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Fractional semilinear Neumann problem with critical nonlinearity

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Abstract: In this paper, we consider the following critical fractional semilinear Neumann problem

$$\begin{cases} (-\Delta)^{1/2}u + \lambda u = u^{\frac{n+1}{n-1}},\ u > 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ is a smooth bounded domain, $\lambda > 0$ and ν is the outward unit normal to $\partial\Omega$. We prove that there exists a constant $\lambda_0 > 0$ such that the above problem admits a minimal energy solution for $\lambda < \lambda_0$. Moreover, if Ω is convex, we show that this solution is constant for sufficiently small λ.

Key words: Fractional Laplacian operator, Neumann boundary condition, critical exponent

1. Introduction

The classical semilinear problem

$$\begin{cases} -\Delta u + \lambda u = u^p,\ u > 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

has been extensively studied in recent years by many authors, where $\lambda > 0$, $p > 1$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and ν is the outward unit normal to $\partial\Omega$. Problem (1) arises from considering steady states of the Keller-Segel system in chemotaxis [22]. When $p < \frac{n+2}{n-2}$ with $n \geq 3$ or $p > 1$ with $n = 1, 2$, Lin et al. [24] obtained the existence of nonconstant solutions for (1), provided λ is sufficiently large, and the only constant solution $u \equiv \lambda^{1/(p-1)}$ for sufficiently small λ. When $p = \frac{n+2}{n-2}$ with $n \geq 3$, Wang [33] showed that problem (1) admits a nonconstant solution for λ suitably large; Adimurthi and Mancini [1] showed that problem (1) admits a minimal-energy solution for $\lambda > 0$, and they also proved that the solution is nonconstant for λ suitably large. For more results in the critical case, we refer to [5, 14, 20, 23, 28] and references therein. In particular, Adimurthi and Yadava [5] testified that the solution given in [1] is constant for λ sufficiently small. For $\lambda > 0$ small, Lin and Ni [23] made the following conjecture:

Lin-Ni's conjecture. For λ small and $p = \frac{n+2}{n-2}$, problem (1) admits only the constant solution.

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We will recall the main results towards proving or disproving Lin-Ni's conjecture as follows. When Ω is a unit ball and λ is sufficiently small, Adimurthi and Yadava [4–6] and Budd et al. [11] proved that any radial solution of (1) must be constant in dimensions $n = 3$ or $n \geq 7$, the conjecture is false for $n = 4, 5, 6$, which reveal that the dimension n has an effect on Lin-Ni's conjecture. When $n = 3$, Zhu [39] and Wei and Xu [37] testified that the conjecture is true for convex domain by using different techniques, and del Pino et al. [15] dealt with the existence of a nontrivial solution with interior bubbling as λ approaches a special positive value. When $n = 3$ or $n \geq 7$, Druet et al. [16] proved that the conjecture is true for the mean convex domains. However, when $n = 5$, Rey and Wei [29] showed that problem (1) has arbitrarily many solutions for any bounded smooth domain, provided that λ is small enough. For any fixed $\lambda > 0$, Wang et al. [34] obtained that there exist infinitely many solutions for some nonconvex domains if $n \geq 3$, and they [35] also proved the existence of infinitely many solutions in some convex domain if $n \geq 4$. Furthermore, when Ω is a ball, they showed that there exist infinitely many nonradially symmetric solutions. When $n = 4$ or 6, Wei et al. [36] proved that problem (1) has a nonconstant solution for any bounded smooth domain, if λ is small enough.

Comparing with problem (1), the following semilinear Dirichlet problem

$$\begin{cases} (-\Delta)^s u = u^p + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has been also studied quite extensively, where $s \in (0, 1]$, $p > 1$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain.

When $s = 1$, $p \in (1, \frac{n+2}{n-2})$ with $n \geq 3$ or $p \in (1, \infty)$ with $n = 1, 2$, Lions [25] proved the existence of positive solutions for (2), provided $\lambda < \lambda_*$, where $\lambda_* > 0$ denotes the first eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$. When $s = 1$, $p = \frac{n+2}{n-2}$, Brezis and Nirenberg [10] obtained that problem (2) admits a positive solution for $n \geq 4$ and $\lambda \in (0, \lambda_*)$, and there is no positive solution of (2) when $\lambda \geq \lambda_*$ or $\lambda \leq 0$ and Ω is a star-shaped domain. Especially, when $\Omega \subset \mathbb{R}^3$ is a ball, they showed that problem (2) has a positive solution if and only if $\lambda \in (\frac{\lambda_*}{4}, \lambda_*)$. When $s \in (0, 1)$, $p = \frac{n+2s}{n-2s}$ with $n \geq 4s$, Barrios et al. [7] (see also Tan [32] for $s = \frac{1}{2}$) obtained that problem (2) has at least one positive solution for $\lambda \in (0, \lambda_*^s)$, and there is no positive solution of (2) with $\lambda \geq \lambda_*^s$. For the study of the Brezis-Nirenberg problem, the readers can refer to [2, 8, 13, 18, 26, 30, 38] and the references therein.

Consider the following fractional semilinear Neumann problem

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p, \ u > 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $s \in (0, 1)$, $\lambda > 0$, $p > 1$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and ν is the outward unit normal to $\partial\Omega$. When $s = \frac{1}{2}$ and $1 < p < \frac{n+1}{n-1}$ with $n \geq 2$, Stinga and Volzone [31] transformed the nonlocal problem (3) to the local problem on a half-cylinder $\mathcal{C} := \Omega \times (0, \infty)$. They proved that (3) has at least one nonconstant solution for λ suitably large, and it has only constant solution for λ sufficiently small. When $s \in (0, 1)$ and $1 < p < \frac{n+2s}{n-2s}$ with $n > 2s$, Ni et al. [27] proved that (3) has at least one nonconstant solution for λ suitably large. When $s \in (0, 1)$ and $p = \frac{n+2s}{n-2s}$, problem (3) involves the fractional critical Sobolev exponent, and it is well known that the Sobolev embedding $H^s(\Omega) \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega)$ is not compact even if Ω is bounded. Thus, the associated energy functional of the local problem does not verify the Palais-Smale condition globally. To the best of our knowledge, we have not found any research on the fractional semilinear Neumann problem (3) with critical Sobolev exponent.

