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On the weak and strong solutions of the velocity-vorticity model of the g-Navier–Stokes equations

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Abstract: In this work, we consider a velocity-vorticity formulation for the g-Navier–Stokes equations. The system is constructed by combining the velocity-pressure system which is included by using the rotational formulation of the nonlinearity and the vorticity equation for the g -Navier–Stokes equations. We prove the existence and uniqueness of weak and strong solutions of this system with the periodic boundary conditions.

Key words: Existence and uniqueness, velocity-vorticity formulation, g-Navier–Stokes Equations

1. Introduction

The velocity-vorticity formulation of the Navier–Stokes equations has been studied extensively by many researchers [1, 7, 10, 19]. The velocity-vorticity model is important to study the properties of the asymptotic behavior of the solutions decay rates [6] and this model is more realistic to understand for strongly rotating flow, and excellent numerical results are given for the system in [1, 10, 15, 18, 19]. In recent years, the velocity-Voigt model has been considered a good approach to study fluid motion. The model has extremely attracted the attention of researchers [3, 11, 28, 29]. Afterward, Larios, Pei, and Rebholz suggest a new model called 3D velocity-vorticity-Voigt (VVV) model in [14]. They prove global well-posedness and regularity of the solutions of the equations which is given of this model. Due to the advantage of the velocity-vorticity models mentioned above, we consider the velocity-vorticity model for the g-Navier–Stokes (gNS) equations:

$$\frac{\partial u}{\partial t} + \nu\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \tag{1}$$

$$\nabla\cdot(gu) = 0 \tag{2}$$

where g is suitable smooth real-valued function. These equations have applications in lakes and shallow waters [4, 16, 17]. For the details of the derivation of the gNS equations, the readers can refer to [23, 25]. Firstly, Hale and Raugel [8, 9] have studied thin domain problems. Then Raugel and Sell [21] and Temam and Ziane [27] have studied Navier–Stokes equations in thin domains which is given by $\Omega_\epsilon = \Omega\times(0,\epsilon)$ where $\Omega\subset\mathbb{R}^2$ and $0<\epsilon<1$. Taking a smooth enough real-valued g function instead of ϵ, Roh derives gNS equations using

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thin domain techniques given in [8, 9, 20]. Roh et al. [2, 13, 24] studied the existence and uniqueness, stability, and long-time behavior of the solutions and attractors of gNS equations. Kaya and Çelebi [12] investigated the existence and uniqueness of g-Kelvin Voigt equations. In this study, we adapt some of the ideas of Larios et al. in [14] for proving the existence and uniqueness of the solutions of the velocity-vorticity model for gNS equations. Of particular interest herein are the velocity-vorticity system of the gNS equations over the two dimensional periodic box $\Omega = (0,1)^2$ in the following form:

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{1}{g}(\nabla g\cdot\nabla)u + w\times u + \triangledown P = f \ , \tag{3}$$

$$\frac{\partial w}{\partial t} - \nu\Delta_g w + \nu\frac{1}{g}(\nabla g\cdot\nabla)w + u\cdot\nabla w = \nabla\times f + w(\frac{\nabla g}{g}\cdot u), \tag{4}$$

$$\nabla\cdot(gu) = 0, \tag{5}$$

$$u(x,0) = u_0, \qquad\qquad w(x,0) = w_0 \tag{6}$$

where we define g-Laplacian operator

$$-\Delta_g u = -\frac{1}{g}(\nabla\cdot g\nabla)u = -\Delta u - \frac{1}{g}(\nabla g\cdot\nabla)u.$$

We consider this problem under periodic boundary conditions. Here we use the equality $\frac{1}{2}\nabla(u\cdot u) = (u\cdot\nabla)u + u\times(\nabla\times u)$ to rewrite nonlinear term in the gNS equations. In this system, u is the fluid velocity, p is the pressure and $P = p + \frac{1}{2}|u|^2$, w which play the role of vorticity, f is the external forcing term. We obtain (4) by taking curl of gNS equations where $w = u_{2x} - u_{1y}$ is a scalar vorticity and $w\times u := (-u_2 w, u_1 w)^T$. Note that the vorticity equation reduces to a scalar equation. This paper is arranged as follows:

In Section 2, we give some necessary notation and preliminary results for the analysis of the system (3) − (6). In Section 3, we establish the existence and uniqueness of the weak and strong solutions of the velocity-vorticity model for the gNS equations.

2. Preliminaries and functional setting

In this part, we introduce some symbols and concepts (see, e.g., [5, 25, 26]).

$L^2(\Omega, g)$ denotes the Hilbert space with the inner product and norm

$$(u,v)_g = \int_\Omega (u\cdot v)\, g dx \text{ and } \|u\|^2_{L^2(\Omega,g)} = (u,u)_g,$$

respectively. The norm in $H^1(\Omega, g)$

$$\|u\|_{H^1(\Omega,g)} = \left[(u,u)_g + \sum_{i=1}^{2}(D_i u, D_i u)_g\right]^{\frac{1}{2}}$$

where $D_i = \frac{\partial}{\partial x_i}$. We consider spaces of real functions defined on $\mathbb{R}^2$ and are periodic with period 1 in x_1 and x_2 directions,

$$u\left(x+e_i\right) = u\left(x\right), \ i = 1,2$$

where e_1 and e_2 are the canonical basis of $\mathbb{R}^2$. In this case, we denote by $C^{\infty}_{per}(\Omega)$ or $L^2_{per}(\Omega, g)$ the space of restrictions to Ω of periodic functions in the sense of above equations. We define spaces in the periodic setting for the gNS equations

$$\mathcal{V}_1 = \left\{ u \in \left(C^{\infty}_{per}(\Omega)\right)^2 : \nabla \cdot gu = 0, \int_{\Omega} u dx = 0 \right\},$$

$$\mathbf{H}_g = \text{the closure of } \mathcal{V}_1 \text{ in } \left(L^2(\Omega, g)\right)^2,$$

$$\mathbf{V}_g = \text{the closure of } \mathcal{V}_1 \text{ in } \left(H^1(\Omega, g)\right)^2,$$

$$\mathbf{H}'_g = \text{ the dual space of } \mathbf{H}_g,$$

$$\mathbf{V}'_g = \text{ the dual space of } \mathbf{V}_g$$

in two dimensions. Vorticity is considered a scalar, we define vorticity space as follows:

$$\mathcal{V}_2 = \left\{ u \in C^{\infty}_{per}(\Omega) : \nabla \cdot gu = 0, \int_{\Omega} u dx = 0 \right\},$$

$$H_g = \text{the closure of } \mathcal{V}_2 \text{ in } L^2(\Omega, g),$$

$$V_g = \text{the closure of } \mathcal{V}_2 \text{ in } H^1(\Omega, g),$$

$$V'_g = \text{ the dual space of } V_g,$$

$$H'_g = \text{ the dual space of } H_g,$$

$$H^{curl}_g = \left\{ f \in H_g : \nabla \times f \in L^2(\Omega, g) \right\}.$$

The inner product and norm in $\mathbf{H}_g$ and H_g are the same of $L^2(\Omega, g)$. The norm in $\mathbf{V}_g$ and V_g are the same as that of $H^1(\Omega, g)$. Let the function $g = g(x_1, x_2)$ be the suitable smooth real-valued function. We assume that g satisfies the following conditions:

(i) $g(x_1, x_2) \in C^{\infty}_{per}(\Omega)$

(ii) $0 < m_0 \leq g(x_1, x_2) \leq M_0$ where m_0 and M_0 are positive constants for all $(x_1, x_2) \in \Omega$

(iii) $\|\nabla g\|_{\infty} = \sup_{(x_1,x_2)\in\Omega} |\nabla g(x_1, x_2)| < \infty$

Two spaces $L^2(\Omega)$ and $L^2(\Omega, g)$ have equivalent norms and the following inequalities are satisfied:

$$\sqrt{m_0}\, \|u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega,g)} \leq \sqrt{M_0}\, \|u\|_{L^2(\Omega)}$$

where m_0 and M_0 are positive constants. Now we define g-Stokes operator and some notations as follows: (see, e.g., [14, 25])

$$A_g u = P_g \left[-\frac{1}{g} (\nabla \cdot g \nabla) u \right].$$

A_g has countable eigenvalues that satisfy the following inequality

$$0 < \lambda_g \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$$

where $\lambda_g = \frac{4\pi^2 m_0}{M_0}$. The Poincaré inequality

$$\sqrt{\lambda_g} \|\varphi\|_{L^2} \leq \|\nabla\varphi\|_{L^2}$$

is satisfied for all $\varphi \in V_g$ where λ_1 is the first eigenvalue of the Stokes operator A_g. Since the operator A_g and P_g are self-adjoint, using integration by parts we have

$$\langle A_g u, u\rangle_g = \int_\Omega (\nabla u : \nabla u)\, g dx = (\nabla u, \nabla u)_g = \|\nabla u\|_{L^2}^2$$

where the operation " : " denoted by " dot product " of two matrices. Now we write the following compact embedding

$$V_g \subset H_g \equiv H_g^{'} \subset V_g^{'}.$$

We define the operator $C_g u = P_g\left[\frac{1}{g}(\nabla g \cdot \nabla)u\right]$ and $\langle C_g u, v\rangle_g = b_g\left(\frac{\nabla g}{g}, u, v\right)$, $P_g : L^2_{per}(\Omega, g) \longrightarrow H_g(\Omega)$ Helmholtz-Leray orthogonal projection. The bilinear operator $B_g : V_g \times V_g \to V_g'$

$$B_g(u, v) = P_g(u \cdot \nabla)v$$

and the trilinear form b_g is defined as

$$b_g(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i (D_i v_j) w_j g dx = (P_g(u \cdot \nabla)v, w)_g, \text{ for all } u, v, w \in V_g.$$

We have the following properties

$$i)\ b_g(u, v, w) = -b_g(u, w, v), \tag{7}$$

$$ii)\ b_g(u, v, v) = 0 \tag{8}$$

for sufficiently smooth functions. Through the paper, c is a constant that changes from line to line. Now we will give the following lemma (see [22, 26]).

