

Extended calculus on $\mathcal{O}(\mathbb{C}_h^{1|1})$

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Abstract: We give an extended calculus over the function algebra on h -deformed superplane. For this, we extend the (h_1, h_2) -deformed differential calculus on the h -deformed superplane by adding inner derivations. We reformulate the results with an R -matrix and present the tensor product realization of the wedge product. We also discuss Cartan calculus via a contraction.

Key words: h -deformation, quantum superplane, differential calculus, inner derivations, lie derivatives, Cartan calculus, contraction

1. Introduction

Noncommutative differential geometry continues to play important roles in different fields of mathematics and mathematical physics in recent years. A differential calculus on an associative (super) algebra is one of the fundamental structures that make up noncommutative geometry. In the language of quantum groups, there are two types of deformations: standard (or q -deformation) and nonstandard (or h -deformation). The q -deformation of Lie groups and algebras are presented in [17] and nonstandard deformation in [24] and [15]. A differential calculus over the quantum hyperplane was introduced in [25]. This calculus is very interesting from the point of view of noncommutative geometry. The natural extension to q -superspaces [22] of differential calculus was introduced in [3–7, 16, 20]. Differential calculus on h -deformed spaces and superspaces was studied in [1, 2, 8–10, 19, 21].

The extended calculus on the quantum plane was introduced in [12] using the approach of [23]. The Cartan calculus on the q -superplane was introduced in [11].

The present paper might be divided into three parts; the first of which is the extension of the differential calculus on the h -deformed superplane $\mathbb{C}_h^{1|1}$ in such a way to include the inner derivations, while the second part is to use the R -matrix of the quantum supergroup $GL_{h,h'}(1|1)$ in the presentation of the differential calculus, and the final part is to upgrade the discussion to the (q, h) -deformed setting.

This paper is organized as follows. In Section 2, we give some information on $\mathcal{O}(\mathbb{C}_h^{1|1})$ from [8, 10]. In Section 3, we present the differential calculus from [10], in a more formal and systematic way, which we will use to construct our notions and we introduce the commutation rules of the inner derivations with functions on the quantum superplane, differential forms and partial differentials. In Section 4, we reformulate the results

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we got in the previous section with an R -matrix and we introduce the tensor product realization of the wedge product. In the last section, we note that a Cartan calculus can be constructed on the superalgebra $\mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ using a contraction procedure.

We denote the degree of a homogeneous element a_i of a \mathbb{Z}_2 -graded vector space by $p(a_i) = i$, where i is in \mathbb{Z}_2 . If $p(a_i) = 0$ then a_i is said to be even and if $p(a_i) = 1$ then a_i is said to be odd. As in the classical case, throughout this paper we will assume that odd (Grassmann) elements are anticommutative among themselves.

2. Review of the superalgebra $\mathcal{O}(\mathbb{C}_h^{1|1})$

Elementary properties of the two-parameter h -deformed superplanes are described in [8, 10]. We briefly mention some concepts as we will need them in this study.

2.1. The algebra of functions on the (h, h') -deformed superplanes

Let us start with the definition of the coordinate ring of $\mathbb{C}_h^{1|1}$. Let $\mathbb{C}\langle x, \theta \rangle$ be a free superalgebra with unit generated by x and θ such that x is of degree (or parity) zero and θ is of degree one. We assume that $h^2 = 0$ and, h and θ are anticommutative.

Definition 2.1 Denote by I_h the two-sided ideal of the free superalgebra $\mathbb{C}\langle x, \theta \rangle$ with elements $x\theta - \theta x - hx^2$ and $\theta^2 + h\theta x$. The \mathbb{Z}_2 -graded, associative, unital algebra

$$\mathcal{O}(\mathbb{C}_h^{1|1}) = \mathbb{C}\langle x, \theta \rangle / I_h$$

is the algebra of polynomials on $\mathbb{C}_h^{1|1}$.

According to this definition, we have [8, 14]

$$x\theta = \theta x + hx^2, \quad \theta^2 = -h\theta x, \tag{1}$$

where $h^2 = 0$.

Interestingly, although the deformation parameter in standard deformation is a nonzero complex number and the generator θ is nilpotent, the deformation parameter in nonstandard deformation is an odd or Grassmann parameter whose square is zero and the generator θ is parafermionic, that is, $\theta^3 = 0$.

Definition 2.2 Let $\Lambda_{h'}(\mathbb{C}_h^{1|1})$ be the superalgebra with φ and y obeying the quadratic relations

$$\varphi^2 = h'\varphi y, \quad \varphi y = y\varphi + h'y^2 \tag{2}$$

where $p(\varphi) = 1$ and $p(y) = 0$, and h' is a Grassmann parameter whose square is zero and anticommuting with φ . We call $\Lambda_{h'}(\mathbb{C}_h^{1|1})$ the exterior algebra of $\mathbb{C}_h^{1|1}$.

Remark 1 In [10], the generators of the superalgebra $\Lambda_{h'}(\mathbb{C}_h^{1|1})$ are defined as differentials of x and θ . Since h and h' are both Grassmann parameters, we assume that $hh' + h'h = 0$ for consistency.

2.2. The quantum supergroup $GL_{h,h'}(1|1)$

Let us consider the free superalgebra $\mathbb{C}\langle a, \beta, \gamma, b \rangle$ where a, b are of degree 0 and β, γ are of degree 1, and write $T = (t_{ij})$ where $t_{ij} \in \{a, \beta, \gamma, b\}$.

Definition 2.3 [8] *A matrix T belongs to $GL_{h,h'}(1|1)$ if and only if the matrix elements of T satisfy the relations*

$$\begin{aligned}
 a\beta &= \beta a - h'(a^2 - \beta\gamma - ab), & b\beta &= \beta b + h'(b^2 + \beta\gamma - ba), \\
 a\gamma &= \gamma a + h(a^2 + \gamma\beta - ab), & b\gamma &= \gamma b - h(b^2 - \gamma\beta - ba), \\
 \beta^2 &= h'\beta(a - b), & \beta\gamma &= -\gamma\beta + (h\beta - h'\gamma)(b - a), \\
 \gamma^2 &= h\gamma(b - a), & ab &= ba + h\beta(a - b) + h'(a - b)\gamma
 \end{aligned} \tag{3}$$

provided that β and γ anticommute with h and h' .

The superalgebra $\mathcal{O}(GL_{h,h'}(1|1))$ is a Hopf superalgebra [8] and it is called the coordinate algebra of the quantum supergroup $GL_{h,h'}(1|1)$.

Theorem 2.4 *Superalgebras $\mathcal{O}(\mathbb{C}_h^{1|1})$ and $\Lambda_{h'}(\mathbb{C}_h^{1|1})$ are left $\mathcal{O}(GL_{h,h'}(1|1))$ -comodule algebras.*

3. Extension of left-covariant differential calculus on $\mathcal{O}(\mathbb{C}_h^{1|1})$

In the first subsection, we will review left-covariant \mathbb{Z}_2 -graded differential calculus on the superalgebra $\mathcal{O}(\mathbb{C}_h^{1|1})$ generated by two generators and quadratic relations [10] with some new theorems and formulas. We start with the definition of a super (or \mathbb{Z}_2 -graded) differential calculus on a superalgebra \mathcal{A} .

Definition 3.1 *A \mathbb{Z}_2 -graded differential calculus over \mathcal{A} is a \mathbb{Z}_2 -graded algebra $\Omega = \bigoplus_{n=0}^{\infty} \Omega^n$ where $\Omega^0 = \mathcal{A}$ and the space Ω^n of n -forms are generated as \mathcal{A} -bimodules via the action of a \mathbb{C} -linear mapping $d : \Omega \rightarrow \Omega$ of degree one such that*

1. $d^2 = 0$,
2. $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{p(\alpha)} \alpha \wedge (d\beta)$ for $\alpha, \beta \in \Omega$.
3. $\Omega^n = \text{Lin}\{a_0 \cdot da_1 \wedge \dots \wedge da_n : a_0, a_1, \dots, a_n \in \mathcal{A}\}$ for $n \in \mathbb{N}$.

3.1. Review of left-covariant differential calculus on $\mathcal{O}(\mathbb{C}_h^{1|1})$

We know, from [10], that there exists a unique left-covariant differential calculus Ω over $\mathcal{O}(\mathbb{C}_h^{1|1})$ with respect to $\mathcal{O}(GL_{h,h'}(1|1))$:

Theorem 3.2 *There is a unique \mathbb{Z}_2 -graded first order differential calculus $\Omega^1(\mathbb{C}_h^{1|1})$ over $\mathcal{O}(\mathbb{C}_h^{1|1})$ which is left-covariant according to $\mathcal{O}(GL_{h,h'}(1|1))$ such that the set $\{dx, d\theta\}$ is a free right $\mathcal{O}(\mathbb{C}_h^{1|1})$ -module basis of $\Omega^1(\mathbb{C}_h^{1|1})$. The bimodule structure for this calculus is determined by the relations*

$$u \cdot dx_j = (-1)^{p(u)p(dx_j)} \sum_i dx_i \cdot \sigma_{ij}(u), \tag{4}$$

where the action of the map $\sigma : \mathcal{O}(\mathbb{C}_h^{1|1}) \rightarrow M_2(\mathcal{O}(\mathbb{C}_h^{1|1}))$ on the generators of $\mathcal{O}(\mathbb{C}_h^{1|1})$ is as follows:

$$\sigma(x) = \begin{bmatrix} (1 - hh')x - h'\theta & hx - hh'\theta \\ -h'x & x - h'\theta \end{bmatrix}, \quad \sigma(\theta) = \begin{bmatrix} \theta - hx & h\theta \\ -(h'\theta + hh'x) & (1 + hh')\theta - hx \end{bmatrix}. \tag{5}$$

Remark 2 We can also define a map $\tau : \mathcal{O}(\mathbb{C}_h^{1|1}) \rightarrow M_2(\mathcal{O}(\mathbb{C}_h^{1|1}))$ by the formulas

$$dx_j \cdot u = \sum_i (-1)^{p(u)p(dx_i)} \tau_{ji}(u) \cdot dx_i,$$

where

$$\tau(x) = \begin{bmatrix} (1 - hh')x + h'\theta & h'x \\ hx + hh'\theta & x + h'\theta \end{bmatrix}, \quad \tau(\theta) = \begin{bmatrix} \theta + hx & h'\theta - hh'x \\ h\theta & (1 + hh')\theta + hx \end{bmatrix}.$$

Theorem 3.3 The maps σ and τ are \mathbb{Z}_2 -graded left-linear homomorphisms such that

$$f_{ij}(uv) = \sum_k (-1)^{p(u)[p(x_j)+p(x_k)]} f_{ik}(u)f_{kj}(v), \quad \forall u, v \in \mathcal{O}(\mathbb{C}_h^{1|1}),$$

for $f \in \{\sigma, \tau\}$.