Motivated by the above work, in this paper, we study the following critical fractional semilinear Neumann problem

$$\begin{cases}(-\Delta)^{\frac{1}{2}}u+\lambda u=u^{\frac{n+1}{n-1}},\ u>0 & \text{in } \Omega,\\ \partial_\nu u=0 & \text{on } \partial\Omega,\end{cases} \tag{4}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 5)$ is a smooth bounded domain, $\lambda>0$, and ν is the outward unit normal to $\partial\Omega$. Our method to overcome the lack of compactness is inspired by the work [1, 3] of Adimurthi, Mancini, and Yadava. That is, using the semigroup language for the extension method as introduced in [12, 31] and variational techniques, we will prove that there exists a constant $\lambda_0>0$ such that the minimizing problem

$$\inf\Big\{\iint_{\mathcal{C}}|\nabla v|^2dxdy+\lambda\int_{\Omega\times\{0\}}|v|^2dx: v\in\mathcal{H}^1(\mathcal{C}),\ \|v\|_{L^{\frac{2n}{n-1}}(\Omega\times\{0\})}=1\Big\}$$

is achieved if $0<\lambda<\lambda_0$. By the Lagrange multiplier rule, we get that problem (4) admits a minimal energy solution. Moreover, inspired by the idea of [5], we will show that this solution is constant, provided $\lambda>0$ is sufficiently small. Since the half-cylinder $\mathcal{C}$ is unbounded and is not a smooth domain, which will cause some difficulties in the proof of Lemma 4.1 below, we use Pohozaev-type identity [21, Lemma 4.1] and even reflection technique to overcome these difficulties. The main result in this paper can be stated as follows.

Theorem 1.1 *There exists a constant $\lambda_0>0$ such that*

(**i**) *problem (4) admits a minimal energy solution for $0<\lambda<\lambda_0$;*

(**ii**) *if Ω is convex, then the solution obtained in* (**i**) *is constant for sufficiently small $\lambda>0$.*

Remark 1 When $s\in(0,1)$, $s\neq\frac{1}{2}$ and $p=\frac{n+2s}{n-2s}$, the existence of solutions for problem (3) remains open.

Indeed, let $X^s(\mathbb{R}^{n+1}_+)$ denote the completion of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ with respect to the norm

$$\|U\|^2_{X^s}=\iint_{\mathbb{R}^{n+1}_+}y^{1-2s}|\nabla U(x,y)|^2dxdy.$$

By [9, Theorem 2.1], for every $U\in X^s(\mathbb{R}^{n+1}_+)$, it holds that

$$S(s,n)\Big(\int_{\mathbb{R}^n}|tr(U)|^{\frac{2n}{n-2s}}dx\Big)^{\frac{n-2s}{n}}\leq\int_{\mathbb{R}^{n+1}_+}y^{1-2s}|\nabla U|^2dxdy, \tag{5}$$

where $tr(U)$ denotes the trace of U on $\mathbb{R}^n\times\{y=0\}$. When $s=\frac{1}{2}$, we denote

$$S_0=\inf\Big\{\frac{\iint_{\mathbb{R}^{n+1}_+}|\nabla w(x,y)|^2dxdy}{(\int_{\mathbb{R}^n}|w(x,0)|^{\frac{2n}{n-1}}dx)^{(n-1)/n}}:\ w\in X^{\frac{1}{2}}(\mathbb{R}^{n+1}_+)\Big\}.$$

From [17, Theorem 1], we know that S_0 is achieved by

$$U_\epsilon(x,y)=\frac{\epsilon^{\frac{n-1}{2}}}{\big(|x|^2+(y+\epsilon)^2\big)^{\frac{n-1}{2}}},\quad \forall\epsilon>0. \tag{6}$$

However, from [7], the extremal function U for the best constant $S(s,n)$ of the trace inequality (5) does not possess an explicit expression if $s \in (0,1)$ and $s \neq \frac{1}{2}$, which may cause Lemma 3.4 below that is needed in the proof of Theorem 1.1 to break down.

The paper is organized as follows. In Section 2, we recall the definition of the spectral Neumann fractional Laplacian $(-\Delta)^{1/2}$ in a bounded domain and some preliminary results. The proof of Theorem 1.1 (**i**) is given in Section 3. The proof of Theorem 1.1 (**ii**) is in Section 4.

2. Preliminaries

In this section, we are devoted to some notations and preliminary results. As in [31], the fractional Neumann Laplacian $(-\Delta)^{1/2}$ in $\mathcal{H}^{1/2}(\Omega)$ is defined as follows. Let $\{\varphi_k\}_{k=0}^{\infty}$ denote the orthonormal basis in $L^2(\Omega)$ formed by the eigenfunctions associated to the eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$ of the Laplacian operator $-\Delta$ in Ω with zero Neumann boundary values on $\partial\Omega$. The Hilbert space $\mathcal{H}^{1/2}(\Omega)$ is defined as follows

$$\mathcal{H}^{1/2}(\Omega) \equiv \mathrm{Dom}\big((-\Delta)^{1/2}\big) := \Big\{u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} \big|\langle u, \varphi_k\rangle_{L^2(\Omega)}\big|^2 < \infty\Big\},$$

endowed with the norm

$$\|u\|^2_{\mathcal{H}^{1/2}(\Omega)} := \|u\|^2_{L^2(\Omega)} + \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} \big|\langle u, \varphi_k\rangle_{L^2(\Omega)}\big|^2.$$

For $u \in \mathcal{H}^{1/2}(\Omega)$, the fractional Neumann Laplacian $(-\Delta)^{1/2}$ is defined by

$$(-\Delta)^{1/2}u(x) = \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} \langle u, \varphi_k\rangle_{L^2(\Omega)}\varphi_k(x) \quad \text{in } \mathcal{H}^{1/2}(\Omega)',$$

where $\mathcal{H}^{1/2}(\Omega)'$ is the dual space of $\mathcal{H}^{1/2}(\Omega)$.

The space $H^{1/2}(\Omega)$ is defined as

$$H^{1/2}(\Omega) := \big\{u \in L^2(\Omega) : \|u\|^2_{H^{1/2}(\Omega)} \overset{\text{def}}{=} \|u\|^2_{L^2(\Omega)} + [u]^2_{H^{1/2}(\Omega)} < \infty\big\},$$

where

$$[u]^2_{H^{1/2}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+1}} dxdy.$$

The Hilbert space $\mathcal{H}^1(\mathcal{C})$ is defined as the completion of $H^1(\mathcal{C})$ under the scalar product

$$(v,w) = \iint_{\mathcal{C}} (\nabla_x v \cdot \nabla_x w + v_y w_y) dxdy + \lambda \int_\Omega (tr_\Omega v)(tr_\Omega w) dx,$$

with the associated norm $\|v\|^2 = (v,v)$.