Lemma 2.1 *The bilinear operator B_g and b_g satisfy the following inequalities in the case $n = 2$:*

$$\left|\langle B_g(u, v), w\rangle_{V_g'}\right| \leq c \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla v\|_{L^2}^{\frac{1}{2}} \|Av\|_{L^2}^{\frac{1}{2}} \|w\|_{L^2} \quad \text{for all } u \in V_g, v \in D(A_g), w \in H_g, \tag{9}$$

$$|b_g(u, v, w)| \leq c \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla v\|_{L^2} \|w\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} \quad \text{for all } u, v, w \in V_g, \tag{10}$$

$$|B_g(u, v)| \leq c \|u\|_{L^2}^{\frac{1}{2}} \|Au\|_{L^2}^{\frac{1}{2}} \|\nabla v\|_{L^2}, \quad \text{for all } u \in D(A_g),\ v \in V_g. \tag{11}$$

3. Existence and uniqueness of weak and strong solutions

In this section, we define weak and strong solutions and give proof of the existence and uniqueness of solutions of the velocity-vorticity model for gNS equations. Using the notation in Section 2, we apply P_g to $(3)-(6)$, the equivalent system of equations is obtained in the following similar to Navier–Stokes equations (see, e.g., [26]).

$$\frac{du}{dt}+\nu A_g u+\nu C_g u+P_g\left(w\times u\right)=P_g f, \tag{12}$$

$$\frac{dw}{dt}+\nu A_g w+\nu C_g w+B_g\left(u,w\right)=P_g\left(\nabla\times f\right)+P_g\left(w\left(\frac{\nabla g}{g}\cdot u\right)\right), \tag{13}$$

$$u(0)=u_0, \qquad\qquad w(0)=w_0. \tag{14}$$

Here we consider the definition of weak solution of the system $(12)-(14)$ in the following form.

Definition 3.1 *Let $T>0$, $f\in L^2(0,T;H_g^{curl})$, $u_0\in\mathbf{H}_g$, and $w_0\in L^2(\Omega,g)$. Then (u,w) is called a weak solution on the time interval $[0,T]$ to system $(3)-(6)$, if $u\in C(0,T;\mathbf{H}_g)\cap L^2(0,T;\mathbf{V}_g)$, $u_t\in L^2(0,T;\mathbf{V}_g^{'})$, $w\in C(0,T;H_g)\cap L^2(0,T;V_g)$, $w_t\in L^2(0,T;V_g^{'})$; moreover, (u,w) satisfies system $(3)-(6)$ in the weak sense following equations*

$$\left(u_t,\boldsymbol{\psi}\right)_g+\nu\left(\nabla u,\nabla\boldsymbol{\psi}\right)_g+\left(w\times u,\boldsymbol{\psi}\right)_g+\nu\left(C_g u,\boldsymbol{\psi}\right)_g=\left(f,\boldsymbol{\psi}\right)_g, \tag{15}$$

$$\left(w_t,\psi\right)_g+\nu\left(\nabla w,\nabla\psi\right)_g+\left(B_g(u,w),\psi\right)+\nu\left(C_g w,\psi\right)_g=\left(\nabla\times f,\psi\right)_g+\left(w\left(\frac{\nabla g}{g}\cdot u\right),\psi\right)_g \tag{16}$$

for any $\boldsymbol{\psi}\in L^2(0,T;\mathbf{V}_g)$ and $\psi\in L^2(0,T;V_g)$. Note that by taking $\boldsymbol{\psi}=\boldsymbol{v}\boldsymbol{\varphi}(t)$ and $\psi=v\varphi(t)$ for $\boldsymbol{v}\in\mathbf{V}_g$, $\boldsymbol{\varphi}\in\left(C_c^1(0,T)\right)^2$ and $v\in V_g$, $\varphi\in C_c^1(0,T)$, it follows that formulation $(12)-(13)$ is equivalent to formulation $(15)-(16)$, interpreted as an operator equation holding in an appropriate distributional sense.

Theorem 3.2 *Let $f\in L^2(0,T;H_g^{curl})$, $u_0\in\mathbf{H}_g$, and $w_0\in L^2(\Omega,g)$. Then the velocity-vorticity system $(3)-(6)$ has a unique global weak solution (u,w) in the sense of Definition 3.1 that satisfies $\nabla\cdot(gw)=0$.*

Proof We prove the theorem using the well-known Feado-Galerkin method [22, 26]. We consider the following finite-dimensional Galerkin ordinary differential equations for $(12)-(13)$.

$$\frac{du_m}{dt}+\nu A_g u_m+\nu C_g u_m+P_m\left(w_m\times u_m\right)=P_m f, \tag{17}$$

$$\frac{dw_m}{dt}+\nu A_g w_m+\nu C_g w_m+B_g\left(u_m,w_m\right)=P_m\left(\nabla\times f\right)+P_m\left(w_m\left(\frac{\nabla g}{g}\cdot u_m\right)\right) \tag{18}$$

with initial data $u_m\left(0\right)=u_{0m}$, $w_m\left(0\right)=w_{0m.}$ We will use $(17)-(18)$ to obtain converges for u_m and w_m. Now using standart technique (see, e.g., [26]), we define an approximate solution for $(12)-(14)$. Since $\mathbf{V}_g$ is separable and $\mathcal{V}_1$ is dense in $\mathbf{V}_g$, there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ which forms a complete orthonormal system in $\mathbf{H}_g$ and a base for $\mathbf{V}_g$. Likewise there exists a sequence $\{y_j\}_{j\in\mathbb{N}}$ which constructs a complete orthonormal system in $L^2(\Omega, g)$ and a base for H_g. Let m be arbitrary fixed positive integer. For each m, we define an approximate solution $(u_m(t), w_m(t))$ of (12) − (14) as follows:

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t)x_j, \qquad w_m(t) = \sum_{j=1}^{m} h_{jm}(t)y_j$$

and

$$\left(u_m^{'}(t), x_i\right)_g + \nu \left(\nabla u_m(t), \nabla x_i\right)_g + \nu b_g \left(\frac{\nabla g}{g}, u_m(t), x_i\right) + \left(w_m(t) \times u_m(t), x_i\right)_g$$

$$= \left(f(t), x_i\right)_g \quad (i = 1, ..., m) \quad t \in [0, T], \tag{19}$$

$$\left(w_m^{'}(t), y_i\right)_g + \nu \left(\nabla w_m(t), \nabla y_i\right)_g + \nu b_g \left(\frac{\nabla g}{g}, w_m(t), y_i\right) + b_g \left(u_m(t), w_m(t), y_i\right)$$

$$= \left(\nabla \times f(t), y_i\right)_g + \left(w_m(t)\left(\frac{\nabla g}{g} \cdot u_m(t)\right), y_i\right)_g \quad (i = 1, ..., m) \;\; t \in [0, T], \tag{20}$$

$$u_m(0) = u_{0m}, \qquad u_m(0) = u_{0m} \tag{21}$$

where u_{0m} and w_{0m} are the orthogonal projections in $\mathbf{H}_g$ and H_g of u_0 and w_0, respectively, onto the space spanned by $x_1, ..., x_m$ and $y_1, ..., y_m$, respectively. The equations (19)−(20) give nonlinear differential equations for g_{jm}, h_{jm}. $j = 1, \ldots, m$ in the following form

$$g_{jm}^{'}(t) = F(g_{jm}(t)),$$

$$h_{jm}^{'}(t) = G(h_{jm}(t))$$

and the condition (21) is equivalent to 2m equations

$$g_{jm}(0) = \left(u_{0m}, x_j\right),$$

$$h_{jm}(0) = \left(w_{0m}, y_j\right).$$

This system forms a nonlinear first order system of ordinary differential equations for the function $g_{jm}(t)$ and $h_{jm}(t)$ and has a maximal solution defined on some interval $[0, t_m]$, $t_m < T$. We shall prove later the priori estimates for the solutions of (19) − (20) so we obtain $t_m = T$ [26]. We multiply (19) and (20) by $g_{jm}(t)$ and $h_{jm}(t)$ respectively, and add these equations for $j = 1, \ldots, m$. We get

$$\left(u_m^{'}(t), u_m(t)\right)_g + \nu \left(\nabla u_m(t), \nabla u_m(t)\right)_g + \nu b_g \left(\frac{\nabla g}{g}, u_m(t), u_m(t)\right) = \left(f(t), u_m(t)\right)_g, \tag{22}$$

$$\left(w_m^{'}(t), w_m(t)\right)_g + \nu \left(\nabla w_m(t), \nabla w_m(t)\right)_g + \nu b_g \left(\frac{\nabla g}{g}, w_m(t), w_m(t)\right) + b_g \left(u_m(t), w_m(t), w_m(t)\right)$$

$$= \left(\nabla \times f(t), w_m(t)\right)_g + \left(w_m(t)(\frac{\nabla g}{g} \cdot u_m(t)), w_m(t)\right)_g. \tag{23}$$