Remark 3 As can be easily shown, the relations (1) are preserved under the action of the maps σ and τ .

To obtain higher order differential forms, we apply the differential d to 1-forms using the fact that the square of d is zero and the Leibnitz rule. Then we have to apply the differential d to both sides of the relations in (4) to obtain the relations satisfied between differentials of the generators of the algebra $\mathcal{O}(\mathbb{C}_h^{1|1})$. However, the expression on the right side of (4) contains the operator σ and we need to establish a relationship between this operator and the differential d . So let us define a map σ^Ω as

$$\sigma^\Omega : \Omega \rightarrow \Omega, \quad \sigma_{jk}^\Omega(du) = d\sigma_{jk}(u), \quad \forall u \in \mathcal{O}(\mathbb{C}_h^{1|1}), \tag{6}$$

where $\sigma_{ij}^\Omega(u) := \sigma_{ij}(u)$ for all $u \in \mathcal{O}(\mathbb{C}_q^{2|1})$. Then we have

Theorem 3.4 The relations between the differentials are as follows

$$du \wedge dx_j = (-1)^{[1+p(du)]p(dx_j)} \sum_i (-1)^{p(dx_i)} dx_i \wedge \sigma_{ij}^\Omega(du). \tag{7}$$

Naturally, the map σ^Ω is also a linear homomorphism acting on 0-forms, with the same properties as the map σ :

Theorem 3.5 The map σ^Ω is a \mathbb{Z}_2 -graded left-linear homomorphism such that

$$\sigma_{ij}^\Omega(du \wedge dv) = \sum_k (-1)^{p(du)[p(dx_k)+p(dx_j)]} \sigma_{ik}^\Omega(du) \wedge \sigma_{kj}^\Omega(dv), \tag{8}$$

for all $u, v \in \mathcal{O}(\mathbb{C}_h^{1|1})$.

Proof One derives from the fact that the differential algebra Ω is associative. Really, from (7), we can write

$$(du \wedge dv) \wedge dx_j = (-1)^{[1+p(dudv)]p(dx_j)} \sum_i (-1)^{p(dx_i)} dx_i \wedge \sigma_{ij}^\Omega(du \wedge dv),$$

for all $u, v \in \mathcal{O}(\mathbb{C}_h^{1|1})$. On the other hand, using $(du \wedge dv) \wedge dx_j = du \wedge (dv \wedge dx_j)$, one obtains

$$\begin{aligned} du \wedge (dv \wedge dx_j) &= (-1)^{[1+p(dv)]p(dx_j)} du \wedge \sum_i (-1)^{p(dx_i)} dx_i \wedge \sigma_{ij}^\Omega(dv) \\ &= (-1)^{[1+p(dv)]p(dx_j)} \sum_{i,k} (-1)^{p(du)p(dx_i)+p(dx_k)} dx_k \wedge \sigma_{ki}^\Omega(du) \wedge \sigma_{ij}^\Omega(dv). \end{aligned}$$

If we substitute k for i and i for k in the second equality, we obtain the equality in (8). □

In the next section, we will also need the action of σ^Ω on the elements such as $u \cdot dv$.

Corollary 3.5.1 *The map σ^Ω together with σ has the following properties*

$$\begin{aligned} \sigma_{ij}^\Omega(u \cdot dv) &= \sum_k (-1)^{p(x_i)+p(x_k)+p(u)[p(x_j)+p(x_k)]} \sigma_{ik}(u) \cdot \sigma_{kj}^\Omega(dv), \\ \sigma_{ij}^\Omega(du \cdot v) &= \sum_k (-1)^{[1+p(du)][p(x_j)+p(x_k)]} \sigma_{ik}^\Omega(du) \cdot \sigma_{kj}(v), \end{aligned} \tag{9}$$

for all $u, v \in \mathcal{O}(\mathbb{C}_h^{1|1})$.

Remark 4 *It is easy to see that the relations (4) are invariant under the action of the map σ^Ω .*

Now, we want to obtain commutation relations of the generators of $\mathcal{O}(\mathbb{C}_h^{1|1})$ with partial derivatives. Therefore, let us first define the partial derivatives of the generators of $\mathcal{O}(\mathbb{C}_h^{1|1})$.

Definition 3.6 *The linear mappings $\partial_x, \partial_\theta : \mathcal{O}(\mathbb{C}_h^{1|1}) \rightarrow \mathcal{O}(\mathbb{C}_h^{1|1})$ defined by*

$$du = dx \partial_x(u) + d\theta \partial_\theta(u), \quad u \in \mathcal{O}(\mathbb{C}_h^{1|1}) \tag{10}$$

are called the partial derivatives of the calculus (Ω, d) , where $p(\partial_x) = 0$ and $p(\partial_\theta) = 1$.

Theorem 3.7 [10] *The partial derivatives with the generators of $\mathcal{O}(\mathbb{C}_h^{1|1})$ obey the following relations*

$$\partial_i \cdot x_k = \delta_{ik} + \sum_j (-1)^{p(x_k)p(\partial_j)} \sigma_{ij}(x_k) \cdot \partial_j, \tag{11}$$

where $\partial_1 := \partial_x$, $\partial_2 := \partial_\theta$ and $x_1 = x$, $x_2 = \theta$. The partial derivatives satisfy the following commutation relations

$$\partial_x \partial_\theta = \partial_\theta \partial_x - h' \partial_x^2, \quad \partial_\theta^2 = h' \partial_x \partial_\theta. \tag{12}$$

Let us denote by $\Omega^1(\mathcal{T}\mathbb{C}_h^{1|1})$ the vector space formed by $\partial_x, \partial_\theta$ satisfying (12). This vector space is called the tangent space.

Corollary 3.7.1 *The set $\{\partial_x, \partial_\theta\}$ is a basis of the right $\mathcal{O}(\mathbb{C}_h^{1|1})$ -module $\Omega^1(\mathcal{T}\mathbb{C}_h^{1|1})$.*

3.2. An exterior differential

We know, from Subsection 3.1, that the cotangent space $\Omega^1(\mathcal{T}^*\mathbb{C}_h^{1|1})$ is a $\mathcal{O}(\mathbb{C}_h^{1|1})$ -bimodule spanned by the basis $\{dx, d\theta\}$ with the relations (4) and that the tangent space $\Omega^1(\mathcal{T}\mathbb{C}_h^{1|1})$ is a $\mathcal{O}(\mathbb{C}_h^{1|1})$ -bimodule spanned by the basis $\{\partial_x, \partial_\theta\}$ with the relations (12). Therefore, we can define an inner product in analogy with the corresponding objects of the theory of ordinary manifolds. The general inner product between $\Omega^1(\mathcal{T}\mathbb{C}_h^{1|1})$ and $\Omega^1(\mathcal{T}^*\mathbb{C}_h^{1|1})$ is of the form

$$\partial_j(dx_k) := \langle \partial_j, dx_k \rangle = \delta_{jk}.$$

In this section, we wish to set up an extended calculus on $\mathcal{O}(\mathbb{C}_h^{1|1})$. Let us start by introducing an exterior derivative operator D that maps k -forms to $(k + 1)$ -forms (with functions being 0-forms) and obeys

$$\begin{aligned} D \circ D &:= D^2 = 0 \\ D(w_1 \wedge w_2) &= (Dw_1) \wedge w_2 + (-1)^{p(w_1)} w_1 \wedge (Dw_2) \\ &\equiv dw_1 \wedge w_2 + (-1)^{p(w_1)} w_1 \wedge dw_2 \end{aligned} \tag{13}$$

where w_i 's are any differential forms. (Actually, we assume that the action of D on w_1 (and then w_2) in (13) is the same as the differential of w_1 , that is, dw_1 .)

The exterior derivative D on the superalgebra $\mathcal{O}(\mathbb{C}_h^{1|1})$ is given by

$$Du \equiv dx \partial_x(u) + d\theta \partial_\theta(u), \tag{14}$$

so that it verifies (13) and the rule

$$D\alpha = d\alpha + (-1)^{p(\alpha)} \alpha D, \tag{15}$$

where α is a differential form on the superspace $\mathbb{C}_h^{1|1}$. In particular,

$$Du_k = du_k + (-1)^{p(u_k)} u_k D. \tag{16}$$

3.3. Inner derivations

To extend the differential calculus on the h -deformed superplane to a larger calculus, it is necessary to add inner derivations to this calculus.

While starting we wish to give some information about inner derivations. An inner derivation is defined to be the contraction of a vector field with a differential form. So, if X is a vector field on a manifold M , then the inner derivation \mathbf{i}_X is a linear operator which transforms k -forms to $(k - 1)$ -forms. The inner derivation is an antiderivation of degree -1 on the exterior algebra and

$$\mathbf{i}_X(\alpha_1 \wedge \alpha_2) = (\mathbf{i}_X\alpha_1) \wedge \alpha_2 + (-1)^{p(\alpha_1)p(\mathbf{i}_X)} \alpha_1 \wedge (\mathbf{i}_X\alpha_2)$$

where α_i ' are any differential form. The action of the inner derivation \mathbf{i}_X on 0- and 1-forms is

$$\mathbf{i}_X(f) = 0, \quad \mathbf{i}_X(df) = X(f).$$

The anticommutativity of forms gives

$$\mathbf{i}_X \circ \mathbf{i}_Y = -\mathbf{i}_Y \circ \mathbf{i}_X, \quad \mathbf{i}_X \circ \mathbf{i}_X := \mathbf{i}_X^2 = 0.$$

From now on, unless we say otherwise, we will write $\mathbf{i}_X \mathbf{i}_Y$ instead of $\mathbf{i}_X \circ \mathbf{i}_Y$.