Referring to [31, Lemma 2.4, Theorem 2.5, and Corollary 2.7], we have the following lemma.

Lemma 2.1 *We have $\mathcal{H}^{1/2}(\Omega) = H^{1/2}(\Omega)$, and there exists a unique bounded linear operator $T : \mathcal{H}^1(\mathcal{C}) \to \mathcal{H}^{1/2}(\Omega)$ such that $Tv(x,y) = v(x,0)$ if $v \in H^1(\mathcal{C})$ and, in particular, $\|Tv\|_{\mathcal{H}^{1/2}(\Omega)} \leq \|v\|$. Furthermore, $T(\mathcal{H}^1(\mathcal{C})) \subset\subset L^q(\Omega)$, for $1 \leq q < 2^* := \frac{2n}{n-1}$, where 2^* denotes the critical fractional Sobolev exponent.*

3. Proof of Theorem 1.1 (i)

In this section we study the existence of minimal energy solution for problem (4). Equivalently, we consider the following problem:

$$\begin{cases} \Delta v = 0, \ v > 0 & \text{in } \mathcal{C}, \\ \partial_\nu v = 0 & \text{on } \partial_L \mathcal{C} := \partial\Omega \times [0,\infty), \\ -v_y + \lambda v = v^{2^*-1} & \text{on } \Omega \times \{0\}. \end{cases} \tag{7}$$

We say that a function $v \in \mathcal{H}^1(\mathcal{C})$ is a weak solution for problem (7) if

$$(v,w) = \int_{\Omega\times\{0\}} v^{2^*-1} w dx, \quad \forall w \in \mathcal{H}^1(\mathcal{C}).$$

The associated energy functional $J_\lambda : \mathcal{H}^1(\mathcal{C}) \to \mathbb{R}$ for (7) is defined as

$$J_\lambda(v) = \frac{1}{2}\|v\|^2 - \frac{1}{2^*}\int_{\Omega\times\{0\}} |v|^{2^*} dx, \quad v \in \mathcal{H}^1(\mathcal{C}).$$

Definition 3.1 *We say that $u = v(\cdot,0)$ is a minimal energy solution of (4) if v is a solution of (7) and satisfies*

$$J_\lambda(v) = \inf\{J_\lambda(w) : \ w \in \mathcal{N}_\lambda\},$$

where

$$\mathcal{N}_\lambda = \Big\{ w \in \mathcal{H}^1(\mathcal{C}) \setminus \{0\} : \ \iint_{\mathcal{C}} |\nabla w|^2 dxdy + \lambda \int_{\Omega\times\{0\}} |w|^2 dx = \int_{\Omega\times\{0\}} |w|^{2^*} dx \Big\}.$$

Now we are ready to demonstrate the following result.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ be a smooth bounded domain. Then there exists a constant $\lambda_0 > 0$ such that, for $\lambda \in (0,\lambda_0)$, problem (7) admits a solution v_0 which satisfies $J_\lambda(v_0) < \frac{S_0^n}{4n}$.*

Proof Motivated by [1, 3, 32], we can prove this theorem directly from the following two lemmas. □

In order to prove Theorem 3.2, we introduce the following functional

$$Q_\lambda(v) = \frac{\iint_{\mathcal{C}} |\nabla v|^2 dxdy + \lambda \int_{\Omega\times\{0\}} |v|^2 dx}{\left(\int_{\Omega\times\{0\}} |v|^{2^*} dx \right)^{2/2^*}}, \quad v \in \mathcal{H}^1(\mathcal{C}),$$

and define

$$S_\lambda := \inf_{v\in\mathcal{H}^1(\mathcal{C})} Q_\lambda(v).$$

Then the following lemma holds.

Lemma 3.3 *For $\lambda > 0$, we have $S_\lambda > 0$. Assume that $S_\lambda < \frac{S_0}{2^{1/n}}$, then there exists a $w \in \mathcal{H}^1(\mathcal{C})$ with $w \geq 0$ such that $S_\lambda = Q_\lambda(w)$. Furthermore, if we define $v_0 = S_\lambda^{\frac{n-1}{2}} w$, then v_0 satisfies (7) with $J_\lambda(v_0) < \frac{S_0^n}{4n}$.*

Proof By Lemma 2.1, there exists a constant $C > 0$ such that

$$\Big(\int_{\Omega\times\{0\}} |v|^{2^*} dx\Big)^{1/2^*} \le C\|v\|, \quad \forall v \in \mathcal{H}^1(\mathcal{C}).$$

By the definition of S_λ, we get that $S_\lambda > 0$.

We choose $\{v_k\} \subset \mathcal{H}^1(\mathcal{C})$ as a minimizing sequence of S_λ with $\|v_k\|_{L^{2^*}(\Omega\times\{0\})} = 1$ (without loss of generality, we may assume $v_k \ge 0$, if not replacing it with $|v_k|$), that is,

$$\|v_k\|^2 = Q_\lambda(v_k) = S_\lambda + o(1) \quad \text{as } k \to \infty;$$

thus, $\{v_k\}$ is bounded in $\mathcal{H}^1(\mathcal{C})$. Then, up to a subsequence, we have $v_k \rightharpoonup w$ in $\mathcal{H}^1(\mathcal{C})$, and $\|w\| \le \liminf\limits_{k\to\infty} \|v_k\| = S_\lambda$. Combining $S_\lambda < \frac{S_0}{2^{1/n}}$ with [33, Lemma 2.1, Theorem 2.1], by a similar discussion as in [32, Proposition 4.4], we obtain that $v_k \to w$ in $L^{2^*}(\Omega\times\{0\})$. Therefore, $\|w\|_{L^{2^*}(\Omega\times\{0\})} = 1$, and $w \ge 0$ is a minimizer of $Q_\lambda(v)$. Thus, there exists $\mu \in \mathbb{R}$ (in fact, $\mu = S_\lambda$) by the Lagrange multiplier rule such that

$$\begin{cases} \Delta w = 0 & \text{in } \mathcal{C}, \\ \partial_\nu w = 0 & \text{on } \partial_L \mathcal{C}, \\ -w_y + \lambda w = \mu w^{2^*-1} & \text{on } \Omega\times\{0\}. \end{cases}$$

Choosing $v_0 = S_\lambda^{\frac{n-1}{2}} w$, then v_0 solves (7). Since $v_0 \in \mathcal{N}_\lambda$, combining $\|w\|^2 = S_\lambda$ with $S_\lambda < \frac{S_0}{2^{1/n}}$, we have

$$\begin{aligned} J_\lambda(v_0) &= \frac{1}{2}\|v_0\|^2 - \frac{1}{2^*}\int_{\Omega\times\{0\}} |v_0|^{2^*} dx = \big(\frac{1}{2} - \frac{1}{2^*}\big)\|v_0\|^2 \\ &= \frac{1}{2n}\|v_0\|^2 = \frac{1}{2n} S_\lambda^{n-1}\|w\|^2 = \frac{1}{2n} S_\lambda^n < \frac{S_0^n}{4n}. \end{aligned}$$

Now the proof is complete.

