\{\|S'\| : \pm T' \leq S' \in \mathcal{P}'\} = \inf\{\|S\| : \pm T \leq S \in \mathcal{P}\} = \|T\|_{\text{r-}\mathcal{P}}.$$

If $\mathcal{P} \subseteq \mathrm{L}(E, F)$ is closed in the operator norm then $\mathcal{P}' \subseteq \mathrm{L}(F', E')$ is also closed in the operator norm. So, the next fact follows from Proposition 1.10.

Corollary 1.11 Let $\mathcal{P}$ be a subspace of $\mathrm{L}(E, F)$ closed in the operator norm. Then r-$\mathcal{P}'(F', E')$ is a Banach space under the enveloping norm.

1.5. In Section 2, we introduce the main definitions and discuss the basic properties of affiliated operators, especially related to enveloping norms. Section 3 is devoted to the domination problem for affiliated operators, under the consideration, with special emphasis on the property (d) and on sequential w-continuity of lattice operations in Banach lattices. For further unexplained terminology and notations, we refer to [1, 2, 4, 6, 12, 13, 24, 29, 32, 33].

2. Affiliated operators and enveloping norms

Several recent papers were devoted to various modifications of limited, Grothendieck, L- and M-weakly compact, and Dunford–Pettis operators, through involving the structure of Banach lattices (see, e.g., [3, 5, 6, 11, 15, 16, 20, 21, 31, 33]). In this section, we show that many of these operators appear as operators affiliated to well-known properties of Banach lattices like the disjoint (dual) Schur property, the disjoint Grothendieck property, the property (d), and the sequential w*-continuity of the lattice operations. In continuation of [4] we shortly discuss the enveloping norms correspondent to these affiliated operators.

2.1. Recall that E (resp. E') has *sequentially* w-*continuous* (resp. *sequentially* w*-*continuous*) *lattice operations* if $(|x_n|)$ is w-null (resp. w*-null) for each w-null (x_n) in E (resp. for each w*-null (x_n) in E').

Proposition 2.1 (see [26, Prop.3.1]) The following are equivalent.

(i) E' has sequentially w*-continuous lattice operations.

(ii) Each order interval in E is limited.

In particular, the dual E' of each discrete Banach lattice E with order continuous norm has sequentially w*-continuous lattice operations [33, Prop.1.1], [26, Cor.3.2]. Under the disjointness assumption on a sequence in E we have the following fact.

Proposition 2.2 (cf. [1, Thm.4.34]) For every disjoint w-null (x_n) in E, the sequence $(|x_n|)$ is also w-null.

This is no longer true for w*-convergence (e.g., the sequence $f_n := e_{2n} - e_{2n+1}$ is disjoint w*-null in c' yet $|f_n|(\mathbf{1}_{\mathbb{N}}) \equiv 2 \not\to 0$ [11, Ex.2.1]). We recall the following properties of Banach lattices.

Definition 2.3 A Banach lattice E has:

a) the *positive Schur property* (briefly, $E \in$ (PSP)) if each w-null sequence in E_+ is norm null (cf. [32]);

b) the *positive disjoint Schur property* (briefly, $E \in$ (PDSP)) if each disjoint w-null sequence in E_+ is norm null;

c) the *disjoint Schur property* (briefly, $E \in$ (DSP)) if each disjoint w-null sequence in E is norm null;

d) the *dual positive Schur property* (briefly, $E \in$ (DPSP)) if each w*-null sequence in E'_+ is norm null [6, Def.3.3];

e) the *dual disjoint Schur property* (briefly, $E \in$ (DDSP)) if each disjoint w*-null sequence in E' is norm null [28, Def.3.2];

f) the *positive Grothendieck property* (briefly, $E \in$ (PGP)) if each w*-null sequence in E'_+ is w-null (cf. [33, p.760]);

g) the *disjoint Grothendieck property* (briefly, $E \in$ (DGP)) if each disjoint w*-null sequence in E' is w-null (cf. [3, Def.1.7.3]);

h) the (swl)-*property* (briefly, $E \in$ (swl)) if $(|x_n|)$ is w-null for each w-null sequence (x_n) in E;