In this and the next two subsections, we will consider the vector fields as the partial derivatives of generators.

Now we will find the relations of the generators x, θ of the algebra $\mathcal{O}(\mathbb{C}_h^{1|1})$ with the inner derivations. The commutation relations (4) satisfied by the generators with their differentials allow us to write the possible relations of both the generators and their differentials with inner derivations as

$$\mathbf{i}_{\partial_k} \cdot u = \sum_j \tilde{\sigma}_{kj}(u) \cdot \mathbf{i}_{\partial_j}, \quad \mathbf{i}_{\partial_k} \cdot du = \mathbf{i}_{\partial_k}(du) + \sum_j \hat{\sigma}_{kj}(du) \cdot \mathbf{i}_{\partial_j} \tag{17}$$

for all $u \in \mathcal{O}(\mathbb{C}_h^{1|1})$, where $\partial_k \in \{\partial_x, \partial_\theta\}$. So our goal will be to express $\tilde{\sigma}$ and $\hat{\sigma}$ in terms of the operators σ and σ^Ω , respectively.

Theorem 3.8 *The relations of the generators with inner derivations are given by the formulas*

$$\mathbf{i}_{\partial_k} \cdot u = \sum_j (-1)^{p(u)p(\mathbf{i}_{\partial_j})} \sigma_{kj}(u) \cdot \mathbf{i}_{\partial_j}, \tag{18}$$

for all $u \in \mathcal{O}(\mathbb{C}_h^{1|1})$. The commutation relations of differentials of the generators with inner derivations are given by the formulas

$$\mathbf{i}_{\partial_k} \cdot du = \mathbf{i}_{\partial_k}(du) + (-1)^{p(\mathbf{i}_{\partial_k})} \sum_j (-1)^{[1+p(du)]p(\mathbf{i}_{\partial_j})} \sigma_{kj}^\Omega(du) \cdot \mathbf{i}_{\partial_j}, \tag{19}$$

for all $u \in \mathcal{O}(\mathbb{C}_h^{1|1})$.

Proof Assuming that there is a sum over the repeated index, if we start from (4) and use relations (17), after a few operations, we write ($\mathbf{i}_k := \mathbf{i}_{\partial_k}$)

$$\begin{aligned} 0 &= \mathbf{i}_k \left(u \cdot dx_j - (-1)^{p(u)p(dx_j)} dx_i \cdot \sigma_{ij}(u) \right) \\ &= \tilde{\sigma}_{km}(u) \cdot \mathbf{i}_m \cdot dx_j - (-1)^{p(u)p(dx_j)} [\delta_{ki} + \hat{\sigma}_{kn}(dx_i) \cdot \mathbf{i}_n] \sigma_{ij}(u) \\ &= \tilde{\sigma}_{km}(u) [\delta_{mj} + \hat{\sigma}_{ms}(dx_j) \mathbf{i}_s] - (-1)^{p(u)p(dx_j)} [\sigma_{kj}(u) + \hat{\sigma}_{kn}(dx_i) \tilde{\sigma}_{nr}(\sigma_{ij}(u)) \mathbf{i}_r] \\ &= \tilde{\sigma}_{kj}(u) - (-1)^{p(u)p(dx_j)} \sigma_{kj}(u) + [\tilde{\sigma}_{km}(u) \hat{\sigma}_{ms}(dx_j) - (-1)^{p(u)p(dx_j)} \hat{\sigma}_{km}(dx_i) \tilde{\sigma}_{ms}(\sigma_{ij}(u))] \mathbf{i}_s. \end{aligned}$$

There are two conclusions we can deduce from here: the first is that

$$\tilde{\sigma}_{ki}(u) = (-1)^{p(u)p(dx_i)} \sigma_{ki}(u) = (-1)^{p(u)p(\mathbf{i}_i)} \sigma_{ki}(u), \quad \forall u \in \mathcal{O}(\mathbb{C}_h^{1|1}).$$

The second is that

$$\hat{\sigma}_{kj}(du) = (-1)^{p(\mathbf{i}_k)+[1+p(du)]p(\mathbf{i}_j)} \sigma_{kj}^\Omega(du), \quad \forall u \in \mathcal{O}(\mathbb{C}_h^{1|1}).$$

Thus, the proof is complete. □

We can check the correctness of the second equality above as follows: we can write

$$\begin{aligned} \mathbf{i}_k(udx_j) &= \sum_m (-1)^{p(u)p(\mathbf{i}_m)} \sigma_{km}(u) \left(\delta_{mj} + \sum_n (-1)^{p(\mathbf{i}_m)+p(x_j)p(\mathbf{i}_n)} \sigma_{mn}^\Omega(dx_j) \mathbf{i}_n \right) \\ &= (-1)^{p(u)p(dx_j)} \sigma_{kj}(u) + \sum_{n,m} (-1)^{p(x_j)p(dx_n)+p(du)p(dx_m)} \sigma_{km}(u) \sigma_{mn}^\Omega(dx_j) \mathbf{i}_n \\ &= (-1)^{p(u)p(dx_j)} \sigma_{kj}(u) + \sum_n (-1)^{p(dx_k)+[p(du)+p(dx_j)]p(dx_n)} \sigma_{kn}^\Omega(udx_j) \mathbf{i}_n \end{aligned}$$

using the first equality in (9). On the other hand, using the second equality in (9) we can write

$$\begin{aligned} \mathbf{i}_k(dx_i \sigma_{ij}(u)) &= \left(\delta_{ki} + (-1)^{p(\mathbf{i}_k)} \sum_m (-1)^{p(x_i)p(\mathbf{i}_m)} \sigma_{km}^\Omega(dx_i) \mathbf{i}_m \right) \sigma_{ij}(u) \\ &= \delta_{ki} \sigma_{ij}(u) + \sum_{n,m} (-1)^{p(dx_k)+p(\sigma_{ij}(u))p(dx_n)+p(x_i)p(dx_m)} \sigma_{km}^\Omega(dx_i) \sigma_{mn}(\sigma_{ij}(u)) \mathbf{i}_n \\ &= \delta_{ki} \sigma_{ij}(u) + \sum_n (-1)^{p(dx_k)+[p(du)+p(dx_j)]p(dx_n)} \sigma_{kn}^\Omega(dx_i \sigma_{ij}(u)) \mathbf{i}_n \end{aligned}$$

and so

$$\mathbf{i}_k \left((-1)^{p(u)p(dx_j)} \sum_i dx_i \sigma_{ij}(u) \right) = (-1)^{p(u)p(dx_j)} \sigma_{kj}(u) + \sum_n (-1)^{p(dx_k)+[p(du)+p(dx_j)]p(dx_n)} \sigma_{kn}^\Omega(udx_j) \mathbf{i}_n.$$

Therefore, $\mathbf{i}_k(u \cdot dx_j) - (-1)^{p(u)p(dx_j)} \mathbf{i}_k(\sum_i dx_i \sigma_{ij}(u)) = 0$.

To find the relations of the partial derivatives with the inner derivations, let us assume the possible relations are of the form

$$\mathbf{i}_{\partial_k} \partial_j = \sum_{m,n} B_{kj}^{mn} \partial_n \mathbf{i}_{\partial_m} \tag{20}$$

where the constants B_{ij}^{kl} possibly depend on h and/or h' .

Theorem 3.9 *The relations between the partial derivatives and the inner derivations are of the form*

$$\begin{aligned} \mathbf{i}_{\partial_x} \partial_x &= (1 - hh') \partial_x \mathbf{i}_{\partial_x} + h (\partial_\theta \mathbf{i}_{\partial_x} - \partial_x \mathbf{i}_{\partial_\theta}), \\ \mathbf{i}_{\partial_x} \partial_\theta &= -\partial_\theta \mathbf{i}_{\partial_x} - h \partial_\theta \mathbf{i}_{\partial_\theta} + h' \partial_x \mathbf{i}_{\partial_x} + hh' \partial_x \mathbf{i}_{\partial_\theta}, \\ \mathbf{i}_{\partial_\theta} \partial_x &= \partial_x \mathbf{i}_{\partial_\theta} + h \partial_\theta \mathbf{i}_{\partial_\theta} + h' \partial_x \mathbf{i}_{\partial_x} - hh' \partial_\theta \mathbf{i}_{\partial_x}, \\ \mathbf{i}_{\partial_\theta} \partial_\theta &= (1 + hh') \partial_\theta \mathbf{i}_{\partial_\theta} - h' (\partial_x \mathbf{i}_{\partial_\theta} + \partial_\theta \mathbf{i}_{\partial_x}). \end{aligned} \tag{21}$$

Proof If we apply the inner derivations \mathbf{i}_{∂_x} and $\mathbf{i}_{\partial_\theta}$ from the left to relations (11), we find

$$\begin{aligned} B_{11}^{11} &= 1 - hh', & B_{12}^{11} &= B_{21}^{11} = -B_{22}^{12} = -B_{22}^{21} = h', & B_{12}^{12} &= -B_{21}^{21} = -1, \\ B_{22}^{22} &= 1 + hh', & B_{11}^{12} &= -B_{11}^{21} = -B_{12}^{22} = B_{21}^{22} = h, & B_{21}^{12} &= -B_{21}^{21} = -hh'. \end{aligned}$$

All other constants are zero. □

Lemma 3.10 *The partial derivatives in terms of the exterior derivative and the inner derivations are expressed below:*

$$\mathbf{i}_{\partial_x} D = \partial_x - D\mathbf{i}_{\partial_x}, \quad \mathbf{i}_{\partial_\theta} D = \partial_\theta + D\mathbf{i}_{\partial_\theta}. \tag{22}$$

Proof To obtain the desired expressions, we apply the inner derivations from the left to $D = \sum du_k \partial_k$ which is given by (14) and use the relations (19) and (21). \square