□

Lemma 3.4 *There exists $\lambda_0 > 0$ such that $S_\lambda < \frac{S_0}{2^{1/n}}$ $(n \ge 5)$, for $0 < \lambda < \lambda_0$.*

Proof Let us now introduce a nonincreasing cut-off function $\phi \in C^\infty(\mathbb{R}^{n+1}_+)$, verifying

$$\phi(x,y) = \begin{cases} 1, & (x,y) \in B^+(0,R/4), \\ 0, & (x,y) \notin \overline{B^+(0,R/2)}, \end{cases}$$

where $B^+(0,R) := \{(x,y) \in \mathbb{R}^{n+1}_+ : \ |(x,y)| < R\}$. Taking R small enough so that $\overline{B^+(0,R/2)} \subset \mathcal{C} \cup (\Omega\times\{0\})$, we will use the function ϕU_ϵ as test function v in the expression for Q_λ above.

Since the boundary $\partial\Omega$ is smooth, then there exists at least one point $x_0 \in \partial\Omega$ such that Ω lies on one side of the tangent plane at x_0 and the mean curvature with respect to the outward unit normal at x_0 is positive. Without loss of generality, we may suppose $x_0 = 0$. Hence, the boundary $\partial\Omega$ near the origin can be represented by

$$\rho(x') := \sum_{i=1}^{n-1} \beta_i x_i^2 + O(|x'|^3), \quad x' = (x_1, ..., x_{n-1}),$$

where $\beta_1, ..., \beta_{n-1}$ are the principal curvatures of $\partial\Omega$ at x_0. Thus, $\rho(x') \geq 0$ and the mean curvature $\frac{2}{n-1}\sum_{i=1}^{n-1}\beta_i > 0$ (for more details see [1, Lemma 2.2]).

Assume a is a suitably small positive constant, and define

$$\begin{aligned}
\Sigma &= \big\{(x', x_n, y) \in B(0,R): \ 0 < x_n < \rho(x'), \ y > 0\big\},\\
\Sigma' &= \big\{(x', x_n) \in B(0,R) \cap \{y=0\}: \ 0 < x_n < \rho(x')\big\};\\
L_a &= \big\{(x,y): \ |x_i| < a, \ 0 < y < a\big\} \subset B^+(0, R/4), \ i = 1, 2, ..., n,\\
L'_a &= \big\{x: \ |x_i| < a\big\} \subset \overline{B^+(0, R/4)} \cap \{y=0\};\\
\Delta_a &= \big\{(x', y): \ x' \in \Delta'_a, \ 0 < y < a\big\},\\
\Delta'_a &= \big\{x' = (x_1, ..., x_{n-1}): \ |x_i| < a\big\}.
\end{aligned}$$

Direct calculations show that, for any $\varepsilon \geq 0$,

$$\int_0^s \frac{1}{(1+t^2)^\varepsilon}dt = s + O(s^3), \tag{8}$$

which will be needed in the following proof.

Claim 1. As $\epsilon \to 0$, we have

$$\iint_{\mathcal{C}} |\nabla(\phi U_\epsilon)|^2 dxdy = \frac{n-1}{2}\int_{\mathbb{R}^n}\frac{1}{(|x|^2+1)^n}dx - \epsilon\omega_{n-1}\frac{n-1}{n-2}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_0^\infty \frac{r^n}{(1+r^2)^n}dr + O(\epsilon^2), \tag{9}$$

where U_ϵ is defined in (6) and ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$.

In fact, by the definition of ϕ, we obtain

$$\iint_{\mathcal{C}} |\nabla(\phi U_\epsilon)|^2 dxdy = \frac{1}{4}\int_{B(0,R)} |\nabla(\phi U_\epsilon)|^2 dxdy - \int_\Sigma |\nabla(\phi U_\epsilon)|^2 dxdy, \tag{10}$$

and

$$\begin{aligned}
\int_{B(0,R)} |\nabla(\phi U_\epsilon)|^2 dxdy =& 2\int_{\mathbb{R}^{n+1}_+} |\nabla(\phi U_\epsilon)|^2 dxdy\\
=& 2\int_{\mathbb{R}^{n+1}_+} \phi^2 |\nabla U_\epsilon|^2 dxdy + O(\epsilon^{n-1})\\
=& 2\int_{\mathbb{R}^{n+1}_+} |\nabla U_\epsilon|^2 dxdy + 2\int_{\mathbb{R}^{n+1}_+} (\phi^2 - 1)|\nabla U_\epsilon|^2 dxdy + O(\epsilon^{n-1})\\
=& 2K_1 + O(\epsilon^{n-1}),
\end{aligned} \tag{11}$$

where

$$K_1 := \int_{\mathbb{R}^{n+1}_+} |\nabla U_\epsilon|^2 dxdy = (n-1)^2 \epsilon^{n-1} \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{1}{\big(|x|^2 + (y+\epsilon)^2\big)^n} dxdy$$
$$= (n-1)^2 \epsilon^{n-1} \int_0^{+\infty} \frac{1}{(y+\epsilon)^n} dy \int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^n} dx$$
$$= (n-1) \int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^n} dx.$$