i) the (sw*l)-*property* (briefly, $E \in$ (sw*l)) if $(|f_n|)$ is w*-null for each w*-null sequence (f_n) in E';

j) the *property* (d) (briefly, $E \in$ (d)) if $(|f_n|)$ is w*-null for each disjoint w*-null sequence (f_n) in E' [15, 33];

k) the bi-*sequence property* (briefly, $E \in$ (bi-sP)) if $f_n(x_n) \to 0$ for each w*-null (f_n) in E'_+ and each disjoint w-null (x_n) in E [6, Def.3.1];

l) the *strong* GP-*property* (briefly, $E \in$ (s-GPP)) if each almost limited subset of E is relatively compact;

m) the *strong* BD-*property* (briefly, $E \in$ (s-BDP)) if each almost limited subset of E is relatively w-compact.

It is well known that (PSP) = (PDSP) = (DSP). Indeed, (PSP) $\subseteq$ (PDSP) holds trivially; (PDSP) $\subseteq$ (DSP) is due to Proposition 2.2; and, for (DSP) $\subseteq$ (PSP) see [32, p.16]. We include a short proof of the following fact.

Lemma 2.4 ([6, Thm.4.2], [33, Prop.2.4]) Let E be a Banach lattice. The following are equivalent:

(i) $E \in$ (bi-sP);

(ii) $E \in$ (Pbi-sP), in the sense that if $f_n(x_n) \to 0$ for each w*-null (f_n) in E'_+ and each disjoint w-null (x_n) in E_+.

(iii) every w*-null sequence (f_n) in E'_+ is uniformly null on each disjoint w-null (x_n) in E_+.

Proof The implication i)$\Longrightarrow$ii) is obvious, whereas ii)$\Longrightarrow$iii) follows from Lemma 1.2 i).

iii)$\Longrightarrow$i) Let (f_n) be w*-null in E'_+ and (x_n) be disjoint w-null in E. By Proposition 2.2, $(x_n^{\pm})$ are both disjoint w-null in E_+. Then (f_n) is uniformly null on both $(x_n^{\pm})$, and hence on $(x_n) = (x_n^+) - (x_n^-)$. By Lemma 1.2 i), $f_n(x_n) \to 0$, as desired. □

2.2. Applying the redistribution between the domain and range (as in Definition 1.5) to properties described in Definition 2.3, we obtain the correspondent affiliated operators as follows.

Definition 2.5 An operator $T : E \to Y$ is:

a) a [PSP]-*operator* if $\|Tx_n\| \to 0$ for each w-null (x_n) in E_+;

b) a [PDSP]-*operator* if $\|Tx_n\| \to 0$ for each disjoint w-null (x_n) in E_+;

c) a [DSP]-*operator* if $\|Tx_n\| \to 0$ for each disjoint w-null (x_n) in E;

d) an [s-GPP]-*operator* if T carries almost limited subsets of E onto relatively compact subsets of Y;

e) an [s-BDP]-*operator* if T carries almost limited subsets of E onto relatively w-compact subsets of Y.

Clearly,

$$\text{[s-GPP]}(E,Y) \subseteq \text{[GPP]}(E,Y) \bigcap \text{[s-BDP]}(E,Y) \quad \text{and} \tag{2.1}$$

$$\text{[s-BDP]}(E,Y) \subseteq \text{[BDP]}(E,Y). \tag{2.2}$$

[DSP]-operators coincide with the almost Dunford–Pettis operators, and hence, by [5, Thm.2.2],

$$\text{[PSP]}(E,Y) = \text{[PDSP]}(E,Y) = \text{[DSP]}(E,Y). \tag{2.3}$$

Definition 2.6 An operator $T : X \to F$ is:

a) a [DPSP]-*operator* if $\|T'f_n\| \to 0$ for each w*-null (f_n) in F'_+;

b) a [DDSP]-*operator* if $\|T'f_n\| \to 0$ for each disjoint w*-null (f_n) in F';

c) a [PGP]-*operator* if $(T'f_n)$ is w-null for each w*-null (f_n) in F'_+;

d) a [DGP]-*operator* if $(T'f_n)$ is w-null for each disjoint w*-null (f_n) in F';

e) an [swl]-*operator* if $(|Tx_n|)$ is w-null for each w-null (x_n) in X.