Theorem 3.11 *The commutation relations between the inner derivations are*

$$\mathbf{i}_{\partial_x}^2 = -h \mathbf{i}_{\partial_\theta} \mathbf{i}_{\partial_x}, \quad \mathbf{i}_{\partial_x} \mathbf{i}_{\partial_\theta} = \mathbf{i}_{\partial_\theta} \mathbf{i}_{\partial_x} - h \mathbf{i}_{\partial_\theta}^2. \tag{23}$$

Proof Possible commutation relations between the inner derivations can be in the form of $\mathbf{i}_{\partial_x} \mathbf{i}_{\partial_\theta} = a \mathbf{i}_{\partial_\theta} \mathbf{i}_{\partial_x} + b \mathbf{i}_{\partial_x}^2 + c \mathbf{i}_{\partial_\theta}^2$ and $\mathbf{i}_{\partial_x} \mathbf{i}_{\partial_x} = k \mathbf{i}_{\partial_x} \mathbf{i}_{\partial_\theta}$ with the constants $a, b, c,$ and k . Therefore, the goal will be achieved when these constants are found. We use Lemma 3.10 for this. For example, one has, by the use of the inverse relations to (21),

$$\begin{aligned} 0 &= D(\mathbf{i}_{\partial_x} \mathbf{i}_{\partial_\theta} - a \mathbf{i}_{\partial_\theta} \mathbf{i}_{\partial_x} - b \mathbf{i}_{\partial_x}^2 - c \mathbf{i}_{\partial_\theta}^2) \\ &= (\partial_x - \mathbf{i}_{\partial_x} D)\mathbf{i}_{\partial_\theta} + a(\partial_\theta - \mathbf{i}_{\partial_\theta} D)\mathbf{i}_{\partial_x} - (-1)^{p(b)}(\partial_x - \mathbf{i}_{\partial_x} D)\mathbf{i}_{\partial_x} + (-1)^{p(c)}(\partial_\theta - \mathbf{i}_{\partial_\theta} D)\mathbf{i}_{\partial_\theta} \\ &= -(\mathbf{i}_{\partial_x} \mathbf{i}_{\partial_\theta} - a \mathbf{i}_{\partial_\theta} \mathbf{i}_{\partial_x} + (-1)^{p(b)} b \mathbf{i}_{\partial_x}^2 + (-1)^{p(c)} c \mathbf{i}_{\partial_\theta}^2)D \\ &\quad + [1 - (1 + hh')a - (-1)^{p(b)} bh + (-1)^{p(c)} ch'] \mathbf{i}_{\partial_\theta} \partial_x \\ &\quad + [(a - 1)h' + (-1)^{p(b)} bhh'] \mathbf{i}_{\partial_x} \partial_x + [(1 - a)h + (-1)^{p(c)} chh'] \mathbf{i}_{\partial_\theta} \partial_\theta \\ &\quad + [1 - hh' - a - (-1)^{p(b)} bh + (-1)^{p(c)} ch'] \mathbf{i}_{\partial_x} \partial_\theta. \end{aligned}$$

So it must be $a = 1, b = 0$ and $c = -h$. The other relation can be obtained by doing similar operations. \square

Remark 5 *Using Lemma 3.10, one can easily see that*

$$D\partial_u = (-1)^{p(\partial_u)} \partial_u D, \tag{24}$$

where $\partial_u \in \{\partial_x, \partial_\theta\}$.

4. The R -matrix formalism

The (h, h') -deformation of $GL(1|1)$ was obtained through a contraction [8]. In this section, we will reformulate all the relations obtained in the previous section using the R matrix found in [8].

We know from Theorem 3.2 that there exist a left-covariant differential calculus over $\mathcal{O}(\mathbb{C}_h^{1|1})$ with respect to the Hopf superalgebra $\mathcal{O}(GL_{h,h'}(1|1))$. So, we can use the R -matrix of the quantum supergroup $GL_{h,h'}(1|1)$ to formulate the calculus.

4.1. Commutation relations of calculus

The relations in (1) can be rewritten as

$$x_i x_j = \sum_{m,n} \hat{R}_{mn}^{ij} x_m x_n \tag{25}$$

with the matrix \hat{R} , where [8]

$$\hat{R}_{h,h'} = \begin{bmatrix} 1 - hh' & -h' & h' & 0 \\ h & -hh' & 1 & h' \\ -h & 1 & -hh' & h' \\ 0 & -h & -h & -1 - hh' \end{bmatrix}.$$

This matrix has the property $\hat{R}_{h,h'}^2 = I$, as well as providing the graded Yang-Baxter equation and braided group relation.

Now, if we apply d to both sides of (25), keeping in mind that some elements of the matrix \hat{R} are *odd*, we can express the relations of generators with their differentials (see, eq. (4)) as

$$x_j \cdot dx_k = (-1)^{p(x_j)} \sum_{m,n} (-1)^{p(\hat{R}_{mn}^{jk})} \hat{R}_{mn}^{jk} dx_m \cdot x_n. \tag{26}$$

From here, we see that

$$dx_j \wedge dx_k = -(-1)^{p(dx_j)} \sum_{m,n} (-1)^{p(dx_m)} \hat{R}_{mn}^{jk} dx_m \wedge dx_n. \tag{27}$$

We can express the relations (11) and (12) in the form

$$\partial_j x_k = \delta_{jk} + \sum_{m,n} (-1)^{p(\partial_j)p(\hat{R}_{jn}^{km})} \hat{R}_{jn}^{km} x_n \partial_m, \tag{28}$$

$$\partial_j \partial_k = \sum_{m,n} (-1)^{p(\partial_j \partial_k)p(\hat{R}_{kj}^{nm})} \hat{R}_{kj}^{nm} \partial_m \partial_n \tag{29}$$

with the help of the \hat{R} -matrix.

4.2. Commutation relations with inner derivations

Proofs of the formulas given below can be made by direct calculations, but it is necessary to do a lot of processing and play with indices.

$$\mathbf{i}_{\partial_j} \cdot x_k = (-1)^{p(x_k)} \sum_{m,n} (-1)^{[1+p(\mathbf{i}_j)]p(\hat{R}_{jn}^{km})} \hat{R}_{jn}^{km} x_n \cdot \mathbf{i}_{\partial_m},$$

$$\mathbf{i}_{\partial_j} \cdot dx_k = \delta_{jk} - (-1)^{p(dx_k)} \sum_{m,n} (-1)^{p(\mathbf{i}_{\partial_j})[1+p(\hat{R}_{jn}^{km})]} \hat{R}_{jn}^{km} dx_n \cdot \mathbf{i}_{\partial_m},$$

$$\mathbf{i}_{\partial_j} \partial_k = (-1)^{p(\partial_k)} \sum_{m,n} (-1)^{[1+p(\mathbf{i}_{\partial_j})+p(\partial_k)]p(\hat{R}_{kj}^{nm})} \hat{R}_{kj}^{nm} \partial_m \mathbf{i}_{\partial_n},$$

$$\mathbf{i}_{\partial_j} \mathbf{i}_{\partial_k} = (-1)^{p(\mathbf{i}_{\partial_k})+1} \sum_{m,n} (-1)^{p(\mathbf{i}_{\partial_n})+[p(\mathbf{i}_{\partial_j})+p(\mathbf{i}_{\partial_k})]p(\hat{R}_{kj}^{nm})} \hat{R}_{kj}^{nm} \mathbf{i}_{\partial_m} \mathbf{i}_{\partial_n}.$$

4.3. Tensor product realization of the wedge

The relations (19) can be used to define the wedge product \wedge of forms as an antisymmetrized tensor product. Since $dx_i \otimes dx_j$ is an element in the tensor space of $\Omega^1(\mathcal{T}^* \mathbb{C}_h^{1|1}) \otimes \Omega^1(\mathcal{T}^* \mathbb{C}_h^{1|1})$, we can define the product of

two forms in terms of tensor products as

$$dx_j \wedge dx_k = dx_j \otimes dx_k - \sum_{m,n} (-1)^{p(dx_j)+p(dx_m)} \hat{R}_{mn}^{jk} dx_m \otimes dx_n.$$

These equations give implicit commutation relations between the dx_k 's. So we have

$$\langle \partial_i, dx_j \wedge dx_k \rangle = \delta_{ij} dx_k - \sum_{m,n} (-1)^{p(dx_j)+p(dx_m)} \hat{R}_{mn}^{ij} \delta_{mi} dx_n.$$

We can also define \mathbf{i}_{X_j} to act on this product by contracting in the first tensor product space, that is,

$$\mathbf{i}_{\partial_k}(dx_i \wedge dx_j) = \delta_{ki} dx_j - (-1)^{p(dx_i)} \sum_{m,n} (-1)^{p(dx_m)+p(\mathbf{i}_{\partial_k})p(\hat{R}_{mn}^{ij})} \hat{R}_{mn}^{ij} \delta_{mk} dx_n.$$

Explicitly,

$$\begin{aligned} \mathbf{i}_{\partial_x}(dx \wedge dx) &= hh'dx - h'd\theta, & \mathbf{i}_{\partial_\theta}(dx \wedge dx) &= h'dx, \\ \mathbf{i}_{\partial_x}(dx \wedge d\theta) &= (1 + hh')d\theta + hdx, & \mathbf{i}_{\partial_\theta}(dx \wedge d\theta) &= dx + h'd\theta, \\ \mathbf{i}_{\partial_x}(d\theta \wedge dx) &= d\theta + hdx, & \mathbf{i}_{\partial_\theta}(d\theta \wedge dx) &= (1 + hh')dx - h'd\theta, \\ \mathbf{i}_{\partial_x}(d\theta \wedge d\theta) &= hd\theta, & \mathbf{i}_{\partial_\theta}(d\theta \wedge d\theta) &= (2 + hh')d\theta - hdx. \end{aligned}$$

A tensor product decomposition of products of inner derivations are in the form

$$\mathbf{i}_{\partial_j} \wedge \mathbf{i}_{\partial_k} = \mathbf{i}_{\partial_j} \otimes \mathbf{i}_{\partial_k} - (-1)^{p(\mathbf{i}_{\partial_k})} \sum_{m,n} (-1)^{p(\mathbf{i}_{\partial_m})+[p(\mathbf{i}_{\partial_j}, \mathbf{i}_{\partial_k})]p(\hat{R}_{kj}^{mn})} \hat{R}_{kj}^{mn} \mathbf{i}_{\partial_n} \otimes \mathbf{i}_{\partial_m}.$$

5. An aspect to the contraction

With the singular limit $q \rightarrow 1$ of a linear transformation from the q -deformed plane, the h -deformed plane can be obtained [2]. This method is known as the contraction procedure [18]. The h -deformation of the superplane and some of the superspaces was also made using the same method (see, for example, [8–10, 14]). Using such a contraction method, a two-parameter differential calculus on the superalgebra $\mathcal{O}(\mathbb{C}_h^{1|1})$ is established in [10]. Our aim in this section is to expand the situation to Cartan calculus. The first two subsections are given to form the background of the next subsection. The information in these subsections is taken from [10] and [11], but presented in a more formal and systematic way.