As for the third integral in (10), by (8), we get

$$C_1(\epsilon) := \int_\Sigma |\nabla(\phi U_\epsilon)|^2 dxdy = \int_{\Sigma \cap L_a} |\nabla(\phi U_\epsilon)|^2 dxdy + \int_{\Sigma \setminus L_a} |\nabla(\phi U_\epsilon)|^2 dxdy$$
$$= \int_{\Sigma \cap L_a} \phi^2 |\nabla U_\epsilon|^2 dxdy + O(\epsilon^{n-1}) = \int_{\Sigma \cap L_a} |\nabla U_\epsilon|^2 dxdy + O(\epsilon^{n-1})$$
$$= (n-1)^2 \epsilon^{n-1} \int_{\Sigma \cap L_a} \frac{1}{\big(|x|^2 + (y+\epsilon)^2\big)^n} dxdy + O(\epsilon^{n-1})$$
$$= (n-1)^2 \epsilon^{n-1} \int_{\Delta_a} dx' dy \int_0^{\rho(x')} \frac{1}{\big(|x'|^2 + (y+\epsilon)^2 + x_n^2\big)^n} dx_n + O(\epsilon^{n-1})$$
$$= (n-1) \epsilon^{n-1} \Big(\sum_{i=1}^{n-1} \beta_i\Big) \int_{\Delta_a} \frac{|x'|^2}{\big(|x'|^2 + (y+\epsilon)^2\big)^n} dx' dy$$
$$+ O\Big(\epsilon^{n-1} \int_{\Delta_a} \frac{|x'|^3}{\big(|x'|^2 + (y+\epsilon)^2\big)^n} dx' dy\Big)$$
$$= (n-1) \epsilon^{n-1} \Big(\sum_{i=1}^{n-1} \beta_i\Big) \int_0^a \int_{\Delta'_a} \frac{|x'|^2}{\big(|x'|^2 + (y+\epsilon)^2\big)^n} dx' dy$$
$$+ O\Big(\epsilon^{n-1} \int_0^a \int_{\Delta'_a} \frac{|x'|^3}{\big(|x'|^2 + (y+\epsilon)^2\big)^n} dx' dy\Big)$$
$$= (n-1) \epsilon^{n-1} \Big(\sum_{i=1}^{n-1} \beta_i\Big) \int_0^a \frac{1}{(y+\epsilon)^{n-1}} dy \int_{\Delta'_{a/(y+\epsilon)}} \frac{|x'|^2}{\big(|x'|^2+1\big)^n} dx'$$
$$+ O\Big(\epsilon^{n-1} \int_0^a \frac{1}{(y+\epsilon)^{n-2}} dy \int_{\Delta'_{a/(y+\epsilon)}} \frac{|x'|^3}{\big(|x'|^2+1\big)^n} dx'\Big),$$

where

$$\int_{\Delta'_{a/(y+\epsilon)}} \frac{|x'|^2}{\big(|x'|^2+1\big)^n} dx' = \int_{|x'|<\frac{a}{\epsilon}} \frac{|x'|^2}{\big(|x'|^2+1\big)^n} dx' - \int_{\{|x'|<\frac{a}{\epsilon}\} \setminus \Delta'_{a/(y+\epsilon)}} \frac{|x'|^2}{\big(|x'|^2+1\big)^n} dx'$$
$$= \omega_{n-1} \int_0^\infty \frac{r^n}{(1+r^2)^n} dr + O(1).$$

Thus,

$$C_1(\epsilon) = \epsilon\omega_{n-1}\frac{n-1}{n-2}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_0^\infty \frac{r^n}{(1+r^2)^n}dr + O(\epsilon^2). \tag{12}$$

Combining (11) with (12), Claim 1 holds.

Claim 2. As $\epsilon \to 0$, we get

$$\int_{\Omega\times\{0\}} |\phi U_\epsilon|^{2^*}dx = \frac{1}{2}\int_{\mathbb{R}^n}\frac{1}{(|x|^2+1)^n}dx - \epsilon\frac{\omega_{n-1}}{n-1}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_0^\infty \frac{r^n}{(1+r^2)^n}dr + O(\epsilon^2). \tag{13}$$

In fact, by the definition of ϕ, we obtain

$$\int_{\Omega\times\{0\}} |\phi U_\epsilon|^{2^*}dx = \frac{1}{2}\int_{B(0,R)\cap\{y=0\}} |\phi U_\epsilon|^{2^*}dx - \int_{\Sigma'}|\phi U_\epsilon|^{2^*}dx, \tag{14}$$

and

$$\begin{aligned}\int_{B(0,R)\cap\{y=0\}} |\phi U_\epsilon|^{2^*}dx &= \int_{\mathbb{R}^n}|\phi(x,0)U_\epsilon(x,0)|^{2^*}dx\\ &= \int_{\mathbb{R}^n}|U_\epsilon(x,0)|^{2^*}dx + \int_{\mathbb{R}^n}(\phi^{2^*}(x,0)-1)|U_\epsilon(x,0)|^{2^*}dx\\ &= K_2 + O(\epsilon^n),\end{aligned} \tag{15}$$

where

$$K_2 := \int_{\mathbb{R}^n}|U_\epsilon(x,0)|^{2^*}dx = \int_{\mathbb{R}^n}\frac{\epsilon^n}{(|x|^2+\epsilon^2)^n}dx = \int_{\mathbb{R}^n}\frac{1}{(|x|^2+1)^n}dx.$$

As for the third integral in (14), by (8), we get

$$\begin{aligned}C_2(\epsilon) :&= \int_{\Sigma'}|\phi U_\epsilon|^{2^*}dx = \int_{\Sigma'\cap L'_a}|\phi U_\epsilon|^{2^*}dx + \int_{\Sigma'\setminus L'_a}|\phi U_\epsilon|^{2^*}dx\\ &= \int_{\Delta'_a}dx'\int_0^{\rho(x')}|U_\epsilon(x,0)|^{2^*}dx_n + O(\epsilon^n)\\ &= \frac{\epsilon^n}{n-1}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_{\Delta'_a}\frac{|x'|^2}{(|x'|^2+\epsilon^2)^n}dx' + O\Big(\epsilon^n\int_{\Delta'_a}\frac{|x'|^3}{(|x'|^2+\epsilon^2)^n}dx'\Big)\\ &= \frac{\epsilon}{n-1}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_{\Delta'_{a/\epsilon}}\frac{|x'|^2}{(|x'|^2+1)^n}dx' + O\Big(\epsilon^2\int_{\Delta'_{a/\epsilon}}\frac{|x'|^3}{(|x'|^2+1)^n}dx'\Big).\end{aligned}$$

Thus,

$$C_2(\epsilon) = \epsilon\frac{\omega_{n-1}}{n-1}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_0^\infty \frac{r^n}{(1+r^2)^n}dr + O(\epsilon^2). \tag{16}$$

Combining (15) with (16), Claim 2 holds.