Using Cauchy-Schwarz and Young inequalities for (22), we obtain the following inequalities:

$$\frac{d}{dt}\left\|u_m(t)\right\|_{L^2}^2+\frac{3\nu}{2}\left\|\nabla u_m(t)\right\|_{L^2}^2\leq\frac{2}{\nu}\left\|f(t)\right\|_{L^2}^2+\left(\frac{\nu}{2}+\frac{2\nu\left\|\nabla g\right\|_\infty^2}{m_0^2}\right)\left\|u_m(t)\right\|_{L^2}^2. \tag{24}$$

From the last inequality, we write:

$$\frac{d}{dt}\left\|u_m(t)\right\|_{L^2}^2\leq\frac{2}{\nu}\left\|f(t)\right\|_{L^2}^2+\alpha\left\|u_m(t)\right\|_{L^2}^2 \tag{25}$$

where $\alpha=\frac{\nu}{2}+\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2}$. Again using Gronwall inequality for (25), we get

$$\sup_{s\in[0,T]}\left\|u_m(s)\right\|_{L^2}^2\leq e^{\alpha T}\left(\left\|u_0\right\|^2+\frac{2}{\nu}\int_0^T\left\|f(s)\right\|_{L^2}^2 ds\right). \tag{26}$$

Let us denote right hand side of (26) by $K_1(T)=e^{\alpha T}\left(\left\|u_0\right\|^2+\frac{2}{\nu}\int_0^T\left\|f(s)\right\|_{L^2}^2 ds\right)$. Using Cauchy-Schwarz and Young inequalities and (10) are applied for (23), we obtain the following inequalities:

$$\frac{d}{dt}\left\|w_m(t)\right\|_{L^2}^2+\nu\left\|\nabla w_m(t)\right\|_{L^2}^2\leq\frac{2}{\nu}\left\|\nabla\times f(t)\right\|_{L^2}^2+\beta\left\|w_m(t)\right\|_{L^2}^2 \tag{27}$$

where $\beta=\left\{\frac{\nu}{2}+\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2}+\frac{2c\|\nabla g\|_\infty^2}{\nu}K_1(T)\right\}$. Neglecting the last term in the left hand side of (27), we obtain

$$\frac{d}{dt}\left\|w_m(t)\right\|_{L^2}^2\leq\frac{2}{\nu}\left\|\nabla\times f(t)\right\|_{L^2}^2+\beta\left\|w_m(t)\right\|_{L^2}^2. \tag{28}$$

Again using Gronwall inequality for (28), we get

$$\sup_{s\in[0,T]}\left\|w_m(s)\right\|_{L^2}^2\leq e^{\beta T}\left(\left\|w_0\right\|^2+\frac{2}{\nu}\int_0^T\left\|\nabla\times f(s)\right\|_{L^2}^2 ds\right). \tag{29}$$

Let us denote right hand side of (29) by $K_2(T)=e^{\beta T}\left(\left\|w_0\right\|^2+\frac{2}{\nu}\int_0^T\left\|\nabla\times f(s)\right\|_{L^2}^2 ds\right)$. From (26) and (29), we imply that the sequences $\{u_m\}_m$ and $\{w_m\}_m$ remain in a bounded set of $L^\infty(0,T;\mathbf{H}_g)$ and $L^\infty(0,T;L^2(\Omega,g))$, respectively. Then we integrate (24) from 0 to T, we get

$$\left\|u_m(T)\right\|_{L^2}^2+\frac{3\nu}{2}\int_0^T\left\|\nabla u_m(t)\right\|_{L^2}^2 dt\leq\left\|u_{0m}\right\|^2+\frac{2}{\nu}\int_0^T\left\|f(t)\right\|_{L^2}^2 dt+\alpha\int_0^T\left\|u_m(t)\right\|_{L^2}^2 dt.$$

Neglecting the first term in the left hand side of above equation, we write

$$\int_0^T\left\|\nabla u_m(t)\right\|_{L^2}^2 dt\leq\frac{2}{3\nu}\left\{\left\|u_0\right\|^2+\frac{2}{\nu}\int_0^T\left\|f(t)\right\|_{L^2}^2 dt+\alpha K_1(T)T\right\}. \tag{30}$$

Let us denote right hand side of (30) by

$$K_3(T) = \frac{2}{3\nu}\left\{\|u_0\|^2 + \frac{2}{\nu}\int_0^T \|f(t)\|_{L^2}^2\,dt + \alpha K_1(T)T\right\}$$

and then we integrate (27) from 0 to T, we get

$$\|w_m(T)\|_{L^2}^2 + \nu\int_0^T \|\nabla w_m(t)\|_{L^2}^2\,dt \le \|w_{0m}\|^2 + \frac{2}{\nu}\int_0^T \|\nabla\times f(t)\|_{L^2}^2\,dt + \beta\int_0^T \|w_m(t)\|_{L^2}^2\,dt.$$

Neglecting the first term in the left hand side of above equation, we have

$$\int_0^T \|\nabla w_m(t)\|_{L^2}^2\,dt \le \frac{1}{\nu}\left\{\|w_0\|^2 + \frac{2}{\nu}\int_0^T \|\nabla\times f(t)\|_{L^2}^2\,dt + \beta K_2(T)T\right\}. \tag{31}$$

Let us denote right hand side of (31) by

$$K_4(T) = \frac{1}{\nu}\left\{\|w_0\|^2 + \frac{2}{\nu}\int_0^T \|\nabla\times f(t)\|_{L^2}^2\,dt + \beta K_2(T)T\right\}.$$

Here using the hypothesis of theorem on data (30) and (31), we imply that the sequences $\{u_m\}_m$ and $\{w_m\}_m$ remain in a bounded set of $L^2(0,T;\mathbf{V}_g)$ and $L^2(0,T;H^1(\Omega,g))$, respectively. Due to the estimates (26) and (30), we assert the existence of elements

$$u_m \in L^\infty(0,T;\mathbf{H}_g)\cap L^2(0,T;\mathbf{V}_g). \tag{32}$$

Due to the estimates (29) and (31), we assert the existence of elements

$$w_m \in L^\infty(0,T;L^2(\Omega,g))\cap L^2(0,T;H^1(\Omega,g)). \tag{33}$$

Multiplying (18) by g and then taking divergence of this equations and denoting $v_m = \nabla\cdot(gw_m)$, we obtain

$$\frac{dv_m}{dt} + B_g(u_m, v_m) = 0. \tag{34}$$

Then, multiplying (34) by v_m and integrating from 0 to T, we get

$$\|v_m(T)\|_{L^2}^2 \le \|v_m(0)\|_{L^2}^2\,. \tag{35}$$

Let us use [22, Lemma7.5] and the initial data $w_0 \in H_g$, then we have $v_m(0)$. This implies $\nabla\cdot(gw_m) = v_m = 0$ in $L^2(0,T;L^2(\Omega,g))$. Since $L^2(0,T;H_g)$ is closed in $L^2(0,T;L^2(\Omega,g))$, we write $\nabla\cdot(gw_0) = 0$. Namely, we get

$$w_m \in L^\infty(0,T;H_g)\cap L^2(0,T;V_g). \tag{36}$$

Now, we will show that $\frac{du_m}{dt}$ is uniformly bounded in $L^2(0,T;\mathbf{V}_g^{'})$. We take the inner product of (17) with $\boldsymbol{\psi} \in L^2(0,T;\mathbf{V}_g)$ for all test functions.

$$\left|\langle P_m(w_m \times u_m), \boldsymbol{\psi}\rangle_{\mathbf{V}_g'}\right| \leq c\,\|u_m\|_{L^2}^{\frac{1}{2}}\,\|\nabla u_m\|_{L^2}^{\frac{1}{2}}\,\|w_m\|_{L^2}\,\|\boldsymbol{\psi}\|_{\mathbf{H}^1},$$

$$\left|\left\langle P_m\left(\frac{1}{g}(\nabla g \nabla)u_m\right), \boldsymbol{\psi}\right\rangle_{\mathbf{V}_g'}\right| \leq c\,\|\nabla g\|_{\infty}\,\|\nabla u_m\|_{L^2}\,\|\boldsymbol{\psi}\|_{\mathbf{H}^1}.$$