In this section, we will denote (q, q') -deformed objects by capital letters. Lower case letters will represent transformed coordinates. The lemmas in each subsection contain the (q, q') -deformed relations. Four parameters, including q and q' , will appear in the relations that emerge in the following theorems. The corresponding relations in Section 3 are obtained in the limits $q \rightarrow 1$ and $q' \rightarrow 1$ of the resulting relations.

5.1. Generators and their differentials

The q -deformed superalgebra of functions on $\mathbb{C}_q^{1|1}$ generated by X and Θ defined by the relations [22]

$$X\Theta = q\Theta X, \quad \Theta^2 = 0, \tag{30}$$

where $q \in \mathbb{C} - \{0\}$. We now introduce new even generator x and odd generator θ in terms of X and Θ by [8, 10]

$$x = X - \frac{h'}{q' - 1} \Theta, \quad \theta = \left(1 + \frac{hh'}{(q - 1)(q' - 1)}\right) \Theta - \frac{h}{q - 1} X, \tag{31}$$

where h and h' are both Grassmann parameters (that is, $hh' = -h'h$ and $h^2 = 0 = h'^2$) and q' is a nonzero complex parameter. Then, in the limits $q \rightarrow 1$ and $q' \rightarrow 1$, the relations (30) changes to the relations (1).

Remark 6 *If we do not get to the limit, we can talk about the two-parameter deformation of the superalgebra $\mathcal{O}(\mathbb{C}^{1|1})$, denoted by $\mathcal{O}(\mathbb{C}_{q,h}^{1|1})$. One of the deformation parameters is a nonzero complex number and the other is a Grassmann parameter:*

$$x\theta = q\theta x + h x^2, \quad \theta^2 = -h\theta x.$$

The q' -deformed exterior algebra $\Lambda_{q'}(\mathbb{C}_q^{1|1})$ of the q -deformed superplane is generated by Φ and Y with the relations [22]

$$\Phi^2 = 0, \quad \Phi Y = q'^{-1} Y \Phi. \tag{32}$$

If we choose $\Phi = dX$ and $Y = d\Theta$, the differential d is uniquely determined by the conditions $d^2 = 0$ and the \mathbb{Z}_2 -graded Leibniz rule and from (31) we write

$$dx = dX + \frac{h'}{q' - 1} d\Theta, \quad d\theta = \left(1 + \frac{hh'}{(q - 1)(q' - 1)}\right) d\Theta + \frac{h}{q - 1} dX. \tag{33}$$

Inserting (33) to (32), we see that the transformed objects obey the relations (2) in the limits $q \rightarrow 1$ and $q' \rightarrow 1$.

Remark 7 *If we do not get to the limit, we can mention the two-parameter deformation of the superalgebra $\Lambda(\mathbb{C}^{1|1})$:*

$$dx \wedge d\theta = q'^{-1}(d\theta \wedge dx + h' d\theta \wedge d\theta), \quad dx \wedge dx = h' d\theta dx.$$

The quantum de Rham complex $\Omega(\mathbb{C}_q^{1|1})$ is formed by adding to the relations (30) and (32) four cross-commutation relations satisfied between the elements of $\mathcal{O}(\mathbb{C}_q^{1|1})$ and $\Lambda_{q'}(\mathbb{C}_q^{1|1})$, which are given by the following lemma.

Lemma 5.1 [11] *The (q, q') -deformed relations of the elements of $\mathcal{O}(\mathbb{C}_q^{1|1})$ and their differentials are as follows:*

$$U \cdot dX_j = (-1)^{p(U)p(dX_j)} \sum_i dX_i \cdot \sigma_{ij}(U), \quad U, X \in \mathcal{O}(\mathbb{C}_q^{1|1}) \tag{34}$$

where

$$\sigma(X) = \begin{bmatrix} qq'X & (qq' - 1)\Theta \\ 0 & qX \end{bmatrix}, \quad \sigma(\Theta) = \begin{bmatrix} q'\Theta & 0 \\ 0 & \Theta \end{bmatrix}. \tag{35}$$

Theorem 5.2 *The generators of $\mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ and their differentials satisfy the following (q, q', h, h') -deformed commutation relations*

$$u \cdot dx_j = (-1)^{p(u)p(dx_j)} \sum_i dx_i \cdot \sigma_{ij}(u), \quad u, x \in \mathcal{O}(\mathbb{C}_{q,h}^{1|1}) \tag{36}$$

where

$$\sigma(x) = \begin{bmatrix} (qq' - hh')x - h'\theta & hx + (qq' - 1 - hh')\theta \\ -qh'x & q(x - h'\theta) \end{bmatrix}, \quad \sigma(\theta) = \begin{bmatrix} q'(\theta - hx) & q'h\theta \\ -(h'\theta + hh'x) & (1 + hh')\theta - hx \end{bmatrix}. \quad (37)$$

Proof Substituting (31) and (33) into (34), we see that the transformed objects satisfy the relations (36). \square

Remark 8 We arrive at the relations (4) in the limits $q \rightarrow 1$ and $q' \rightarrow 1$.

5.2. The partial derivatives

If we introduce relations of the coordinates of the q -deformed superplane with the partial derivatives after obtaining the derivatives ∂_x and ∂_θ in terms of ∂_X and ∂_Θ , we can find the relations (11) (and also (12)) given in Theorem 3.7.

Lemma 5.3 [11] *The relations between the generators of $\mathcal{O}(\mathbb{C}_q^{1|1})$ and partial derivatives are as follows*

$$\partial_i \cdot X_k = \delta_{ik} + \sum_j (-1)^{p(X_k)p(\partial_j)} \sigma_{ij}(X_k) \cdot \partial_j, \quad (38)$$

where $\partial_1 := \partial_X$, $\partial_2 := \partial_\Theta$ and $X_1 = X$, $X_2 = \Theta$.

Theorem 5.4 *The (q, q', h, h') -deformed commutation relations of the generators of the superalgebra $\mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ with partial derivatives are as follows*

$$\partial_i \cdot x_k = \delta_{ik} + \sum_j (-1)^{p(x_k)p(\partial_j)} \sigma_{ij}(x_k) \cdot \partial_j \quad (39)$$

with the matrices σ in (37).

Proof We can transform the partial derivatives ∂_x and ∂_θ of the elements of $\mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ in terms of ∂_X and ∂_Θ of the generators of $\mathcal{O}(\mathbb{C}_q^{1|1})$ as

$$\partial_x = \left(1 - \frac{hh'}{(q-1)(q'-1)}\right) \partial_X + \frac{h}{q-1} \partial_\Theta, \quad \partial_\theta = \partial_\Theta - \frac{h'}{q'-1} \partial_X. \quad (40)$$

Substituting (40) together with (31) into (38), we get the relations (39). \square

Remark 9 *Relations (39) contain four parameters, and we reached the relations (11) in the limits $q \rightarrow 1$ and $q' \rightarrow 1$.*

5.3. The exterior differentials

In the q -deformed case, it is known that the exterior differential can be written in terms of differentials and partial derivatives as follows:

$$d_q f = (dX \partial_X + d\Theta \partial_\Theta)(f)$$

where f is a differentiable function. If (33) and (40) are considered, then the differential $d_h := d$ preserves its form

$$df = (dx\partial_x + d\theta\partial_\theta)(f).$$

Let us introduce an exterior derivative D_q which maps k -forms to $(k + 1)$ -forms and obeys the rules $D_q^2 = 0$ and

$$\begin{aligned} D_q(\alpha_1 \wedge \alpha_2) &= (D_q\alpha_1) \wedge \alpha_2 + (-1)^{p(\alpha_1)} \alpha_1 \wedge (D_q\alpha_2) \\ &\equiv d_q\alpha_1 \wedge \alpha_2 + (-1)^{p(\alpha_1)} \alpha_1 \wedge d_q\alpha_2 \end{aligned} \tag{41}$$

where α_i 's are any differential forms. The exterior derivative D_q on the superalgebra $\mathcal{O}(\mathbb{C}_q^{1|1})$ is given by

$$D_q f \equiv (dX \partial_X + d\Theta \partial_\Theta)(f),$$

so that it verifies (41) and the rule $D_q\alpha = d\alpha + (-1)^{p(\alpha)} \alpha D_q$, where α is a differential form on $\mathbb{C}_q^{1|1}$. In particular, $D_q X_k = dX_k + (-1)^{p(X_k)} X_k D_q$.

As a note, the case for the operator d above is also valid for the operator D_q . That is, the operator $D_h := D$ also preserves with the same form as D_q .

5.4. The inner derivations

As we mentioned in Section 3, we use partial derivatives of generators as vector fields. We connected the coordinates of the h -deformed superplane to the coordinates of the q -deformed superplane with a singular transformation above. Hence, both differentials and partial derivatives of generators are affected by this situation. Naturally, inner derivations and Lie derivatives will also be affected.