Claim 3. As $\epsilon \to 0$, we have

$$\int_{\Omega\times\{0\}} |\phi U_\epsilon|^2 dx = \frac{1}{2}\epsilon \int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^{n-1}} dx + O(\epsilon^2) \quad \text{for } n \geq 5. \tag{17}$$

In fact, by the definition of ϕ, we obtain

$$\int_{\Omega\times\{0\}} |\phi U_\epsilon|^2 dx = \frac{1}{2}\int_{B(0,R)\cap\{y=0\}} |\phi U_\epsilon|^2 dx - \int_{\Sigma'} |\phi U_\epsilon|^2 dx, \tag{18}$$

and

$$\begin{aligned}\int_{B(0,R)\cap\{y=0\}} |\phi U_\epsilon|^2 dx &= \int_{\mathbb{R}^n} |\phi(x,0)U_\epsilon(x,0)|^2 dx \\ &= \int_{\mathbb{R}^n} |U_\epsilon(x,0)|^2 dx + \int_{\mathbb{R}^n} (\phi^2(x,0)-1)|U_\epsilon(x,0)|^2 dx \\ &= K_3(\epsilon) + O(\epsilon^{n-1}),\end{aligned} \tag{19}$$

where

$$K_3(\epsilon) := \int_{\mathbb{R}^n} |U_\epsilon(x,0)|^2 dx = \int_{\mathbb{R}^n} \frac{\epsilon^{n-1}}{(|x|^2+\epsilon^2)^{n-1}} dx = \epsilon \int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^{n-1}} dx. \tag{20}$$

As for the third integral in (18), by (8), we get

$$\begin{aligned}\int_{\Sigma'} |\phi U_\epsilon|^2 dx &= \int_{\Sigma'\cap L'_a} |\phi U_\epsilon|^2 dx + \int_{\Sigma'\setminus L'_a} |\phi U_\epsilon|^2 dx \\ &= \int_{\Delta'_a} dx' \int_0^{\rho(x')} |U_\epsilon(x,0)|^2 dx_n + O(\epsilon^{n-1}) \\ &= \frac{\epsilon^{n-1}}{n-1}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_{\Delta'_a} \frac{|x'|^2}{(|x'|^2+\epsilon^2)^{n-1}} dx' + O\Big(\epsilon^{n-1}\int_{\Delta'_a} \frac{|x'|^3}{(|x'|^2+\epsilon^2)^{n-1}} dx'\Big) \\ &= \frac{\epsilon^2}{n-1}\Big(\sum_{i=1}^{n-1}\beta_i\Big)\int_{\Delta'_{a/\epsilon}} \frac{|x'|^2}{(|x'|^2+1)^{n-1}} dx' + O\Big(\epsilon^3\int_{\Delta'_{a/\epsilon}} \frac{|x'|^3}{(|x'|^2+1)^{n-1}} dx'\Big) \\ &= O(\epsilon^2) \quad \text{for } n \geq 5.\end{aligned} \tag{21}$$

Combining (19) with (21), Claim 3 holds.

From (9), (13), (17), and $S_0 = K_1/K_2^{2/2^*}$, we obtain that

$$
\begin{aligned}
Q_\lambda(\phi U_\epsilon) =& \frac{\iint_{\mathcal{C}} |\nabla(\phi U_\epsilon)|^2 dxdy + \lambda \int_{\Omega\times\{0\}} |\phi U_\epsilon|^2 dx}{\left(\int_{\Omega\times\{0\}} |\phi U_\epsilon|^{2^*} dx\right)^{2/2^*}} \\
=& \frac{\frac{1}{2}K_1 - C_1(\epsilon) + \frac{1}{2}\lambda K_3(\epsilon) + O(\epsilon^2)}{\left(\frac{1}{2}K_2 - C_2(\epsilon) + O(\epsilon^n)\right)^{2/2^*}} \\
=& \frac{\frac{1}{2}K_1 - C_1(\epsilon) + \frac{1}{2}\lambda K_3(\epsilon) + O(\epsilon^2)}{(\frac{1}{2})^{2/2^*} K_2^{2/2^*} \left(1 - \frac{2}{K_2}C_2(\epsilon) + O(\epsilon^n)\right)^{2/2^*}} \\
=& \frac{S_0}{2^{1/n}} + 2^{\frac{n-1}{n}} S_0 \Big(\frac{n-1}{n}\frac{C_2(\epsilon)}{K_2} - \frac{C_1(\epsilon)}{K_1} + \frac{\lambda K_3(\epsilon)}{2K_1} \Big) + O(\epsilon^2) \\
=& \frac{S_0}{2^{1/n}} + 2^{\frac{n-1}{n}} S_0 \frac{C_2(\epsilon)}{K_1} \Big(\frac{(n-1)^2}{n} - \frac{C_1(\epsilon)}{C_2(\epsilon)} + \frac{\lambda}{2}\frac{K_3(\epsilon)}{C_2(\epsilon)} \Big) + O(\epsilon^2).
\end{aligned}
$$

By (12), (16), and (20), we get that

$$
\lim_{\epsilon\to 0} \frac{C_1(\epsilon)}{C_2(\epsilon)} = \frac{(n-1)^2}{n-2},
$$

and

$$
\lim_{\epsilon\to 0} \frac{K_3(\epsilon)}{C_2(\epsilon)} = \lim_{\epsilon\to 0} \frac{K_3'(\epsilon)}{C_2'(\epsilon)} = \frac{\int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^{n-1}} dx}{\frac{\omega_{n-1}}{n-1}\left(\sum_{i=1}^{n-1}\beta_i\right)\int_0^\infty \frac{r^n}{(1+r^2)^n} dr} \overset{\text{def}}{=} \widetilde{C} = \widetilde{C}(n,\beta_i) > 0.
$$

Let

$$
A_\lambda := \lim_{\epsilon\to 0} \Big(\frac{(n-1)^2}{n} - \frac{C_1(\epsilon)}{C_2(\epsilon)} + \frac{\lambda}{2}\frac{K_3(\epsilon)}{C_2(\epsilon)} \Big) = -\frac{2(n-1)^2}{n(n-2)} + \frac{\widetilde{C}}{2}\lambda.
$$

It is clear that there exists $\lambda_0 > 0$ such that $A_\lambda < 0$ for $0 < \lambda < \lambda_0$. Thus, we obtain that

$$
S_\lambda \le Q_\lambda(\phi U_\epsilon) < \frac{S_0}{2^{1/n}},
$$

provided $\lambda \in (0, \lambda_0)$ and $\epsilon > 0$ is small enough. Now the proof is complete. □

Proof of Theorem 1.1 (i). Taking $u_0 = v_0(\cdot, 0)$, where v_0 is the positive solution of problem (7) given in Theorem 3.2, then u_0 is a positive solution of problem (4). It remains to prove that v_0 is a minimal energy solution of problem (4). Indeed, by $v_0 \in \mathcal{N}_\lambda$, we have

$$
J_\lambda(v_0) \ge \inf\{J_\lambda(\widetilde{w}) : \ \widetilde{w} \in \mathcal{N}_\lambda\}. \tag{22}
$$