Thus, $P_m(w_m \times u_m)$ and $C_g u_m$ are uniformly bounded $L^2(0,T;\mathbf{V}_g^{'})$ and it is easily seen that f and $A_g u_m$ are bounded in $L^2(0,T;\mathbf{V}_g^{'})$ uniformly in m. We obtain that

$$\frac{du_m}{dt} \text{ is uniformly bounded in } L^2(0,T;\mathbf{V}_g^{'}). \tag{37}$$

Now, we will show that $\frac{dw_m}{dt}$ is uniformly bounded in $L^2(0,T;V_g^{'})$. We take the inner product of (18) with $\psi \in L^2(0,T;V_g)$ for all test functions.

$$\left|\left\langle P_m w_m\left(\frac{\nabla g}{g}\cdot u_m\right), \psi\right\rangle_{V_g'}\right| \leq c\,\|\nabla g\|_{\infty}\,\|w_m\|_{L^2}^{\frac{1}{2}}\,\|\nabla w_m\|_{L^2}^{\frac{1}{2}}\,\|u_m\|_{L^2}^{\frac{1}{2}}\,\|\nabla u_m\|_{L^2}^{\frac{1}{2}}\,\|\psi\|_{H^1},$$

$$\left|\langle B_g(u_m, w_m), \psi\rangle_{V_g'}\right| \leq c\,\|g\|_{\infty}\,\|u_m\|_{L^2}^{\frac{1}{2}}\,\|\nabla u_m\|_{L^2}^{\frac{1}{2}}\,\|\nabla w_m\|_{L^2}\,\|\psi\|_{L^2}^{\frac{1}{2}}\,\|\nabla\psi\|_{L^2}^{\frac{1}{2}},$$

$$\left|\left\langle P_m\left(\frac{1}{g}(\nabla g \cdot \nabla)w_m\right), \psi\right\rangle_{V_g'}\right| \leq c\,\|\nabla g\|_{\infty}\,\|\nabla w_m\|_{L^2}\,\|\psi\|_{H^1}.$$

Thus, $P_m w_m\left(\frac{\nabla g}{g}\cdot u_m\right)$, $B_g(u_m, w_m)$ and $C_g w_m$ are uniformly bounded $L^2(0,T;V_g^{'})$ and it is easily seen that $\nabla \times f$ and $A_g w_m$ are bounded in $L^2(0,T;V_g^{'})$, uniformly in m, we obtain that

$$\frac{dw_m}{dt} \text{ is uniformly bounded in } L^2(0,T;V_g^{'}). \tag{38}$$

From (32) and (37) and Aubin compactness theorem (see, e.g., [22, 26]), we conclude that there is a subsequence (which we shall relabel) $u_m(t)$ and a function $u(t)$ such that

$$u_m \rightharpoonup u \text{ weakly in } L^2(0,T;\mathbf{V}_g), \tag{39}$$

$$u_m \rightharpoonup u \text{ weak-* in } L^\infty(0,T;\mathbf{H}_g). \tag{40}$$

(39), (40), and (37) give us

$$u_m \to u \text{ strongly in } L^2(0,T;\mathbf{H}_g). \tag{41}$$

Similarly, from (36) and (38) and Aubin compactness theorem, there is a subsequence (which we shall relabel) $w_m(t)$ and a function $w(t)$ such that

$$w_m \rightharpoonup w \text{ weakly in } L^2(0,T;V_g), \tag{42}$$

$$w_m \rightharpoonup w \text{ weak-* in } L^\infty(0,T;H_g). \tag{43}$$

(42), (43), and (38) give us

$$w_m \to w \text{ strongly in } L^2(0,T;H_g). \tag{44}$$

In order to pass to the limit, we consider the scalar functions $\boldsymbol{\psi} \in C_c^1([0,T];\mathbf{V}_g)$ and $\psi \in C_c^1([0,T];V_g)$. Now taking inner product of (17) and (18) by $\boldsymbol{\psi}$ and ψ, respectively, and then integrating in time over $[0,t]$, $0<t<T$ and using integration by parts, we have

$$-\int_0^t (u_m,\boldsymbol{\psi}_t)_g\,ds + (u_m(t),\boldsymbol{\psi}(t))_g - (u_m(0),\boldsymbol{\psi}(0))_g + \nu\int_0^t \left(A_g^{\frac12}u, A_g^{\frac12}\boldsymbol{\psi}\right)_g ds$$

$$+\int_0^t (u_m\times w_m, P_m\boldsymbol{\psi})_g\,ds + \nu\int_0^t b_g\left(\frac{\nabla g}{g},u_m,\boldsymbol{\psi}\right)ds = \int_0^t (f,\boldsymbol{\psi})_g\,ds, \tag{45}$$

$$-\int_0^t (w_m,\psi_t)_g\,ds + (w_m(t),\psi(t))_g - (w_m(0),\psi(0))_g$$

$$+\nu\int_0^t \left(A_g^{\frac12}w, A_g^{\frac12}\psi\right)_g ds + \nu\int_0^t b_g\left(\frac{\nabla g}{g},w_m,\psi\right)ds + \int_0^t b_g(u_m,w_m,\psi)ds$$

$$=\int_0^t (\nabla\times f,\psi)_g\,ds + \int_0^t \left(w_m\left(\frac{\nabla g}{g}\cdot u_m\right),P_m\psi\right)_g ds. \tag{46}$$

Using the standard arguments, as in theory of Navier–Stokes equations (see, e.g., [22, 26]), we can pass to limit (45) and (46). Considering the selection of $\boldsymbol{\psi}$ and ψ in (45) and (46), respectively, we have expressed the convergence of each term as in the following. The sequence $u_m(t)$ converges weakly in $L^2(0,T;\mathbf{H}_g)$ and this implies that

$$(u_m(t),\boldsymbol{\psi}(t)) \to (u(t),\boldsymbol{\psi}(t)),$$

$$(u_m(0),\boldsymbol{\psi}(t)) \to (u(0),\boldsymbol{\psi}(t))$$

for a.e. $t\in[0,T]$. Similarly, the sequence $w_m(t)$ converges weakly in $L^2(0,T;H_g)$ and this implies that

$$(w_m(t),\psi(t)) \to (w(t),\psi(t)),$$

$$(w_m(0),\psi(t)) \to (w(0),\psi(t))$$

for a.e. $t \in [0,T]$. Concerning the nonlinear term in (45), we get

$$\left|\int_0^t (u_m \times w_m, P_m\ \boldsymbol{\psi})_g\, ds - \int_0^t (u \times w, \boldsymbol{\psi})_g\, ds\right|$$

$$\leq c\, \|\boldsymbol{\psi}\|_{L^\infty(0,t;L^2)} \|\boldsymbol{\psi}\|_{L^\infty(0,t;H^1)} \|w_m\|_{L^2(0,t;H^1)} \|u_m - u\|_{L^2(0,t;L^2)}$$

$$+c\, \|\boldsymbol{\psi}\|_{L^\infty(0,t;L^2)} \|\boldsymbol{\psi}\|_{L^\infty(0,t;H^1)} \|w_m - w\|_{L^2(0,t;L^2)} \|u_m\|_{L^2(0,t;H^1)}$$

$$+c\, \|P_m\boldsymbol{\psi}-\boldsymbol{\psi}\|_{L^\infty(0,t;L^2)} \|P_m\boldsymbol{\psi}-\boldsymbol{\psi}\|_{L^\infty(0,t;H^1)} \|w_m\|_{L^2(0,t;L^2)} \|u_m\|_{L^2(0,t;H^1)}.$$

This nonlinear term converges to 0 in view of (41) and (44) and the uniform bounds on u_m and w_m. On the other hand, we have

$$\left|\int_0^t b_g\left(\frac{\nabla g}{g}, u_m, P_m\boldsymbol{\psi}\right) ds - \int_0^t b_g\left(\frac{\nabla g}{g}, u, \boldsymbol{\psi}\right) ds\right| \leq c\, \|\nabla g\|_\infty \|P_m\boldsymbol{\psi}-\boldsymbol{\psi}\|_{L^\infty(0,t;H^1)} \|u_m\|_{L^2(0,t;L^2)}$$

$$+ c\, \|\nabla g\|_\infty \|u_m - u\|_{L^2(0,t;L^2)} \|\boldsymbol{\psi}\|_{L^2(0,t;H^1)}$$

which converges to 0 in view of (41) and using [22, Lemma 7.5]. Now for the other term in (46) as in the following

$$\left|\int_0^t b_g(u_m, w_m, P_m\psi) ds - \int_0^t b_g(u, w, \psi) ds\right|$$

$$\leq c\, \|P_m\psi - \psi\|_{L^\infty(0,t;L^2)} \|w_m\|_{L^2(0,t;H^1)} \|u_m\|_{L^2(0,t;H^1)}$$

$$+c\, \|u_m\|_{L^2(0,t;H^1)} \|w_m - w\|_{L^2(0,t;L^2)} \|\psi\|_{L^\infty(0,t;H^1)}$$

$$+c\, \|\psi\|_{L^\infty(0,t;H^1)} \|w_m\|_{L^2(0,t;H^1)} \|u_m - u\|_{L^2(0,t;L^2)}$$