[DDSP]-operators coincide with the almost limited operators, whereas [DGP]-operators agree with the almost Grothendieck operators.

Theorem 2.7 $(\text{[DPSP]}(X,F))' \cup (\text{[DDSP]}(X,F))' \subseteq \text{[PSP]}(F',X')$.

Proof Let (f_n) be disjoint w-null in F'_+. Then (f_n) is disjoint w*-null in F'_+. If $T \in \text{[DPSP]}(X,F)$ or $T \in \text{[DDSP]}(X,F)$ then in both cases $\|T'f_n\| \to 0$. Thus $T' \in \text{[PDSP]}(F',X')$, and hence $T' \in \text{[PSP]}(F',X')$ by (2.3). □

Definition 2.8 An operator $T : E \to F$ is called:

a) a [dswl]-*operator* if $(|Tx_n|)$ is w-null for each disjoint w-null (x_n);

b) an [sw*l]-*operator* if $(|T'f_n|)$ is w*-null for each w*-null (f_n) in F';

c) a [d]-*operator* if $(|T'f_n|)$ is w^*-null for each disjoint w^*-null (f_n) in F';

d) a [bi-sP]-*operator* if $f_n(Tx_n) \to 0$ for each w^*-null (f_n) in F'_+ and each disjoint w-null (x_n) in E;

e) a [Pbi-sP]-*operator* if $f_n(Tx_n) \to 0$ for each w^*-null (f_n) in F'_+ and each disjoint w-null (x_n) in E_+.

Theorem 2.9 For a Banach lattice F the following are hold.

i) $F \in (\mathrm{d})$ iff $\text{r-[d]}(E,F) = \mathrm{L}_r(E,F)$ for every E.

ii) F' has sequentially w^*-continuous lattice operations iff
$[\mathrm{sw}^*\mathrm{l}](E,F) = \mathrm{L}_r(E,F)$ for every E.

Proof i) For the necessity, let E be a Banach lattice. It is enough to prove $\mathrm{L}_+(E,F) \subseteq [\mathrm{d}](E,F)$. So, let $0 \le T : E \to F$ and (f_n) be disjoint w^*-null in F'. Since $F \in (\mathrm{d})$ then $(|f_n|)$ is w^*-null, and then $(T'|f_n|)$ is w^*-null in E'. It follows from $|T'f_n| \le T'|f_n|$ that $(|T'f_n|)$ is w^*-null, and hence $T \in [\mathrm{d}](E,F)$. The sufficiency is immediate since $I_F \in [\mathrm{d}](F,F)$ implies $F \in (\mathrm{d})$.
ii) Just remove the disjointness condition on (f_n) in the proof of i). □

The next result shows that [Pbi-sP]-operators agree with [bi-sP]-opeators.

Theorem 2.10 $[\text{bi-sP}](E,F) = [\text{Pbi-sP}](E,F)$.

Proof Clearly, $[\text{bi-sP}](E,F) \subseteq [\text{Pbi-sP}](E,F)$. Let $T \in [\text{Pbi-sP}](E,F)$, (f_n) be w^*-null in F'_+, and (x_n) be disjoint w-null in E. By Proposition 2.2, $(|x_n|)$ is disjoint w-null in E. Since $T \in [\text{Pbi-sP}](E,F)$, $f_n(T|x_n|) \to 0$. It follows from $|f_n(Tx_n)| \le f_n(T|x_n|)$ that $f_n(Tx_n) \to 0$, and hence $T \in [\text{bi-sP}](E,F)$. □

Theorem 2.11 Let $T \in \mathrm{L}(E,F)$. The following hold.

i) T is a [d]-operator iff T is almost o-limited.

ii) T is an $[\mathrm{sw}^*\mathrm{l}]$-operator iff T is o-limited.