Meanwhile, if $\widetilde{w} \in \mathcal{N}_\lambda$, by the definition of S_λ, we obtain $\|\widetilde{w}\|^2 \ge S_\lambda \|\widetilde{w}\|^2_{L^{2^*}(\Omega\times\{0\})} = S_\lambda \|\widetilde{w}\|^{\frac{4}{2^*}}$. Thus,

$$
\frac{1}{2n} S_\lambda^n \le \frac{1}{2n}\|\widetilde{w}\|^2 = J_\lambda(\widetilde{w}), \quad \widetilde{w} \in \mathcal{N}_\lambda.
$$

Lemma 3.3 implies that $v_0 = S_\lambda^{\frac{n-1}{2}} w \in \mathcal{N}_\lambda$ and $\|w\|^2 = S_\lambda$. Hence,

$$J_\lambda(v_0) = \frac{1}{2n}\|v_0\|^2 = \frac{1}{2n} S_\lambda^{n-1}\|w\|^2 = \frac{1}{2n} S_\lambda^n \le J_\lambda(\widetilde{w}), \quad \forall \widetilde{w} \in \mathcal{N}_\lambda. \tag{23}$$

By (22) and (23), we get

$$J_\lambda(v_0) = \inf\{J_\lambda(\widetilde{w}) : \ \widetilde{w} \in \mathcal{N}_\lambda\}.$$

Therefore, u_0 is a minimal energy solution of (4). Now the proof is complete.

4. Proof of Theorem 1.1 (ii)

In this section, we will prove that the solution obtained in Theorem 1.1 (**i**) is constant for sufficiently small λ, if $\Omega \subset \mathbb{R}^n$ $(n \ge 5)$ is a bounded smooth convex domain. Let ϵ, $\mu > 0$, and define

$$A_{\mu,\epsilon} = \Big\{(u,\lambda) : \ u = v(\cdot, 0), \ J_\lambda(v) < (1-\epsilon)\frac{S_0^n}{4n}, \text{ where } v \text{ satisfies (7) for some } \lambda \le \mu\Big\}.$$

Lemma 4.1 *Assume* $\{(u_k, \lambda_k)\} \subset A_{\mu,\epsilon}$ *with* $\lambda_k \to 0$ $(k \to \infty)$*, then* $\lim_{k\to\infty} |u_k|_{L^\infty(\Omega)} = 0$*, provided that* Ω *is a bounded smooth convex domain.*

Proof Let $v_k(x,y)$ be the solution of (7) corresponding to each $u_k(x)$ with $v_k(x,0) = u_k(x)$ and $\lambda = \lambda_k$. That is, $v_k(x,y)$ verifies

$$\begin{cases} \Delta v_k = 0, \ v_k > 0 & \text{in } \mathcal{C}, \\ \partial_\nu v_k = 0 & \text{on } \partial_L \mathcal{C}, \\ -(v_k)_y + \lambda_k v_k = v_k^{2^*-1} & \text{on } \Omega \times \{0\}, \end{cases}$$

where $\lambda_k > 0$ for $k \in \mathbb{N}$, and Ω is a bounded smooth convex domain. We break the proof into the following Steps.

Step 1. We claim that $\{u_k\}$ is bounded in $\mathcal{H}^{1/2}(\Omega)$ and up to a subsequence, we obtain that

$$\begin{aligned} v_k &\rightharpoonup \tilde{v} && \text{in } \mathcal{H}^1(\mathcal{C}), \\ v_k(x,0) &\to \tilde{v}(x,0) && \text{in } L^q(\Omega), \ 1 \le q < 2^*, \\ v_k(x,0) &\to \tilde{v}(x,0) && \text{a.e. in } \Omega. \end{aligned} \tag{24}$$

Indeed, we have that

$$\begin{aligned} J_{\lambda_k}(v_k) =& \frac{1}{2}\Big(\iint_{\mathcal{C}} |\nabla v_k|^2 dxdy + \lambda_k \int_{\Omega\times\{0\}} |v_k|^2 dx\Big) - \frac{1}{2^*}\int_{\Omega\times\{0\}} |v_k|^{2^*} dx \\ =& \Big(\frac{1}{2} - \frac{1}{2^*}\Big)\Big(\iint_{\mathcal{C}} |\nabla v_k|^2 dxdy + \lambda_k \int_{\Omega\times\{0\}} |v_k|^2 dx\Big) \\ <& (1-\epsilon)\frac{S_0^n}{4n}; \end{aligned}$$

thus,

$$\int_{\Omega\times\{0\}} |v_k|^{2^*} dx = \iint_{\mathcal{C}} |\nabla v_k|^2 dxdy + \lambda_k \int_{\Omega\times\{0\}} |v_k|^2 dx < (1-\epsilon)\frac{S_0^n}{2}. \tag{25}$$

By Hölder's inequality, there exists a $C>0$ such that

$$\int_{\Omega\times\{0\}} |v_k|^2 dx \le C\Big(\int_{\Omega\times\{0\}} |v_k|^{2^*} dx\Big)^{\frac{2}{2^*}}. \tag{26}$$

Combining (25), (26) with $\lambda_k \to 0$ $(k\to\infty)$, we get the boundedness of $\{v_k\}$ in $\mathcal{H}^1(\mathcal{C})$. Thus, $\{u_k\}$ is bounded in $\mathcal{H}^{1/2}(\Omega)$ by Lemma 2.1 and (24) holds.

Step 2. We claim that $\tilde{v}\equiv 0$ and $\|v_k\|\to 0$ $(k\to\infty)$.

In fact, $\tilde{v}$ satisfies

$$\begin{cases} \Delta\tilde{v}=0,\ \tilde{v}\ge 0 & \text{in } \mathcal{C},\\ \partial_\nu \tilde{v}=0 & \text{on } \partial_L\mathcal{C},\\ -\tilde{v}_y=\tilde{v}^{2^*-1} & \text{on } \Omega\times\{0\}. \end{cases}$$