which converges to 0 in view of (41) and (44) and also using [22, Lemma 7.5]. And then we consider the last term in the left hand side of (46), we write

$$\left|\int_0^t \left(w_m\left(\frac{\nabla g}{g}\cdot u_m\right), P_m\psi\right)_g ds - \int_0^t \left(w\left(\frac{\nabla g}{g}\cdot u\right), \psi\right)_g ds\right|$$

$$\leq c\, \|\nabla g\|_\infty \|\psi\|_{L^\infty(0,t;L^2)} \|\psi\|_{L^\infty(0,t;H^1)} \|w_m\|_{L^2(0,t;H^1)} \|u_m\|_{L^2(0,t;H^1)}$$

$$+ c\, \|\nabla g\|_\infty \|\psi\|_{L^\infty(0,t;L^2)} \|\psi\|_{L^\infty(0,t;H^1)} \|w_m - w\|_{L^2(0,t;L^2)} \|u_m\|_{L^2(0,t;H^1)}$$

$$+ c\, \|\nabla g\|_\infty \|P_m\psi - \psi\|_{L^2(0,t;L^2)} \|w_m\|_{L^2(0,t;H^1)} \|u\|^{\frac{1}{2}}_{L^\infty(0,t;L^2)} \|u\|^{\frac{1}{2}}_{L^\infty(0,t;H^1)}$$

which converges to 0 in view of (41) and (44) and also using [22, Lemma 7.5] and the uniform bounds on w_m and u_m.

Similarly, we will show to pass to the limit for the equation (34). Taking inner product of the equation (34) by $\psi \in C_c^1([0,T];V_g)$ and then integrating over $[0,t]$, $0<t<T$ and using integration by parts we get

$$-\int_0^t (v_m,\psi_t)_g\,ds + (v_m(t),\psi(t))_g - (v_m(0),\psi(0))_g + \int_0^t \langle B_g(u_m,v_m),P_m\psi)\rangle\,ds = 0. \tag{47}$$

For the nonlinear term in (47), we write

$$\begin{aligned}&\left|\int_0^t \langle B_g(u_m,v_m),P_m\psi)\rangle\,ds - \int_0^t \langle B_g(u,v),\psi)\rangle\,ds\right|\\ &\le c\,\|P_m\psi-\psi\|_{L^\infty(0,t;L^2)}\,\|u_m\|_{L^2(0,t;H^1)}\,\|v_m\|_{L^2(0,t;H^1)}\\ &+c\,\|\psi\|_{L^\infty(0,t;H^1)}\,\|v\|_{L^2(0,t;H^1)}\,\|u_m-u\|_{L^2(0,t;L^2)}\\ &+c\,\|u_m\|_{L^2(0,t;H^1)}\,\|v_m-v\|_{L^2(0,t;L^2)}\,\|\psi\|_{L^\infty(0,t;H^1)}\,.\end{aligned}$$

Using the results about convergences in above, we will pass to the limit in (47) then we get the following weak formulation for $v=\nabla\cdot(gw)$ in view of (34)

$$\langle v_t,\psi\rangle + \langle B_g(u,v),\psi\rangle = 0.$$

Now all the above convergences are valid if we take $\boldsymbol{\psi} = \ \boldsymbol{v}\boldsymbol{\varphi}(t)$ and $\psi = v\varphi(t)$ where $\boldsymbol{v}\in \mathbf{V}_g$, $\boldsymbol{\varphi}\in \left(C^1(0,T)\right)^2$ and $v\in V_g$, $\varphi\in C^1(0,T)$. In particular, the convergences are valid for all $\boldsymbol{\varphi}\in\left(C_c^1[0,T]\right)^2$ and $\varphi\in C_c^1[0,T]$; thus, (15) and (16) hold in the sense of distributions, which in turn implies that (12) and (13) are valid as an equation of operators. In other words, (3) – (6) hold in the weak sense by Lemma 1.2 in [26]. Furthermore, it is easy to show that the initial condition is satisfied in a weak sense similar to the problem for Navier–Stokes equations in [26]. So we prove the existence of weak solutions of our system in the means of Definition 3.1.

Now we prove the uniqueness of the weak solutions. We suppose (u_1,w_1) and (u_2,w_2) are two weak solutions to system (3) – (6) with the same initial condition and forcing term f. Let $u=u_1-u_2$ and $w=w_1-w_2$. Then we obtain

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{1}{g}(\nabla g\cdot\nabla)u + w\times u_1 + w_2\times u + \nabla P = 0, \tag{48}$$

$$\frac{\partial w}{\partial t} - \nu\Delta_g w + \nu\frac{1}{g}(\nabla g\cdot\nabla)w + u_1\cdot\nabla w - u\cdot\nabla w_2 = w\left(\frac{\nabla g}{g}\cdot u_1\right) - w_2\left(\frac{\nabla g}{g}\cdot u\right), \tag{49}$$

$$\nabla\cdot(gu)=0\ , \qquad \nabla\cdot(gw)=0, \tag{50}$$

$$u(x,0)=0, \qquad w(x,0)=0 \tag{51}$$

when (48) and (49) are multiplied by $u(t)$ and $w(t)$ respectively, and add these equations, we get

$$\frac{d}{dt}\left(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)+2\nu\|\nabla u(t)\|_{L^2}^2+2\nu\|\nabla w(t)\|_{L^2}^2$$

$$\leq -2\int_\Omega (w(t)\times u_1(t))ugdx+2\nu\left|b_g\left(\frac{\nabla g}{g},u(t),u(t)\right)\right|+2\nu\left|b_g\left(\frac{\nabla g}{g},w(t),w(t)\right)\right|$$

$$+2\left|b_g\left(u(t),w_2(t),w(t)\right)\right|+2\int_\Omega w(t)\left(\frac{\nabla g}{g}\cdot u_1(t)\right)w(t)gdx+2\int_\Omega w_2(t)\left(\frac{\nabla g}{g}\cdot u(t)\right)w(t)gdx. \tag{52}$$

Cauchy-Schwarz and Young inequalities are applied for each term in the right hand side of (52), we write

$$\frac{d}{dt}\left(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)+\frac{\nu}{2}\|\nabla u(t)\|_{L^2}^2$$

$$\leq\left\{\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2}+\frac{2c}{\nu^3}\|w_2(t)\|_{L^2}^2\|\nabla w_2(t)\|_{L^2}^2+\frac{2c}{\nu}\|\nabla g\|_\infty^2\|w_2(t)\|_{L^2}^2\right\}\|u(t)\|_{L^2}^2$$

$$+\left\{\frac{2c}{\nu}\|\nabla u_1(t)\|_{L^2}^2+\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2}+\frac{2c}{\nu}\|\nabla g\|_\infty^2\|u_1(t)\|_{L^2}^2\right\}\|w(t)\|_{L^2}^2.$$

When necessary arrangements are made, we obtain

$$\frac{d}{dt}\left(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)\leq K_5\left(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right) \tag{53}$$

where

$$K_5=\max\left\{\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2}+\frac{2c}{\nu^3}\|w_2(t)\|_{L^2}^2\|\nabla w_2(t)\|_{L^2}^2+\frac{2c}{\nu}\|\nabla g\|_\infty^2\|w_2(t)\|_{L^2}^2,\right.$$

$$\left.\frac{2c}{\nu}\|\nabla u_1(t)\|_{L^2}^2+\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2}+\frac{2c}{\nu}\|\nabla g\|_\infty^2\|u_1(t)\|_{L^2}^2\right\}.$$

By using Gronwall inequality, we conclude that

$$\left(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)\leq e^{K_5(t)}\left(\|u(0)\|_{L^2}^2+\|w(0)\|_{L^2}^2\right).$$

Thus, $u(t)=0$ and $w(t)=0$ for all $t\geq 0$ since we have $u(0)=0$ and $w(0)=0$. Thus, the theorem is proved. □

Definition 3.3 *Let $T > 0$, $f \in L^2(0,T;H_g^{curl})$, $u_0 \in \mathbf{V}_g$, and $w_0 \in V_g$. The pair of functions (u,w) is called a strong solutions on the time interval $[0,T]$ to system $(3)-(6)$, if it is a weak solution as in Definition 3.1 and satisfies additionaly $u \in L^2(0,T;D(A_g)) \cap L^\infty(0,T;\mathbf{V}_g)$, and $w \in L^2(0,T;D(A_g)) \cap L^\infty(0,T;V_g)$.*

Now we will give theorem the existence of the strong solutions for the system $(3)-(6)$.