Proof i) For the necessity, let $T \in [\mathrm{d}](E,F)$. Suppose $x \in E_+$ and (f_n) is disjoint w^*-null in F'. By the assumption, $(|T'f_n|)$ is w^*-null, and hence $|T'f_n|x \to 0$. By the Riesz–Kantorovich formula, $|T'f_n|x = \sup\{|(T'f_n)y| : |y| \le x\} \to 0$, and hence $(T'f_n)$ is uniformly null on each $[0,x]$. Thus $T \in \text{a-o-Lm}(E,F)$.

For the sufficiency, let $T \in \text{a-o-Lm}(E,F)$. Suppose (f_n) is disjoint w^*-null in F'. In order to prove $T \in [\mathrm{d}](E,F)$, we need to show that $(|T'f_n|) \xrightarrow{\mathrm{w}^*} 0$. It is enough to prove that $|T'f_n|x \to 0$ for each $x \in E_+$. Let $x \in E_+$. By the assumption, $\sup\{|(T'f_n)y| : |y| \le x\} \to 0$. Therefore, the Riesz–Kantorovich formula implies $|T'f_n|x \to 0$, and hence $T \in [\mathrm{d}](E,F)$.

ii) Just remove the disjointness condition on (f_n) in the proof of i). □

2.3. Affiliated operators from the previous subsection form vector spaces, which are complete under the operator norm; the details are included in the next lemma.

Lemma 2.12 The following sets of affiliated operators are vector spaces that are complete in the operator norm.

i) [PSP](E,Y).

ii) [DPSP](X,F) and [DDSP](X,F).

iii) [PGP](X,F) and [DGP](X,F).

iv) [swl](X,F) and [dswl](E,F).

v) [sw*l](E,F) and [d](E,F).

vi) [bi-sP](E,F).

vii) [GPP](X,Y) and [s-GPP](E,Y).

viii) [BDP](X,Y) and [s-BDP](E,Y).

Proof We skip trivial checking that all sets of affiliated operators in the lemma are vector spaces. It remains to show that each space of affiliated operators under consideration is a closed in the operator norm subspace of the correspondent space of all linear operators. As arguments here are straightforward and standard, we present them in some basic cases.

i) Let $[\mathrm{PSP}](E,Y) \ni T_k \stackrel{\|\cdot\|}{\to} T \in \mathrm{L}(E,Y)$. Let (x_n) be w-null in E_+. We need to show $\|Tx_n\| \to 0$. Let $\varepsilon > 0$. Pick some $k \in \mathbb{N}$ with $\|T - T_k\| \le \varepsilon$. Since $T_k \in \mathrm{PSP}(E,Y)$, there exists n_0 such that $\|T_k x_n\| \le \varepsilon$ for $n \ge n_0$. Take $M \in \mathbb{R}$ satisfying $\|x_n\| \le M$ for all $n \in \mathbb{N}$. Since

$$\|Tx_n\| = \|(T - T_k)x_n + T_k x_n\| \le \|T - T_k\| \cdot \|x_n\| + \|T_k x_n\| \le \varepsilon(M+1)$$

for $n \ge n_0$, and since $\varepsilon > 0$ is arbitrary, $\|Tx_n\| \to 0$.

ii) As the case of [DDSP](X,F) is similar, we confine ourselves to considering [DPSP](X,F).

Let $[\mathrm{DPSP}](X,F) \ni T_k \stackrel{\|\cdot\|}{\to} T \in \mathrm{L}(X,F)$, and let (f_n) be w*-null in F'_+. In order to show $(T'f_n)$ is norm null, let $\varepsilon > 0$ and pick k with $\|T' - T'_k\| \le \varepsilon$. Since $T_k \in [\mathrm{DPSP}](X,F)$, there exists n_0 with $\|T'_k f_n\| \le \varepsilon$ for all $n \ge n_0$. As (f_n) is w*-null, there exists $M \in \mathbb{R}$ satisfying $\|f_n\| \le M$ for all $n \in \mathbb{N}$. Since

$$\|T'f_n\| \le \|T'f_n - T'_k f_n\| + \|T'_k f_n\| \le \|T'_k - T'\|\|f_n\| + \varepsilon \le \varepsilon(M+1)$$

for $n \ge n_0$. It follows $\|T'f_n\| \to 0$, as desired.

iii) As the case of [DGP](X,F) is similar, we consider [PGP](X,F) only.