Let us introduce the inner derivations \mathbf{i}_{∂_x} and $\mathbf{i}_{\partial_\theta}$ in terms of \mathbf{i}_{∂_X} and $\mathbf{i}_{\partial_\Theta}$ by

$$\mathbf{i}_{\partial_x} = \left(1 - \frac{hh'}{(q-1)(q'-1)}\right) \mathbf{i}_{\partial_X} + \frac{h}{q-1} \mathbf{i}_{\partial_\Theta}, \quad \mathbf{i}_{\partial_\theta} = \mathbf{i}_{\partial_\Theta} - \frac{h'}{q'-1} \mathbf{i}_{\partial_X}. \tag{42}$$

We will see below that although the expressions in (42) in the limits $q \rightarrow 1$ and $q' \rightarrow 1$ have no limits, the resulting relations are well defined.

Lemma 5.5 [11] (i) *The relations of the generators with the inner derivations are given by*

$$\begin{aligned} \mathbf{i}_{\partial_X} \cdot X &= qq'X \cdot \mathbf{i}_{\partial_X} + (qq' - 1)\Theta \cdot \mathbf{i}_{\partial_\Theta}, & \mathbf{i}_{\partial_\Theta} \cdot X &= qX \cdot \mathbf{i}_{\partial_\Theta}, \\ \mathbf{i}_{\partial_X} \cdot \Theta &= -q'\Theta \cdot \mathbf{i}_{\partial_X}, & \mathbf{i}_{\partial_\Theta} \cdot \Theta &= \Theta \cdot \mathbf{i}_{\partial_\Theta}. \end{aligned} \tag{43}$$

(ii) *The relations of differentials of the generators with the inner derivations are given by*

$$\begin{aligned} \mathbf{i}_{\partial_X} \cdot dX &= 1 - dX \cdot \mathbf{i}_{\partial_X} + (q^{-1}q'^{-1} - 1)d\Theta \cdot \mathbf{i}_{\partial_\Theta}, & \mathbf{i}_{\partial_X} \cdot d\Theta &= q^{-1}d\Theta \cdot \mathbf{i}_{\partial_X}, \\ \mathbf{i}_{\partial_\Theta} \cdot d\Theta &= 1 + q^{-1}q'^{-1}d\Theta \cdot \mathbf{i}_{\partial_\Theta}, & \mathbf{i}_{\partial_\Theta} \cdot dX &= q'^{-1}dX \cdot \mathbf{i}_{\partial_\Theta}. \end{aligned} \tag{44}$$

(iii) *The relations of the partial derivatives with the inner derivations are given by*

$$\begin{aligned} \mathbf{i}_{\partial_X} \cdot \partial_X &= q^{-1}q'^{-1}\partial_X \cdot \mathbf{i}_{\partial_X}, & \mathbf{i}_{\partial_\Theta} \cdot \partial_X &= q^{-1}\partial_X \cdot \mathbf{i}_{\partial_\Theta}, \\ \mathbf{i}_{\partial_X} \cdot \partial_\Theta &= -q'^{-1}[\partial_\Theta \cdot \mathbf{i}_{\partial_X} + (q' - q^{-1})\partial_X \cdot \mathbf{i}_{\partial_\Theta}], & \mathbf{i}_{\partial_\Theta} \cdot \partial_\Theta &= \partial_\Theta \cdot \mathbf{i}_{\partial_\Theta}. \end{aligned} \tag{45}$$

Theorem 5.6 (i) The (q, q', h, h') -deformed relations between the generators and the inner derivations are given, in a compact form, by

$$\mathbf{i}_{\partial_k} \cdot u = \sum_j (-1)^{p(u)p(\mathbf{i}_{\partial_j})} \sigma_{kj}(u) \cdot \mathbf{i}_{\partial_j}, \quad u \in \mathcal{O}(\mathbb{C}_{q,h}^{1|1}) \tag{46}$$

with the matrices σ in (37).

(ii) The (q, q', h, h') -deformed relations of the differentials with the inner derivations are given, in a compact form, by

$$\mathbf{i}_{\partial_k} \cdot dx_m = \delta_{km} + (qq')^{-1} (-1)^{p(\mathbf{i}_{\partial_k})} \sum_j (-1)^{[1+p(dx_m)]p(\mathbf{i}_{\partial_j})} d\sigma_{kj}(x_m) \cdot \mathbf{i}_{\partial_j} \tag{47}$$

for all $x_m \in \mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ with the matrices σ in (37).

(iii) The (q, q', h, h') -deformed commutation relations of the partial derivatives with the inner derivations are given by

$$\begin{aligned} \mathbf{i}_{\partial_x} \cdot \partial_x &= q^{-1} q'^{-1} [(1 - hh') \partial_x \cdot \mathbf{i}_{\partial_x} + h(\partial_\theta \cdot \mathbf{i}_{\partial_x} - q' \partial_x \cdot \mathbf{i}_{\partial_\theta})], \\ \mathbf{i}_{\partial_x} \cdot \partial_\theta &= (qq')^{-1} [-q \partial_\theta \cdot \mathbf{i}_{\partial_x} + qh' \partial_x \cdot \mathbf{i}_{\partial_\theta} - h \partial_\theta \cdot \mathbf{i}_{\partial_\theta} + hh' \partial_x \cdot \mathbf{i}_{\partial_\theta}] + [1 - (qq')^{-1}] \partial_x \cdot \mathbf{i}_{\partial_\theta}, \\ \mathbf{i}_{\partial_\theta} \cdot \partial_x &= q^{-1} (\partial_x \cdot \mathbf{i}_{\partial_\theta} + q'^{-1} h' \partial_x \cdot \mathbf{i}_{\partial_x} + h \partial_\theta \cdot \mathbf{i}_{\partial_\theta} - q'^{-1} hh' \partial_\theta \cdot \mathbf{i}_{\partial_x}), \\ \mathbf{i}_{\partial_\theta} \cdot \partial_\theta &= (1 + q^{-1} q'^{-1} hh') \partial_\theta \cdot \mathbf{i}_{\partial_\theta} - q^{-1} q'^{-1} h' (\partial_x \cdot \mathbf{i}_{\partial_\theta} + q \partial_\theta \cdot \mathbf{i}_{\partial_x}). \end{aligned} \tag{48}$$

Proof (i) Inserting (31) and (42) to (43), directly only after long calculations we obtain explicit relations as follows:

$$\begin{aligned} \mathbf{i}_{\partial_x} \cdot x &= [(qq' - hh')x - h'\theta] \cdot \mathbf{i}_{\partial_x} + (hx - hh'\theta) \cdot \mathbf{i}_{\partial_\theta} + (qq' - 1)\theta \cdot \mathbf{i}_{\partial_\theta}, \\ \mathbf{i}_{\partial_x} \cdot \theta &= -q'(\theta - hx) \cdot \mathbf{i}_{\partial_x} + q'h\theta \cdot \mathbf{i}_{\partial_\theta}, \\ \mathbf{i}_{\partial_\theta} \cdot x &= q(x - h'\theta) \cdot \mathbf{i}_{\partial_\theta} - qh'x \cdot \mathbf{i}_{\partial_x}, \\ \mathbf{i}_{\partial_\theta} \cdot \theta &= [(1 + hh')\theta - hx] \cdot \mathbf{i}_{\partial_\theta} + (h'\theta + hh'x) \cdot \mathbf{i}_{\partial_x}. \end{aligned}$$

These relations are the same as the relations in (46). In the limits $q \rightarrow 1$ and $q' \rightarrow 1$, we get the relations (18).

(ii) Inserting (42) and (33) to (44), we obtain explicit relations as follows:

$$\begin{aligned} \mathbf{i}_{\partial_x} \cdot dx &= 1 - [(1 - q^{-1} q'^{-1} hh')dx + q^{-1} q'^{-1} h'd\theta] \cdot \mathbf{i}_{\partial_x} + q^{-1} q'^{-1} (hdx + hh'd\theta) \cdot \mathbf{i}_{\partial_\theta} \\ &\quad + (q^{-1} q'^{-1} - 1)d\theta \cdot \mathbf{i}_{\partial_\theta}, \\ \mathbf{i}_{\partial_x} \cdot d\theta &= q^{-1} (d\theta + hdx) \cdot \mathbf{i}_{\partial_x} + q^{-1} h d\theta \cdot \mathbf{i}_{\partial_\theta}, \\ \mathbf{i}_{\partial_\theta} \cdot dx &= q'^{-1} (dx + h'd\theta) \cdot \mathbf{i}_{\partial_\theta} + q'^{-1} h' dx \cdot \mathbf{i}_{\partial_x}, \\ \mathbf{i}_{\partial_\theta} \cdot d\theta &= 1 + (qq')^{-1} [(1 + hh')d\theta + hdx] \cdot \mathbf{i}_{\partial_\theta} - (qq')^{-1} (h'd\theta - hh'dx) \cdot \mathbf{i}_{\partial_x}. \end{aligned}$$

These relations are the same as the relations in (47). In the limits $q \rightarrow 1$ and $q' \rightarrow 1$, we get the relations (19).