Theorem 3.4 *Let the initial data $u_0 \in \mathbf{V}_g$, $w_0 \in V_g$, and $f \in L^2(0,T;H_g^{curl})$. Then, there exists a unique strong solution (u,w) in the sense of Definition 3.3.*

Proof Now we multiply the Galerkin approximation (17) by $A_g u_m$ and integrate by over Ω, we will show priori estimates for the higher order regularity of the solution of $(3)-(6)$. Hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_m(t)\|_{L^2}^2 + \nu\|A_g u_m(t)\|_{L^2}^2 \leq -\nu b_g\left(\frac{\nabla g}{g}, u_m(t), A_g u_m(t)\right)$$

$$-\left(w_m \times u_m, A_g u_m(t)\right)_g + \left(f, A_g u_m(t)\right)_g.$$

Here using the Cauchy-Schwarz, Young inequalities, and (9), we get

$$\frac{d}{dt}\|\nabla u_m(t)\|_{L^2}^2 + \frac{\nu}{2}\|A_g u_m(t)\|_{L^2}^2 \leq \frac{1}{\nu}\|f\|_{L^2}^2$$

$$+\left(\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{c}{\nu^3}\|w_m(t)\|_{L^2}^2\|\nabla w_m(t)\|_{L^2}^2\right)\|\nabla u_m(t)\|_{L^2}^2. \tag{54}$$

Neglecting the second term in the left hand side of (54), we write

$$\frac{d}{dt}\|\nabla u_m(t)\|_{L^2}^2 \leq \frac{1}{\nu}\|f\|_{L^2}^2 + p_1\|\nabla u_m(t)\|_{L^2}^2$$

where

$$p_1 = \frac{2\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{c}{\nu^3}\|w_m(t)\|_{L^2}^2\|\nabla w_m(t)\|_{L^2}^2.$$

Using Gronwall's lemma, we have

$$\|\nabla u_m(t)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 \exp\left(\int_0^t p_1(\tau)d\tau\right) + \frac{1}{\nu}\int_0^t \|f(s)\|_{L^2}^2 \exp\left(\int_0^t p_1(\tau)d\tau\right) ds.$$

Similarly, we multiply (18) by $A_g w_m$ and integrate by over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla w_m(t)\|_{L^2}^2 + \frac{\nu}{4}\|A_g w_m(t)\|_{L^2}^2 = -\nu b_g\left(\frac{\nabla g}{g}, w_m(t), A_g w_m(t)\right) - b_g\left(u_m(t), w_m(t), A_g w_m(t)\right)$$

$$+\left(w_m(t)\left(\frac{\nabla g}{g}\cdot u_m(t)\right), A_g w_m(t)\right)_g + \left(\nabla \times f, A_g w_m(t)\right)_g.$$

Using the Cauchy-Schwarz, Young inequalities and the inequality of (9), we see that

$$\frac{d}{dt}\|\nabla w_m(t)\|_{L^2}^2+\frac{\nu}{2}\|A_g u_m(t)\|_{L^2}^2\le\frac{4}{\nu}\|\nabla\times f\|_{L^2}^2$$

$$+\left(\frac{2c\nu\|\nabla g\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}\|u_m(t)\|_{L^2}^2\|\nabla u_m(t)\|_{L^2}^2\right.$$

$$\left.+\frac{2c\|\nabla g\|_\infty^2}{\nu m_0^2}\|\nabla u_m(t)\|_{L^2}^2\right)\|\nabla w_m(t)\|_{L^2}^2. \tag{55}$$

Neglecting the second term in the left hand side of (55), we obtain

$$\frac{d}{dt}\|\nabla w_m(t)\|_{L^2}^2\le\frac{4}{\nu}\|\nabla\times f\|_{L^2}^2+p_2\|\nabla w_m(t)\|_{L^2}^2$$

where

$$p_2=\frac{2c\nu\|\nabla g\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}\|u_m(t)\|_{L^2}^2\|\nabla u_m(t)\|_{L^2}^2+\frac{2c\|\nabla g\|_\infty^2}{\nu m_0^2}\|\nabla u_m(t)\|_{L^2}^2.$$

By using Gronwall inequality, we get

$$\|\nabla w_m(t)\|_{L^2}^2\le\|\nabla w_0\|_{L^2}^2\exp\left(\int_0^t p_2(\tau)d\tau\right)+\frac{4}{\nu}\int_0^t\|\nabla\times f(s)\|_{L^2}^2\exp\left(\int_0^t p_2(\tau)d\tau\right)ds.$$

To obtain a priori estimates for strong solution, we use the following estimates which are given in Theorem 3.2;

$$\sup_{s\in[0,T]}\|u_m(s)\|_{L^2}^2\le K_1,\qquad \int_0^T\|\nabla u_m(t)\|_{L^2}^2\,dt\le K_3,$$

$$\sup_{s\in[0,T]}\|w_m(s)\|_{L^2}^2\le K_2,\qquad \int_0^T\|\nabla w_m(t)\|_{L^2}^2\,dt\le K_4$$

where the constants $K_1=K_1(u_0,\nu,f,T),\ K_2=K_2(w_0,\nu,f,T),\ K_3=K_3(u_0,\nu,f,T),\ K_4=K_4(w_0,\nu,f,T)$. Therefore, we obtain

$$\int_0^t p_1(\tau)d\tau\le\frac{2\nu T\|\nabla g\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}\sup_{t\in[0,T]}\|w_m(t)\|_{L^2}^2\int_0^T\|\nabla w_m(\tau)\|_{L^2}^2\,d\tau\le\frac{2\nu T\|\nabla g\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}K_2K_4=K_6$$

and

$$\int_0^t p_2(\tau)d\tau\le\frac{2c\nu T\|\nabla g\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}\sup_{t\in[0,T]}\|u_m(t)\|_{L^2}^2\int_0^T\|\nabla u_m(\tau)\|_{L^2}^2\,d\tau+\frac{4c\|\nabla g\|_\infty^2}{\nu m_0^2}\int_0^T\|\nabla u_m(\tau)\|_{L^2}^2\,d\tau$$

$$\le\frac{2c\nu T\|\nabla g\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}K_1K_3+\frac{4c\|\nabla g\|_\infty^2}{\nu m_0^2}K_3=K_7.$$

Moreover, we have

$$\sup_{s\in[0,T]} \|\nabla u_m(t)\|_{L^2}^2 \le e^{K_6} \|\nabla u_0\|_{L^2}^2 + \frac{1}{\nu} T e^{K_6} \|f\|_{L^2(0,T;L^2)}^2 = K_8 \tag{56}$$

and

$$\sup_{s\in[0,T]} \|\nabla w_m(t)\|_{L^2}^2 \le e^{K_7} \|\nabla w_0\|_{L^2}^2 + \frac{4}{\nu} T e^{K_7} \|\nabla \times f\|_{L^2(0,T;L^2)}^2 = K_9. \tag{57}$$

Then we integrate (54) and (55) from 0 to T, we get

$$\|\nabla u_m(T)\|_{L^2}^2 + \frac{\nu}{2}\int_0^T \|A_g u_m(t)\|_{L^2}^2 \le \|\nabla u_0\|_{L^2}^2 + \frac{1}{\nu}\|f\|_{L^2(0,T;L^2)}^2 + \frac{2\nu\|\nabla g\|_\infty^2}{m_0^2} K_3 + \frac{c}{\nu^3} K_2 K_4 K_3$$

and

$$\|\nabla w_m(T)\|_{L^2}^2 + \frac{\nu}{4}\int_0^T \|A_g w_m(t)\|_{L^2}^2 \le \|\nabla w_0\|_{L^2}^2 + \frac{4}{\nu}\|\nabla \times f\|_{L^2(0,T;L^2)}^2$$

$$+ \frac{2\nu\|\nabla g\|_\infty^2}{m_0^2} K_4 + \frac{c}{\nu^3} K_3 K_1 K_4 + \frac{4c\|\nabla g\|_\infty^2}{\nu m_0^2} K_3 K_4.$$

Neglecting the first term in the left hand side of above equations, we write

$$\int_0^T \|A_g u_m(t)\|_{L^2}^2 \le \frac{2}{\nu}\left(\|\nabla u_0\|_{L^2}^2 + \frac{1}{\nu}\|f\|_{L^2(0,T;L^2)}^2 + \frac{2\nu\|\nabla g\|_\infty^2}{m_0^2} K_3 + \frac{c}{\nu^3} K_2 K_4 K_3\right) = K_{10}, \tag{58}$$

$$\int_0^T \|A_g w_m(t)\|_{L^2}^2 \le \frac{4}{\nu}\left(\|\nabla w_0\|_{L^2}^2 + \frac{4}{\nu}\|\nabla \times f\|_{L^2(0,T;L^2)}^2\right.$$

$$\left.+\frac{2\nu\|\nabla g\|_\infty^2}{m_0^2} K_4 + \frac{c}{\nu^3} K_3 K_1 K_4 + \frac{4c\|\nabla g\|_\infty^2}{\nu m_0^2} K_3 K_4\right)$$

$$= K_{11} \tag{59}$$

where K_{10} and K_{11} are the positive constants. From above estimates,

$$u_m \in L^\infty(0,T;\mathbf{V}_g) \cap L^2(0,T;D(A_g)),$$

$$w_m \in L^\infty(0,T;V_g) \cap L^2(0,T;D(A_g)).$$

Now we note that

$$\int_0^T \|D_t u_m(t)\|_{L^2}^2 dt \le \int_0^T \|w_m(t) \times u_m(t)\|_{L^2}^2 dt + \nu\int_0^T \|A_g u_m(t)\|_{L^2}^2 dt + \nu\int_0^T \|C_g u_m(t)\|_{L^2}^2 dt + \int_0^T \|f(t)\|_{L^2}^2 dt$$

and

$$\int_0^T \|D_t w_m(t)\|_{L^2}^2\,dt \le \int_0^T \|B_g(u_m(t), w_m(t))\|_{L^2}^2\,dt + \nu \int_0^T \|A_g w_m(t)\|_{L^2}^2\,dt$$

$$+ \nu \int_0^T \|C_g w_m(t)\|_{L^2}^2\,dt + \int_0^T \left\| w_m(t)\left(\frac{\nabla g}{g}\cdot u_m(t)\right)\right\|_{L^2}^2 dt + \int_0^T \|\nabla \times f(t)\|_{L^2}^2\,dt.$$