Let $[\mathrm{PGP}](X,F) \ni T_k \stackrel{\|\cdot\|}{\to} T \in \mathrm{L}(X,F)$, and let (f_n) be w*-null in F'_+. In order to show that $(T'f_n)$ is w-null, pick a $g \in F''$, and let $\varepsilon > 0$. Fix any k with $\|T' - T'_k\| \le \varepsilon$. Since $T_k \in [\mathrm{PGP}](X,F)$, there exists n_0 with $|g(T'_k f_n)| \le \varepsilon$ for all $n \ge n_0$. Let $M \in \mathbb{R}$ be such $\|f_n\| \le M$ for all $n \in \mathbb{N}$. Because of

$$|g(T'f_n)| \le |g(T'f_n - T'_k f_n)| + |g(T'_k f_n)| \le$$

$$\|g\|\|T' - T'_k\|\|f_n\| + \varepsilon \le (\|g\|M + 1)\varepsilon$$

for $n \ge n_0$, and since $\varepsilon > 0$ is arbitrary, it follows $g(T'f_n) \to 0$. Since $g \in F''$ is arbitrary, $T \in [\mathrm{PGP}](X,F)$.

iv) We consider the space [swl](X,F) only. The case of [dswl](E,F) is similar.

Let $[\text{swl}](X,F) \ni T_k \stackrel{\|\cdot\|}{\to} T \in \mathrm{L}(X,F)$ and let (x_n) be w-null in X. We need to show $|Tx_n| \stackrel{\mathrm{w}}{\to} 0$ in F. Let $f \in F'$. There exists an $M \in \mathbb{R}$ with $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Take some $\varepsilon > 0$ and pick $k \in \mathbb{N}$ with $\|T - T_k\| \leq \varepsilon$. Choose n_0 such that $|f|(|T_k x_n|) \leq \varepsilon$ for all $n \geq n_0$. Then

$$|f(|Tx_n|)| \leq |f|(|(T - T_k)x_n + T_k x_n|) \leq$$

$$\|f\| \cdot \|T - T_k\| \cdot M + |f|(|T_k x_n|) \leq \varepsilon(\|f\|M + 1).$$

Since $\varepsilon > 0$ is arbitrary, $f(|Tx_n|) \to 0$; and, since $f \in F'$ is arbitrary, $|Tx_n| \stackrel{\mathrm{w}}{\to} 0$.

v) We consider the space $[\mathrm{d}](E,F)$ only. The case of $[\mathrm{sw}^*\mathrm{l}](E,F)$ is similar.

Let $[\mathrm{d}](E,F) \ni T_k \stackrel{\|\cdot\|}{\to} T \in \mathrm{L}(E,F)$, and let (f_n) be disjoint w*-null in F'. We need to show $(|T'f_n|) \stackrel{\mathrm{w}^*}{\to} 0$. It is enough to prove that $|T'f_n|x \to 0$ for each $x \in E_+$. Let $x \in E_+$ and $\varepsilon > 0$. Pick $k \in \mathbb{N}$ with $\|T' - T'_k\| \leq \varepsilon$. By the assumption, $|T'_k f_n|x \to 0$. So, let $n_0 \in \mathbb{N}$ be such that $|T'_k f_n|x \leq \varepsilon$ whenever $n \geq n_0$. As (f_n) is w*-null, there exists $M \in \mathbb{R}$ with $\|f_n\| \leq M$ for all $n \in \mathbb{N}$. By the Riesz–Kantorovich formula, for $n \geq n_0$,

$$|T'f_n|x = \sup\{|(T'f_n)y| : |y| \leq x\} \leq$$

$$\sup\{|((T' - T'_k)f_n)y| : |y| \leq x\} + \sup\{|(T'_k f_n)y| : |y| \leq x\} \leq$$

$$\sup\{\|T' - T'_k\| \cdot \|f_n\| \cdot \|y\| : |y| \leq x\} + |T'_k f_n|x \leq \varepsilon(M\|x\| + 1).$$