(iii) Inserting (40) and (42) to (45), we obtain the desired relations. □

Lemma 5.7 [11] *The q -deformed relations between the inner derivations are given by*

$$\mathbf{i}_{\partial_X}^2 = 0, \quad \mathbf{i}_{\partial_X} \mathbf{i}_{\partial_\Theta} = q \mathbf{i}_{\partial_\Theta} \mathbf{i}_{\partial_X}. \tag{49}$$

Theorem 5.8 *The (q, h) -deformed commutation relations between the inner derivations are given by*

$$\mathbf{i}_{\partial_x}^2 = -h \mathbf{i}_{\partial_\theta} \mathbf{i}_{\partial_x}, \quad \mathbf{i}_{\partial_x} \mathbf{i}_{\partial_\theta} = q \mathbf{i}_{\partial_\theta} \mathbf{i}_{\partial_x} - h \mathbf{i}_{\partial_\theta}^2. \tag{50}$$

Proof Inserting (42) to (49), we obtain the desired relations. □

Remark 10 *The dependence of the inner derivations and the partial derivatives are the same as the standard deformed forms:*

$$\mathbf{i}_{\partial_j} D = \partial_j + q^{-1} q'^{-1} (-1)^{p(i_{\partial_j})} D \mathbf{i}_{\partial_j}.$$

5.5. The Lie derivatives

Let us introduce the Lie derivatives \mathcal{L}_{∂_x} and $\mathcal{L}_{\partial_\theta}$ in terms of \mathcal{L}_{∂_X} and $\mathcal{L}_{\partial_\Theta}$ by

$$\mathcal{L}_{\partial_x} = \left(1 - \frac{hh'}{(q-1)(q'-1)} \right) \mathcal{L}_{\partial_X} + \frac{h\mathcal{L}_{\partial_\Theta}}{q-1}, \quad \mathcal{L}_{\partial_\theta} = \mathcal{L}_{\partial_\Theta} - \frac{h'\mathcal{L}_{\partial_X}}{q'-1}. \tag{51}$$

Then, one can easily show that the Lie derivatives \mathcal{L}_{∂_x} and $\mathcal{L}_{\partial_\theta}$ preserve their undeformed forms:

$$\mathcal{L}_{\partial_j} = \mathbf{i}_{\partial_j} D - (-1)^{p(i_{\partial_j})} D \mathbf{i}_{\partial_j}.$$

Lemma 5.9 [11] (i) *The commutation relations of the Lie derivatives with the generators of $\mathcal{O}(\mathbb{C}_q^{1|1})$ are as follows:*

$$\begin{aligned} \mathcal{L}_{\partial_X} \cdot X &= 1 + qq'X \cdot \mathcal{L}_{\partial_X} + (qq' - 1)[\Theta \cdot \mathcal{L}_{\partial_\Theta} + dX \cdot \mathbf{i}_{\partial_X} + (1 - q^{-1}q'^{-1})d\Theta \cdot \mathbf{i}_{\partial_\Theta}], \\ \mathcal{L}_{\partial_X} \cdot \Theta &= q'\Theta \cdot \mathcal{L}_{\partial_X} + (q^{-1} - q')d\Theta \cdot \mathbf{i}_{\partial_X}, \\ \mathcal{L}_{\partial_\Theta} \cdot X &= qX \cdot \mathcal{L}_{\partial_\Theta} + (q'^{-1} - q)dX \cdot \mathbf{i}_{\partial_\Theta}, \\ \mathcal{L}_{\partial_\Theta} \cdot \Theta &= 1 - \Theta \cdot \mathcal{L}_{\partial_\Theta} + (q^{-1}q'^{-1} - 1)d\Theta \cdot \mathbf{i}_{\partial_\Theta}. \end{aligned} \tag{52}$$

(ii) *The relations between the Lie derivatives and the differentials are as follows:*

$$\begin{aligned} \mathcal{L}_{\partial_X} \cdot dX &= dX \cdot \mathcal{L}_{\partial_X} + (1 - q^{-1}q'^{-1})d\Theta \cdot \mathcal{L}_{\partial_\Theta}, & \mathcal{L}_{\partial_X} \cdot d\Theta &= q^{-1}d\Theta \cdot \mathcal{L}_{\partial_X}, \\ \mathcal{L}_{\partial_\Theta} \cdot dX &= -q'^{-1}dX \cdot \mathcal{L}_{\partial_\Theta}, & \mathcal{L}_{\partial_\Theta} \cdot d\Theta &= q^{-1}q'^{-1}d\Theta \cdot \mathcal{L}_{\partial_\Theta}. \end{aligned} \tag{53}$$

(iii) *The relations between the Lie derivatives and the partial derivatives are as follows:*

$$\begin{aligned} \mathcal{L}_{\partial_X} \cdot \partial_X &= \partial_X \cdot \mathcal{L}_{\partial_X}, & \mathcal{L}_{\partial_X} \cdot \partial_\Theta &= q\partial_\Theta \cdot \mathcal{L}_{\partial_X} + (1 - qq')\partial_x \cdot \mathcal{L}_{\partial_\theta}, \\ \mathcal{L}_{\partial_\Theta} \cdot \partial_X &= q'\partial_X \cdot \mathcal{L}_{\partial_\Theta}, & \mathcal{L}_{\partial_\Theta} \cdot \partial_\Theta &= -qq'\partial_\Theta \cdot \mathcal{L}_{\partial_\Theta}. \end{aligned} \tag{54}$$

Theorem 5.10 (i) The (q, q', h, h') -deformed relations between the generators of the superalgebra $\mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ and the Lie derivatives are given, in a compact form, by

$$\mathcal{L}_{\partial_k} \cdot x_m = \delta_{km} + \sum_j (-1)^{p(x_m)p(\mathcal{L}_{\partial_j})} \sigma_{kj}(x_m) \cdot \mathcal{L}_{\partial_j} + (q^{-1}q'^{-1} - 1)(-1)^{p(\mathbf{i}_{\partial_k})} \sum_j \sigma_{kj}^\Omega(x_m) \cdot \mathbf{i}_{\partial_j} \quad (55)$$

for all $x_m \in \mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ with the matrices σ in (37).

(ii) The (q, q', h, h') -deformed relations of the differentials with the Lie derivatives are given, in a compact form, by

$$\mathcal{L}_{\partial_k} \cdot dx_m = (qq')^{-1}(-1)^{p(\mathcal{L}_{\partial_k})} \sum_j (-1)^{[1+p(dx_m)]p(\mathcal{L}_{\partial_j})} \sigma_{kj}^\Omega(x_m) \cdot \mathcal{L}_{\partial_j}, \quad (56)$$

for all $x_m \in \mathcal{O}(\mathbb{C}_{q,h}^{1|1})$ with the matrices σ in (37).

(iii) The (q, q', h, h') -deformed commutation relations of the partial derivatives with the Lie derivatives are given by

$$\begin{aligned} \mathcal{L}_{\partial_x} \cdot \partial_x &= (1 + hh')\partial_x \cdot \mathcal{L}_{\partial_x} - h(q' \partial_x \cdot \mathcal{L}_{\partial_\theta} - \partial_\theta \cdot \mathcal{L}_{\partial_x}), \\ \mathcal{L}_{\partial_x} \cdot \partial_\theta &= \partial_\theta \cdot \mathcal{L}_{\partial_x} - h' \partial_x \cdot \mathcal{L}_{\partial_\theta} + h \partial_\theta \cdot \mathcal{L}_{\partial_\theta} - hh' \partial_x \cdot \mathcal{L}_{\partial_\theta}, \\ \mathcal{L}_{\partial_\theta} \cdot \partial_x &= -q' \partial_x \cdot \mathcal{L}_{\partial_\theta} + h' \partial_x \cdot \mathcal{L}_{\partial_x} + q' h \partial_\theta \cdot \mathcal{L}_{\partial_\theta} - hh' \partial_\theta \cdot \mathcal{L}_{\partial_x}, \\ \mathcal{L}_{\partial_\theta} \cdot \partial_\theta &= -(qq' - hh')\partial_\theta \cdot \mathcal{L}_{\partial_\theta} + h' (\partial_x \cdot \mathcal{L}_{\partial_\theta} + qh' \partial_\theta \cdot \mathcal{L}_{\partial_x}). \end{aligned} \quad (57)$$

Proof (i) Substituting (31) and (51) into (52), directly only after long calculations, we obtain explicit relations as follows:

$$\begin{aligned} \mathcal{L}_{\partial_x} \cdot x &= 1 + [(qq' - hh')x - h'\theta] \cdot \mathcal{L}_{\partial_x} + (qq' - 1 - hh')\theta \cdot \mathcal{L}_{\partial_\theta} \\ &\quad + (q^{-1}q'^{-1} - 1)(hdx \cdot \mathbf{i}_{\partial_\theta} - h'd\theta \cdot \mathbf{i}_{\partial_x} + hh'dx \cdot \mathbf{i}_{\partial_\theta} + hh'd\theta \cdot \mathbf{i}_{\partial_x}) \\ &\quad + (qq' - 1)[dx \cdot \mathbf{i}_{\partial_x} + (1 - q^{-1}q'^{-1})d\theta \cdot \mathbf{i}_{\partial_\theta}], \\ \mathcal{L}_{\partial_x} \cdot \theta &= q'(\theta - hx) \cdot \mathcal{L}_{\partial_x} - q'h\theta \cdot \mathcal{L}_{\partial_\theta} + (q^{-1} - q')[h(dx \cdot \mathbf{i}_{\partial_x} + d\theta \cdot \mathbf{i}_{\partial_\theta}) + d\theta \cdot \mathbf{i}_{\partial_x}], \\ \mathcal{L}_{\partial_\theta} \cdot x &= -qh'x \cdot \mathcal{L}_{\partial_x} + q(x - h'\theta) \cdot \mathcal{L}_{\partial_\theta} + (q'^{-1} - q)[h'(dx \cdot \mathbf{i}_{\partial_x} + d\theta \cdot \mathbf{i}_{\partial_\theta}) + dx \cdot \mathbf{i}_{\partial_\theta}], \\ \mathcal{L}_{\partial_\theta} \cdot \theta &= 1 - (hh'x + h'\theta) \cdot \mathcal{L}_{\partial_x} + [hx - (1 + hh')\theta] \cdot \mathcal{L}_{\partial_\theta} \\ &\quad + (q^{-1}q'^{-1} - 1)[(hh'dx - h'd\theta) \cdot \mathbf{i}_{\partial_x} + (hdx + (1 + hh')d\theta) \cdot \mathbf{i}_{\partial_\theta}]. \end{aligned}$$