The boundedness of the terms in the right hand side of the inequalities is shown below using Cauchy-Schwarz and (9) and (11).

$$\int_0^T \|w_m(t)\times u_m(t)\|_{L^2}^2\,dt \le c\int_0^T \|u_m(t)\|_{L^2}\|\nabla u_m(t)\|_{L^2}\|w_m(t)\|_{L^2}\|\nabla u_m(t)\|_{L^2}\,dt$$

$$\le c\left(\sup_{t\in[0,T]}\|u_m(t)\|_{L^2}^2\int_0^T \|\nabla u_m(t)\|_{L^2}^2\,dt\right)^{\frac{1}{2}}\left(\sup_{t\in[0,T]}\|w_m(t)\|_{L^2}^2\int_0^T \|\nabla w_m(t)\|_{L^2}^2\,dt\right)^{\frac{1}{2}}$$

$$\le c\sqrt{K_1K_3}\sqrt{K_2K_4} = K_{12}.$$

Therefore, $w_m(t)\times u_m(t)$ belongs to the space $L^2(0,T;\mathbf{H}_g)$.

$$\int_0^T \|C_g u_m(t)\|_{L^2}^2\,dt \le c\,\|\nabla g\|_\infty^2 \int_0^T \|\nabla u_m(t)\|_{L^2}^2\,dt \le c\,\|\nabla g\|_\infty^2 K_3 = K_{13}.$$

Hence, $C_g u_m(t)$ also belongs to the space $L^2(0,T;\mathbf{H}_g)$. As a result,

$$\frac{du_m}{dt} \text{ belongs to } L^2(0,T;\mathbf{H}_g). \tag{60}$$

Similarly,

$$\int_0^T \|B_g(u_m(t), w_m(t)\|_{L^2}^2\,dt \le c\int_0^T \|u_m(t)\|_{L^2}\|A_g u_m(t)\|_{L^2}\|\nabla w_m(t)\|_{L^2}^2\,dt$$

$$\le c\left(\sup_{t\in[0,T]}\|u_m(t)\|_{L^2}^2\int_0^T \|A_g u_m(t)\|_{L^2}^2\,dt\right)^{\frac{1}{2}}\left(\sup_{t\in[0,T]}\|\nabla w_m(t)\|_{L^2}^2\int_0^T \|\nabla w_m(t)\|_{L^2}^2\,dt\right)^{\frac{1}{2}}$$

$$\le c\sqrt{K_1K_{10}}\sqrt{K_9K_4} = K_{14}.$$

Therefore, $B_g\left(u_m(t), w_m(t)\right)$ belongs to $L^2(0,T;H_g)$.

$$\int_0^T \left\| w_m(t)\left(\frac{\nabla g}{g}\cdot u_m(t)\right)\right\|_{L^2}^2 dt \le c\left\|\nabla g\right\|_\infty^2 \int_0^T \left\|w_m(t)\right\|_{L^2}\left\|\nabla w_m(t)\right\|_{L^2}\left\|u_m(t)\right\|_{L^2}\left\|\nabla u_m(t)\right\|_{L^2} dt$$

$$\le c\left\|\nabla g\right\|_\infty^2 \left(\sup_{t\in[0,T]}\left\|w_m(t)\right\|_{L^2}^2 \int_0^T \left\|\nabla w_m(t)\right\|_{L^2}^2 dt\right)^{\frac{1}{2}} \left(\sup_{t\in[0,T]}\left\|u_m(t)\right\|_{L^2}^2 \int_0^T \left\|\nabla u_m(t)\right\|_{L^2}^2 dt\right)^{\frac{1}{2}}$$

$$\le c\left\|\nabla g\right\|_\infty^2 \sqrt{K_2K_4}\sqrt{K_1K_3} = K_{16}.$$

Hence, $w_m(t)\left(\frac{\nabla g}{g}\cdot u_m(t)\right)$ also belongs to the space $L^2(0,T;H_g)$. As a result,

$$\frac{dw_m}{dt} \text{ belongs to } L^2(0,T;H_g). \tag{61}$$

Thus, using standard techniques in [26], from the above estimates, we conclude that there exist a subsequence of u_m and w_m which are convergent to u and w such that

$$u \in L^\infty(0,T;\mathbf{V}_g)\cap L^2(0,T;D(A_g)),$$

$$w \in L^\infty(0,T;V_g)\cap L^2(0,T;D(A_g)).$$

Using (60), (61), and the last two results $u \in C([0,T];\mathbf{V}_g)$ and $w \in C([0,T];V_g)$ (by [22, theorem 7.7]) which makes the initial conditions meaningful. Therefore, we prove the existence of strong solutions.

Now we show that strong solutions continuously depend on the initial conditions and the given data as well as the uniqueness of the strong solutions. We suppose (u_1,w_1) and (u_2,w_2) are two strong solutions to system (3) – (6) with $(u_1(0),w_1(0))$ and $(u_2(0),w_2(0))$ initial conditions and forcing terms f_1, f_2. Let $u = u_1 - u_2$, $w = w_1 - w_2$. $u(0) = u_1(0) - u_2(0)$, $w(0) = w_1(0) - w_2(0)$ and $f = f_1 - f_2$. Then we obtain the following system:

$$\frac{du}{dt} + \nu A_g u + \nu C_g u + w\times u_1 + w_2 \times u = f, \tag{62}$$

$$\frac{dw}{dt} + \nu A_g w + \nu C_g w + B_g(u_1,w) - B_g(u,w_2) = \nabla\times f + w\left(\frac{\nabla g}{g}\cdot u_1\right) - w_2\left(\frac{\nabla g}{g}\cdot u\right). \tag{63}$$

Now (62) and (63) are multiplied by $A_g u(t)$ and $A_g w(t)$ respectively, we get

$$\frac{d}{dt}\left\|\nabla u(t)\right\|_{L^2}^2 + 2\nu\left\|A_g u(t)\right\|_{L^2}^2 \le 2\nu\left|b_g\left(\frac{\nabla g}{g}, u(t), A_g u(t)\right)\right| + \left|2(w(t)\times u_1(t), A_g u(t))_g\right|$$

$$+ \left|2(w_2(t)\times u(t), A_g u(t))_g\right| + 2(f(t), A_g u(t))_g$$

and

$$\frac{d}{dt}\left\|\nabla w(t)\right\|_{L^2}^2+2\nu\left\|A_g w(t)\right\|_{L^2}^2+2\nu b_g\left(\frac{\nabla g}{g},w(t),A_g w(t)\right)+2b_g\left(u_1(t),w(t),A_g w(t)\right)-2b_g\left(u(t),w_2(t),A_g w(t)\right)$$

$$=\left(\nabla\times f,A_g w_m(t)\right)_g+\left(w(t)\left(\frac{\nabla g}{g}\cdot u_1(t)\right),A_g w(t)\right)_g-\left(w_2(t)\left(\frac{\nabla g}{g}\cdot u(t)\right),A_g w(t)\right)_g.$$

Here Cauchy-Schwarz and Young inequalities are applied for each term in the right hand side of these equations, we write

$$\frac{d}{dt}\left\|\nabla u(t)\right\|_{L^2}^2\le\left(\frac{\nu\left\|\nabla g\right\|_\infty^2}{m_0^2}+\frac{2c}{\nu}\left\|\nabla w_2(t)\right\|_{L^2}^2\right)\left\|\nabla u(t)\right\|_{L^2}^2$$

$$+\frac{2c}{\nu}\left\|\nabla u_1(t)\right\|_{L^2}^2\left\|\nabla w(t)\right\|_{L^2}^2+\frac{2}{\nu}\left\|f\right\|_{L^2}^2 \tag{64}$$

and

$$\frac{d}{dt}\left\|\nabla w(t)\right\|_{L^2}^2+\frac{\nu}{2}\left\|A_g w(t)\right\|_{L^2}^2\le\left(\frac{4c}{\nu}\left\|A_g w_2(t)\right\|_{L^2}^2+\frac{4c\left\|\nabla g\right\|_\infty^2}{\nu}\left\|\nabla w_2(t)\right\|_{L^2}^2\right)\left\|\nabla u(t)\right\|_{L^2}^2$$

$$+\left(\frac{4\nu\left\|\nabla g\right\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}\left\|u_1(t)\right\|_{L^2}^2\left\|\nabla u_1(t)\right\|_{L^2}^2\right.$$

$$\left.+\frac{4c\left\|\nabla g\right\|_\infty^2}{\nu}\left\|\nabla u_1(t)\right\|_{L^2}^2\right)\left\|\nabla w(t)\right\|_{L^2}^2+\frac{4}{\nu}\left\|\nabla\times f\right\|_{L^2}^2. \tag{65}$$