Therefore $|T'f_n|x \to 0$, and hence $T \in [\mathrm{d}](E,F)$.

vi) Let $[\text{bi-sP}](E,F) \ni T_k \stackrel{\|\cdot\|}{\to} T \in \mathrm{L}(E,F)$. Let (f_n) be w*-null in F'_+ and let (x_n) be disjoint w-null in E. We need to show $f_n(Tx_n) \to 0$. Pick $M \in \mathbb{R}$ such that $\|f_n\| \leq M$ and $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Take some $\varepsilon > 0$. Pick $k \in \mathbb{N}$ with $\|T - T_k\| \leq \varepsilon$. Since $T_k \in [\text{bi-sP}](E,F)$, there exists $n_0 \in \mathbb{N}$ such that $|f_n(T_k x_n)| \leq \varepsilon$ for $n \geq n_0$. Then

$$|f_n(Tx_n)| \leq |f_n((T - T_k)x_n)| + |f_n(T_k x_n)| \leq$$

$$\|f_n\| \cdot \|T - T_k\| \cdot \|x_n\| + \varepsilon \leq (M^2 + 1)\varepsilon \quad (\forall n \geq n_0).$$

Since $\varepsilon > 0$ is arbitrary, $f_n(Tx_n) \to 0$.

vii) As the case of $[\mathrm{GPP}](X,Y)$ is similar, we consider $[\text{s-GPP}](E,Y)$ only.

Let $[\text{s-GPP}](E,Y) \ni T_k \stackrel{\|\cdot\|}{\to} T \in \mathrm{L}(E,Y)$, and let $A \subseteq E$ be a-limited. We need to show that $T(A)$ is relatively compact. Since a-limited sets are bounded, there exists $M \in \mathbb{R}$ with $\|x\| \leq M$ for all $x \in A$. Choose $\varepsilon > 0$ and pick a $k \in \mathbb{N}$ such that $\|T - T_k\| \leq \varepsilon$. Then

$$Tx = T_k x + (T - T_k)x \in T_k(A) + \|T - T_k\| \cdot \|x\| B_Y = T_k(A) + \varepsilon M \cdot B_Y$$

for all $x \in A$, and hence $T(A) \subseteq T_k(A) + \varepsilon M \cdot B_Y$. By the assumption, $T_k(A)$ is relatively compact. Since $\varepsilon > 0$ is arbitrary, $T(A)$ is totally bounded and hence is relatively compact, as desired.

viii) As the case of $[\text{BDP}](X,Y)$ is similar, we consider $[\text{s-BDP}](E,Y)$ only.

Let $[\text{s-BDP}](E,Y) \ni T_k \overset{\|\cdot\|}{\to} T \in \mathrm{L}(E,Y)$, and let $A \subseteq E$ be a-limited. We need to show that $T(A)$ is relatively w-compact. Since a-limited sets are bounded, there exists $M \in \mathbb{R}$ such that $\|x\| \le M$ for all $x \in A$. Take $\varepsilon > 0$ and pick any $k \in \mathbb{N}$ with $\|T - T_k\| \le \varepsilon$. Then $T(A) \subseteq T_k(A) + \varepsilon M \cdot B_Y$, as above in vii). By the assumption, $T_k(A)$ is relatively w-compact. Since $\varepsilon > 0$ is arbitrary, $T(A)$ is relatively w-compact by the Grothendieck result [1, Thm.3.44].

□

The next result follows from Theorem 1.10 and Lemma 2.12.

Theorem 2.13 The spaces r-$[\text{PSP}](E,F)$, r-$[\text{DPSP}](E,F)$, r-$[\text{DDSP}](E,F)$, r-$[\text{PGP}](E,F)$, r-$[\text{DGP}](E,F)$, r-$[\text{swl}](E,F)$, r-$[\text{dswl}](E,F)$, r-$[\text{sw}^*\text{l}](E,F)$, r-$[\text{d}](E,F)$, r-$[\text{bi-sP}](E,F)$, r-$[\text{GPP}](E,F)$, r-$[\text{s-GPP}](E,F)$, r-$[\text{BDP}](E,F)$, and r-$[\text{s-BDP}](E,F)$ are all Banach spaces under their enveloping norms.