(ii) Substituting (33) and (51) into (53), we obtain explicit relations as follows:

$$\begin{aligned} \mathcal{L}_{\partial_x} \cdot dx &= (1 - q^{-1}q'^{-1}hh')dx \cdot \mathcal{L}_{\partial_x} + q^{-1}q'^{-1}[h'd\theta \cdot \mathcal{L}_{\partial_x} - hdx \cdot \mathcal{L}_{\partial_\theta} - hh'd\theta \cdot \mathcal{L}_{\partial_\theta}] \\ &\quad + (1 - q^{-1}q'^{-1})d\theta \cdot \mathcal{L}_{\partial_\theta}, \\ \mathcal{L}_{\partial_x} \cdot d\theta &= q^{-1}d\theta \mathcal{L}_{\partial_x} + q^{-1}hdx \cdot \mathcal{L}_{\partial_\theta} + q^{-1}hd\theta \cdot \mathcal{L}_{\partial_\theta}, \\ \mathcal{L}_{\partial_\theta} \cdot dx &= -q'^{-1}dx \cdot \mathcal{L}_{\partial_\theta} - q'^{-1}hdx \cdot \mathcal{L}_{\partial_x} - q'^{-1}hd\theta \cdot \mathcal{L}_{\partial_\theta}, \\ \mathcal{L}_{\partial_\theta} \cdot d\theta &= q^{-1}q'^{-1}[(1 + hh')d\theta \cdot \mathcal{L}_{\partial_\theta} - hd\theta \cdot \mathcal{L}_{\partial_x} + h'dx \mathcal{L}_{\partial_\theta} + hh'dx \cdot \mathcal{L}_{\partial_x}]. \end{aligned}$$

(iii) Substituting (40) and (51) into (54), directly only after long calculations, we obtain desired relations. \square

Lemma 5.11 [11] *The relations between Lie derivatives are given by the formulas*

$$\mathcal{L}_{\partial_x} \mathcal{L}_{\partial_\theta} = q'^{-1} \mathcal{L}_{\partial_\theta} \mathcal{L}_{\partial_x}, \quad \mathcal{L}_{\partial_\theta}^2 = 0. \tag{58}$$

Theorem 5.12 *The (q', h') -deformed relations between the Lie derivatives are given by*

$$\mathcal{L}_{\partial_x} \mathcal{L}_{\partial_\theta} = q'^{-1} (\mathcal{L}_{\partial_\theta} \mathcal{L}_{\partial_x} - h' \mathcal{L}_{\partial_x}^2), \quad \mathcal{L}_{\partial_\theta}^2 = h' \mathcal{L}_{\partial_x} \mathcal{L}_{\partial_\theta}. \tag{59}$$

Proof Inserting (51) to (58), directly only after long calculations, we obtain desired relations. □

Lemma 5.13 [11] *The relations between the inner derivations and the Lie derivatives are given by the formulas*

$$\begin{aligned} \mathcal{L}_{\partial_x} \mathbf{i}_{\partial_x} &= \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_x}, & \mathcal{L}_{\partial_x} \mathbf{i}_{\partial_\theta} &= q \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_x} + (qq' - 1) \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_\theta} \\ \mathcal{L}_{\partial_\theta} \mathbf{i}_{\partial_x} &= -q' \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_\theta}, & \mathcal{L}_{\partial_\theta} \mathbf{i}_{\partial_\theta} &= qq' \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_\theta}. \end{aligned} \tag{60}$$

The proof of the following theorem follows from (51) and (60).

Theorem 5.14 *The (q, q', h, h') -deformed relations between the inner derivations and the Lie derivatives are given by*

$$\begin{aligned} \mathcal{L}_{\partial_x} \mathbf{i}_{\partial_x} &= (1 - hh') \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_x} + h(q' \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_\theta} + \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_x}), \\ \mathcal{L}_{\partial_x} \mathbf{i}_{\partial_\theta} &= q \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_x} - qh' \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_\theta} - h \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_\theta} + (qq' - 1 + hh') \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_\theta}, \\ \mathcal{L}_{\partial_\theta} \mathbf{i}_{\partial_x} &= -q' \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_\theta} + h' \mathbf{i}_{\partial_x} \mathcal{L}_{\partial_x} - q' h \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_\theta} - hh' \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_x}, \\ \mathcal{L}_{\partial_\theta} \mathbf{i}_{\partial_\theta} &= (qq' + hh') \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_\theta} - h' (\mathbf{i}_{\partial_x} \mathcal{L}_{\partial_\theta} - q \mathbf{i}_{\partial_\theta} \mathcal{L}_{\partial_x}). \end{aligned} \tag{61}$$

6. Discussion

We know that none of the transformations (31), (33), (40), (42), and (51) exist within the limits $q \rightarrow 1$ and $q' \rightarrow 1$. However, all relations obtained using those transformations are well behaved. In the limits $q \rightarrow 1$ and $q' \rightarrow 1$, the relations (55)-(57), (59), and (61) are dropped due to Remark 10, leaving only an extended calculus containing the inner derivations on $\mathcal{O}(\mathbb{C}_h^{1|1})$.

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References

- [1] Aghamohammadi A. The 2-parametric extension of h -deformation of $GL(2)$, and the differential calculus on its quantum plane. *Modern Physics Letters A* 1993; 8 (27): 2607-2613. <https://doi.org/10.1142/S0217732393002981>
- [2] Aghamohammadi A, Khorrami M, Shariati A. Jordanian deformation of $SL(2)$ as a contraction of its Drinfeld-Jimbo deformation. *Journal of Physics A Mathematical and General* 1995; 28 (8): L225-L231. <https://doi.org/10.1088/0305-4470/28/8/001>
- [3] Bruce AJ, Dublij S. Double-graded quantum superplane. *Reports on Mathematical Physics* 2020; 86 (3): 383-400. [https://doi.org/10.1016/S0034-4877\(20\)30089-6](https://doi.org/10.1016/S0034-4877(20)30089-6)

- [4] Celik S. Differential geometry of the q -superplane. *Journal of Physics A Mathematical and General* 1998; 31 (48): 9695-9701. <https://doi.org/10.1088/0305-4470/31/48/006>
- [5] Celik S. Differential geometry of the Z_3 -graded quantum superplane. *Journal of Physics A Mathematical and General* 2002; 35 (19): 4257-4268. <https://doi.org/10.1088/0305-4470/35/19/307>
- [6] Celik S. Bicovariant differential calculus on the quantum superspace $\mathbb{R}_q(1|2)$. *Journal of Algebra and its Applications* 2016; 15 (9): 1650172, 17 pp. <https://doi.org/10.1142/S0219498816501723>
- [7] Celik S. Covariant differential calculi on quantum symplectic superspace $SP_q^{1|2}$. *Journal of Mathematical Physics* 2017; 58 (2): 023508, 15 pp. <https://doi.org/10.1063/1.4975924>
- [8] Celik S. Two-parametric extension of h -deformation of $GL(1|1)$. *Letters in Mathematical Physics* 1997; 42 (12): 299-308. <https://doi.org/10.1023/A:1007372809419>
- [9] Celik S, Celik SA, Arik M. Differential calculus on the h -superplane. *Journal of Mathematical Physics* 1998; 39 (6): 3426-3436. <https://doi.org/10.1063/1.532438>
- [10] Celik S, Celik SA. Two-parameter differential calculus on the h -superplane. *Journal of Mathematical Physics* 1999; 40 (11): 5998-6008. <https://doi.org/10.1063/1.533066>
- [11] Celik S. Cartan calculi on the quantum superplane. *Journal of Mathematical Physics* 2006; 47 (8): 083501, 16 pp. <https://doi.org/10.1063/1.2227638>
- [12] Chryssomalakos C, Schupp P, Zumino B. Induced extended calculus on the quantum plane. *St Petersburg Mathematical Journal* 1995; 6 (3): 685-696.
- [13] Chung WS. Comment on the solutions of the graded Yang-Baxter equation. *Journal of Mathematical Physics* 1994; 35 (5): 2484-2489. <https://doi.org/10.1063/1.530870>
- [14] Dabrowski L, Parashar P. h -deformation of $GL(1|1)$. *Letters in Mathematical Physics* 1996; 38 (10): 331-336. <https://doi.org/10.1007/BF00398357>
- [15] Ewen H, Ogievetsky O, Wess J. Quantum matrices in two dimensions. *Letters in Mathematical Physics* 1991; 22 (8): 297-305. <https://doi.org/10.1007/BF00405605>
- [16] Fakhri H, Laheghi S. Left-covariant first order differential calculus on quantum Hopf supersymmetry algebra. *Journal of Mathematical Physics* 2021; 62 (3): 031702, 11 pp. <https://doi.org/10.1063/5.0031135>
- [17] Faddeev LD, Reshetikhin NY, Takhtajan LA. Quantization of Lie groups and Lie algebras. *Leningrad Mathematical Journal* 1990; 1: 193-225. <https://doi.org/10.1016/B978-0-12-400465-8.50019-5>
- [18] Inonu E, Wigner EP. On the contraction of groups and their representations. *Proceedings of the National Academy of Sciences of the USA* 1953; 39 (6): 510-524. <https://doi.org/10.1073/pnas.39.6.510>
- [19] Karimipour V. The quantum de Rham complexes associated with $SL_h(2)$. *Letters in Mathematical Physics* 1994; 30 (2): 87-98. <https://doi.org/10.1007/BF00939696>
- [20] Kobayashi T, Uematsu T. Differential calculus on the quantum superspace and deformation of phase space. *Zeitschrift für Physik C Particles and Fields* 1992; 56 (6): 193-199. <https://doi.org/10.1007/BF01555514>
- [21] Kobayashi T. Differential calculi on h -deformed Bosonic and Fermionic quantum planes. *Modern Physics Letters A* 1993; 8 (39): 3741-3748. <https://doi.org/10.1142/S0217732393003470>
- [22] Manin YI. Multiparametric quantum deformation of the general linear supergroup. *Communications in Mathematical Physics* 1989; 123 (3): 163-175. <https://doi.org/10.1007/BF01244022>
- [23] Schupp P, Watts P, Zumino B. Differential geometry on linear quantum groups. *Letters in Mathematical Physics* 1992; 25 (6): 139-147. <https://doi.org/10.1007/BF00398310>
- [24] Zakrzewski S. A Hopf star-algebra of polynomials on the quantum $SL(2, \mathbb{R})$ for a unitary R -matrix. *Letters in Mathematical Physics* 1991; 22 (8): 287-289. <https://doi.org/10.1007/BF00405603>

- [25] Wess J, Zumino B. Covariant differential calculus on the quantum hyperplane. Nuclear Physics B 1990; 18 (2): 302-312. [https://doi.org/10.1016/0920-5632\(91\)90143-3](https://doi.org/10.1016/0920-5632(91)90143-3)