Adding (64) and (65) and also making necessary arrangements, we obtain

$$\frac{d}{dt}\left(\left\|\nabla u(t)\right\|_{L^2}^2+\left\|\nabla w(t)\right\|_{L^2}^2\right)\le L\left(\left\|\nabla u(t)\right\|_{L^2}^2+\left\|\nabla u(t)\right\|_{L^2}^2\right)+\frac{2}{\nu}\left\|f\right\|_{L^2}^2+\frac{4}{\nu}\left\|\nabla\times f\right\|_{L^2}^2$$

where

$$L=\max\left\{\frac{4c}{\nu}\left\|A_g w_2(t)\right\|_{L^2}^2+\frac{4c\left\|\nabla g\right\|_\infty^2}{\nu}\left\|\nabla w_2(t)\right\|_{L^2}^2+\frac{\nu\left\|\nabla g\right\|_\infty^2}{m_0^2}+\frac{2c}{\nu}\left\|\nabla w_2(t)\right\|_{L^2}^2,\right.$$

$$\left.\frac{4\nu\left\|\nabla g\right\|_\infty^2}{m_0^2}+\frac{c}{\nu^3}\left\|u_1(t)\right\|_{L^2}^2\left\|\nabla u_1(t)\right\|_{L^2}^2+\frac{4c\left\|\nabla g\right\|_\infty^2}{\nu}\left\|\nabla u_1(t)\right\|_{L^2}^2+\frac{2c}{\nu}\left\|\nabla u_1(t)\right\|_{L^2}^2\right\}.$$

By using Gronwall inequality, we conclude that

$$\|\nabla u(T)\|_{L^2}^2 + \|\nabla w(T)\|_{L^2}^2 \leq \exp\left(\int_0^T L(\tau)d\tau\right)\left(\|\nabla u(0)\|_{L^2}^2 + \|\nabla w(0)\|_{L^2}^2\right)$$

$$+\left(\frac{2T}{\nu}\|f\|_{L^2}^2 + \frac{4T}{\nu}\|\nabla\times f\|_{L^2}^2\right)\exp\left(\int_0^T L(\tau)d\tau\right).$$

Here if $f_1, f_2 \in L^2(0,T;H_g^{curl})$, $u_1(0), u_2(0) \in \mathbf{V}_g$, $w_1(0), w_2(0) \in V_g$, then we can conclude that strong solutions continuously depend on the initial conditions and the given data. Moreover, the strong solutions are unique when $u_1(0) = u_2(0)$, $w_1(0) = w_2(0)$ and $f_1 = f_2$. □

References

[1] Akbas M, Rebholz LG, Zerfas C. Optimal vorticity accuracy in an efficient velocity–vorticity method for the 2D Navier–Stokes equations. Calcolo 2018; 55 (1): 1-29. https://doi.org/10.1007/s10092-018-0246-7

[2] Bae HO, Roh J. Existence of solutions of the g-Navier–Stokes equations. Taiwanese Journal of Mathematics 2004; 8 (1): 85-102. https://doi.org/10.11650/twjm/1500558459

[3] Baranovskii ES. Strong solutions of the incompressible Navier–Stokes–Voigt model. Mathematics 2020; 8 (2): 181. https://doi.org/10.3390/math8020181

[4] Camassa R, Holm DD, Levermore CD. Long-time effects of bottom topography in shallow water. Physica D: Nonlinear Phenomena 1996; 98 (2-4): 258-286. https://doi.org/10.1016/0167-2789(96)00117-0

[5] Constantin P, Foias C. Navier–Stokes equations. University of Chicago Press, 1988.

[6] Gallay T, Wayne CE. Long time asymptotic of the Navier–Stokes and vorticity equations on $\mathbb{R}^3$. Philosophical Transactions of the Royal Society of London. Series A: Mathematical Physical Engineering Sciences 2002; 360 (1799): 2155-2188. https://doi.org/10.1098/rsta.2002.1068

[7] Gatski TB. Review of incompressible fluid flow computations using the vorticity-velocity formulation. Applied Numerical Mathematics 1991; 7 (3): 227-239. https://doi.org/10.1016/0168-9274(91)90035-X

[8] Hale JK, Raugel G. Partial differential equations on thin domains. In Mathematics in Science and Engineering 1992; (186): 63-97. https://doi.org/10.1016/S0076-5392(08)63376-7

[9] Hale JK, Raugel G. Reaction-diffusion equation on thin domains. Journal de mathématiques pures et appliquées. 1992; 71 (1): 33-95.

[10] Heister T, Olshanskii MA, Rebhols LG. Unconditional long-time stability of a velocity vorticity method for the 2D Navier–Stokes equations. Numerische Mathematik 2017; 135 (1): 143-167. https://doi.org/10.1007/s00211-016-0794-1

[11] Kalantarov VK, Titi ES. Global attractors and determining modes for the 3D Navier–Stokes Voight equations. Chinese Annals of Mathematics, Series B 2009; 30 (6): 697-714. https://doi.org/10.1007/s11401-009-0205-3

[12] Kaya M, Çelebi AO. Existence of weak solutions of the g Kelvin-Voight equation. Mathematical and computer modelling 2009; 49 (3-4): 497-504. https://doi.org/10.1016/j.mcm.2008.03.005

[13] Kwak M, Kwean H, Roh J. The dimension of attractor of the 2D g Navier–Stokes equations. Journal of Mathematical Analysis and Applications 2006; 315 (2): 436-461. https://doi.org/10.1016/j.jmaa.2005.04.050

[14] Larios A, Pei Y, Rebholz LG. Global well-posedness of the velocity-vorticity-Voigt model of the 3D Navier–Stokes equations. Journal of Differential Equations 2019; 266 (5); 2435-2465. https://doi.org/10.1016/j.jde.2018.08.033

[15] Lee HK, Olshanskii MA, Rebholz LG. On error analysis for the 3D Navier–Stokes equations in velocity-vorticity-helicity form. SIAM Journal on Numerical Analysis 2011; 49 (2): 711-732. https://doi.org/10.1137/10080124X

[16] Levermore CD, Oliver M, Titi ES. Global well-posedness for models of shallow water in a basin with a varying bottom Indiana University Mathematics Journal 1996; 45 (2): 479-510.

[17] Levermore CD, Oliver M, Titi ES. Global well-posedness for the lake equations Physica D: Nonlinear Phenomena 1996; 98 (2-4): 492-509. https://doi.org/10.1016/0167-2789(96)00108-X

[18] Majda AJ, Bertozzi AL, Ogawa A. Vorticity and incompressible flow. Cambridge texts in applied mathematics. Applied Mechanics Reviews 2002; 55 (4): 229-247. https://doi.org/10.1115/1.1483363

[19] Olshanskii MA, Rebholz LG. Velocity-vorticity-helicity formulation and a solver for the Navier–Stokes equations. Journal of Computational Physics 2010; 229 (11): 4291-4303. https://doi.org/10.1016/j.jcp.2010.02.012

[20] Raugel G, Sell GR. Navier–Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions. Journal of the American Mathematical Society 1993; 6 (3): 503-568. https://doi.org/10.1090/S0894-0347-1993-1179539-4

[21] Raugel G, Sell GR. Navier–Stokes equations on thin 3D domains. II. Global regularity of spatially periodic solutions. Pitman Research Notes in Mathematics Series 1994; 254-247.

[22] Robinson JC. Infinite-Dimensional Dynamical Systems: An introduction to dissipative parabolic PDEs and the theory of global attractors. Cambridge texts in applied mathematics. Cambridge University Press 2003; 56 (4): 207-208, 239-256. https://doi.org/10.1115/1.1579456

[23] Roh J. Derivation of the g Navier Stokes Equations. Journal of the Chungcheong Mathematical Society 2006; 19 (3): 213-218.

[24] Roh J. Dynamics of the Navier Stokes equations. Journal of Differential Equations 2005; 211 (2): 452-484. https://doi.org/10.1016/j.jde.2004.08.016

[25] Roh J. g-Navier Stokes Equations. PhD Thesis, University of Minnesota, 2001.

[26] Temam R. Navier–Stokes equations. Theory and numerical analysis. American Mathematical Society press. (343) 2001; 161-163, 252-290.

[27] Temam R, Ziane M. Navier–Stokes equations in three-dimensional thin domains with various boundary conditions. Advances in Differential Equations 1996; 1 (4): 499-546.

[28] Ustiuzhaninova A, Turbin M. Feedback Control Problem for Modified Kelvin-Voigt Model. Journal of Dynamical and Control Systems 2021; 1 (16): https://doi.org/10.1007/s10883-021-09539-0

[29] Zvyaginz V, Turbin M. Optimal feedback control problem for inhomogeneous Voigt fluid motion model. Journal of Fixed Point Theory and Applications 2021; 1(38): https://doi.org/10.1007/s11784-020-00838-w