3. Domination for affiliated operators

Here we gather domination results for the defined above affiliated operators. Some of them already appeared in the literature, the others seem new.

3.1. It turns out that the property (d) and the sequential w-continuity (w^*-continuity) of lattice operations play an important role in the domination problem.

Lemma 3.1 *The following are equivalent.*

i) $E \in (\text{d})$.

ii) *Each order interval in* E *is* a-*limited.*

Proof i)$\Longrightarrow$ii) It suffices to show that intervals $[-a,a]$ are a-limited. Let $a \in E_+$, and let (f_n) be disjoint w^*-null in E'. We need to show that (f_n) is uniformly null on $[-a,a]$. By Lemma 1.2, it is enough to show that $f_n(a_n) \to 0$ for each sequence (a_n) in $[-a,a]$. So, let (a_n) be in $[-a,a]$. Since $E \in (\text{d})$ then $(|f_n|)$ is w^*-null in E'_+, and hence $f_n(a) \to 0$. It follows from $-f_n(a) \le f_n(a_n) \le f_n(a)$ for all $n \in \mathbb{N}$ that $f_n(a_n) \to 0$. By Lemma 1.2, (f_n) is uniformly null on $[-a,a]$, as desired.

ii)$\Longrightarrow$i) Let (f_n) be disjoint w^*-null in E'. We need to show that $(|f_n|)$ is w^*-null. Pick an $a \in E_+$. By the assumption, (f_n) is uniformly null on $[-a,a]$, and in view of the Riesz–Kantorovich formula, $|f_n|a = \sup\limits_{y \in [-a,a]} |f_n(y)| \to 0$. Since $a \in E_+$ is arbitrary, $(|f_n|)$ is w^*-null, as desired. □

The [s-GPP]-operators do not satisfy the domination property in the strong sense that even an operator which is dominated by a rank one operator need not to be a [GPP]-operator.

Example 3.2 (cf. [1, Ex.5.30]) Define operators $T, S : L^1[0,1] \to \ell^\infty$ by $T(f) := (\int_0^1 f(t)dt)_{k=1}^\infty$, and $S(f) := (\int_0^1 f(t) r_k^+(t)dt)_{k=1}^\infty$, where $r_k(t) = \operatorname{sgn}\sin(2^k \pi t)$ are the Rademacher functions on $[0,1]$. Then T is rank one, and hence $T \in [\text{s-GPP}](L^1[0,1], \ell^\infty)$. Moreover, $0 \le S \le T$, yet S is not a [GPP]-operator. To see this, consider the sequence $(r_n(t))$ in the order interval $[-\mathbb{1}, \mathbb{1}]$, that is an a-limited subset of $L^1[0,1]$, e.g.

by Lemma 3.1. The sequence $(Sr_n) = (\frac{1}{2}e_n)$, where e_n is the n-th unite vector in ℓ^∞, has no norm convergent subsequences, and hence $S \notin [\text{GPP}](L^1[0,1], \ell^\infty)$.

We do not know whether or not the operator S in Example 3.2 is a [BDP]-operator. The proof of the following result of [26] consists in removing the disjointness condition in the proof of Lemma 3.1.

Proposition 3.3 The following are equivalent.

(i) E' has sequentially w*-continuous lattice operations.

(ii) Each order interval in E is limited.

3.2. Here we gather several (partially positive) domination results.

Theorem 3.4 Let E and F be Banach lattices. The following spaces of operators satisfy the domination property.

i) $[\text{PSP}](E,F)$.

ii) $[\text{DPSP}](E,F)$.

iii) $[\text{DDSP}](E,F)$, under the assumption $F \in (\text{d})$.

iv) $[\text{PGP}](E,F)$.

v) $[\text{DGP}](E,F)$, under the assumption $F \in (\text{d})$.

vi) $[\text{dswl}](E,F)$.

vii) $[\text{swl}](E,F)$, under the assumption that E has sequentially w-continuous lattice operations.

viii) $[\text{sw}^*\text{l}](E,F)$, under the assumption that F' has sequentially w*-continuous lattice operations.

ix) $[\text{d}](E,F)$, under the assumption $F \in (\text{d})$.

x) $[\text{bi-sP}](E,F)$.

Proof As above, we restrict ourselves to basic cases.

i) Let $0 \le S \le T \in [\text{PSP}](E,F)$ and let (x_n) be w-null in E_+. Since $T \in [\text{PSP}](E,F)$ then $\|Tx_n\| \to 0$. It follows from $0 \le Sx_n \le Tx_n$ that $\|Sx_n\| \to 0$, and hence $S \in [\text{PSP}](E,F)$.

ii) Let $0 \le S \le T \in [\text{DPSP}](E,F)$ and let (f_n) be w*-null in F'_+. Since $T \in [\text{DPSP}](E,F)$ then $\|T'f_n\| \to 0$. It follows from $0 \le S' \le T'$ that $0 \le S'f_n \le T'f_n$, and hence $\|S'f_n\| \to 0$. Thus, $S \in [\text{DPSP}](E,F)$.

iii) As [DDSP]-operators agree with almost limited operators, we refer for the proof to [15, Cor.3].

iv) Let $0 \le S \le T \in [\text{PGP}](E,F)$, and (f_n) be w*-null in F'_+. In order to prove $S \in [\text{PGP}](E,F)$, it suffices to prove $g(S'f_n) \to 0$ for all $g \in E'_+$. Let $g \in E'_+$. Since $T \in [\text{PGP}](E,F)$, $g(T'f_n) \to 0$. It follows from $0 \le g(S'f_n) \le g(T'f_n)$ that $g(S'f_n) \to 0$, as desired.

v) As [DGP]-operators agree with almost Grothendieck operators, we refer for the proof to [21, Prop.3.7].

vi) Let $0 \leq S \leq T \in [\text{dswl}](E,F)$, and let (x_n) be disjoint w-null in E. In order to prove $S \in [\text{dswl}](E,F)$, it suffices to prove $f(|Sx_n|) \to 0$ for all $f \in F'_+$. So, let $f \in F'_+$. By Proposition 2.2, $(|x_n|)$ is w-null. Since $T \in [\text{dswl}](E,F)$ then $(T|x_n|) = (|T(|x_n|)|)$ is w-null, and hence $f(T|x_n|) \to 0$. It follows from $|Sx_n| \leq S|x_n| \leq T|x_n|$ that $f(|Sx_n|) \to 0$ as desired.

vii) Let $0 \leq S \leq T \in [\text{swl}](E,F)$, and (x_n) be w-null in E. It suffices to prove $f(|Sx_n|) \to 0$ for all $f \in F'_+$. Let $f \in F'_+$. By the assumption, $(|x_n|)$ is w-null. Since $T \in [\text{swl}](E,F)$, $f(T|x_n|) = f(|Tx_n|) \to 0$. In view of $|Sx_n| \leq S|x_n| \leq T|x_n|$, $f(|Sx_n|) \to 0$, and hence $S \in [\text{swl}](E,F)$.

viii) It follows from Theorem 2.9 ii).

ix) It follows from Theorem 2.9 i).

x) Let $0 \leq S \leq T \in [\text{bi-sP}](E,F)$. Let (f_n) be w*-null in F'_+, and let (x_n) be disjoint w-null in E. In order to prove $S \in [\text{bi-sP}](E,F)$, it suffices to prove $f_n(Sx_n) \to 0$. By Proposition 2.2, $(|x_n|)$ is disjoint w-null in E, and, since $T \in [\text{bi-sP}](E,F)$, then $f_n(T|x_n|) \to 0$. It follows from $|f_n(Sx_n)| \leq f_n(S|x_n|) \leq f_n(T|x_n|)$ that $f_n(Sx_n) \to 0$, and hence $S \in [\text{bi-sP}](E,F)$.

□

In view of [1, Thm.4.31], the next fact follows from Theorem 3.4 vii).

Corollary 3.5 *Let E be an* AM*-space, and let* $0 \leq S \leq T \in [\text{swl}](E,F)$. *Then* $S \in [\text{swl}](E,F)$.

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