By a similar discussion as in [7, Proposition 5.1] and [9, Theorem 4.7], we have $\tilde{v}\in L^\infty(\mathcal{C})$. Since Ω is a bounded smooth convex domain, then $\tilde{v}\equiv 0$ by Pohozaev-type identity [21, Lemma 4.1]. Combining (24) with Hölder inequality, we get, for $\lambda_0>0$,

$$\overline{\lim_{k\to\infty}} J_{\lambda_0/2}(v_k) = \overline{\lim_{k\to\infty}}\Big(J_{\lambda_k}(v_k) + \frac{\lambda_0-\lambda_k}{2}\|v_k(x,0)\|_{L^2(\Omega)}\Big) \le (1-\epsilon)\frac{S_0^n}{4n},$$

and

$$\begin{aligned} |\langle J'_{\lambda_0/2}(v_k),\varphi\rangle| &= \Big|\langle J'_{\lambda_k}(v_k),\varphi\rangle + (\lambda_0-\lambda_k)\int_{\Omega\times\{0\}} v_k(x,0)\varphi dx\Big| \\ &\le C|\lambda_0-\lambda_k|\cdot\|v_k(x,0)\|_{L^2(\Omega)}\|\varphi\| \\ &\to 0 \quad \text{as } k\to\infty,\ \forall\ \varphi\in\mathcal{H}^1(\mathcal{C}). \end{aligned}$$

Thus, we get a Palais-Smale sequence $\{v_k\}$ of $J_{\lambda_0/2}$ on $(-\infty, S_0^n/4n)$. Taking ideas from [32, Lemma 5.1], we know that $J_{\lambda_0/2}$ verifies the local Palais-Smale condition on $(-\infty, S_0^n/4n)$. Up to a subsequence, we have $\|v_k\|\to 0$ $(k\to\infty)$.

Step 3. We claim that

$$\overline{\lim_{k\to\infty}} |u_k|_{L^\infty(\Omega)} < \infty. \tag{27}$$

Suppose that (27) is false. Then we can assume that there exists a sequence $\{P_k\}\subset\overline{\Omega}$ such that

$$M_k := \sup_{\Omega} u_k = u_k(P_k)\to\infty,\ P_k\to P\in\overline{\Omega} \quad \text{as } k\to\infty.$$

By Hopf's maximum lemma, the maximum of $v_k(x,y)$ can lie only on $\overline{\Omega}\times\{0\}$; thus, we get

$$\sup_{\overline{\mathcal{C}}} v_k = v_k(P_k,0) = M_k,$$

where $\overline{\mathcal{C}} := \overline{\Omega} \times [0, \infty)$. Let t_k satisfy

$$M_k \cdot t_k^{\frac{n-1}{2}} = 1.$$

Up to a subsequence, one of the following holds:

$$\lim_{k\to\infty} \frac{d(P_k, \partial\Omega)}{t_k} = \infty, \tag{28}$$

$$\lim_{k\to\infty} \frac{d(P_k, \partial\Omega)}{t_k} < \infty. \tag{29}$$

Suppose that (28) holds. Since $t_k \to 0$, there exists a $k_0 > 0$ such that $B(P_k, t_k R) \subset \Omega$ for every $R > 0$ and $k \geq k_0$. In this case, let $B_k(R) = B_0(R) = B(0, R)$.

Suppose that (29) holds. Let $Q_k \in \partial\Omega$ satisfy $d(P_k, Q_k) = d(P_k, \partial\Omega)$. Then there exists a k_0 such that, for $k \geq k_0$, $B(z_k, t_k R) \subset \Omega$, where $z_k = P_k + Rt_k\nu_k$ and ν_k is the outward unit normal at Q_k. Let $B_k(R) = B(\nu_k R, R)$ and $B_0(R) = B(\nu_0 R, R)$, where $\nu_0 = \lim\limits_{k\to\infty} \nu_k$.

For $k > k_0$, we define w_k as

$$w_k(x, y) = t_k^{\frac{n-1}{2}} v_k(P_k + t_k x, t_k y),$$

then w_k verifies

$$\begin{cases} \Delta w_k = 0, \ 0 < w_k \leq 1 & \text{in } B_k(R) \times (0, \infty), \\ -(w_k)_y = w_k^{2^*-1} - \lambda_k t_k w_k & \text{on } B_k(R) \times \{0\}, \\ w_k(0) = 1. \end{cases} \tag{30}$$

We now study problem (30) restricted on $B_k^{n+1}(R) \cap \{y \geq 0\}$, where $B_k^{n+1}(R)$ is the open ball in $\mathbb{R}^{n+1}$ with radius R centered at $(v_k R, 0)$, and extend w_k to the ball $B_k^{n+1}(R)$ by even reflection:

$$w_{k,ev}(x, y) = \begin{cases} w_k(x, y) & \text{for } y \geq 0, \\ w_k(x, -y) & \text{for } y \leq 0. \end{cases}$$

Then $w_{k,ev}$ satisfies

$$\begin{cases} \Delta w_{k,ev} = 0, \ 0 < w_{k,ev} \leq 1 & \text{in } B_k^{n+1}(R), \\ w_{k,ev}(0) = 1. \end{cases}$$

By elliptic regularity [19] and $0 < w_{k,ev} \leq 1$ in $B_k^{n+1}(R)$, we get that $w_{k,ev} \in C^{1,\alpha}(\overline{B_k^{n+1}(R)})$, for some $\alpha \in (0, 1)$. Up to a subsequence, we have $w_{k,ev} \to w_0$ in $C^1(\overline{B_0^{n+1}(R)})$, where $B_0^{n+1}(R) := B((v_0 R, 0), R)$. Thus, $w_{k,ev}(x, y) = w_k(x, y) \to w_0$ in $C^1(\overline{B_0^{n+1}(R)} \cap \{y \geq 0\})$, and w_0 verifies

$$\begin{cases} \Delta w_0 = 0, \ 0 \leq w_0 & \text{in } B_0^{n+1}(R) \cap \{y > 0\}, \\ -(w_0)_y = w_0^{2^*-1} & \text{on } B_0^{n+1}(R) \cap \{y = 0\}, \\ w_0(0) = 1. \end{cases} \tag{31}$$

On the other hand, from Step 2, we have

$$\begin{aligned}\iint_{B_k^{n+1}(R)\cap\{y\geq 0\}}|\nabla w_{k,ev}|^2dxdy &= \iint_{B_k^{n+1}(R)\cap\{y\geq 0\}}|\nabla w_k|^2dxdy\\ &\leq \iint_{\mathcal{C}}|\nabla v_k|^2dxdy\\ &\to 0 \quad \text{as } k\to\infty;\end{aligned}$$

thus, $\nabla w_0\equiv 0$ in $\overline{B_0^{n+1}(R)}\cap\{y\geq 0\}$. Since $w_0(0)=1$, we get $w_0\equiv 1$ in $\overline{B_0^{n+1}(R)}\cap\{y\geq 0\}$, which contradicts the second equation of (31). Thus, (27) holds.

In conclusion, from Step 3 and [31, Theorem 3.5 (3)], we have $\overline{\lim\limits_{k\to\infty}}|u_k|_{C^{0,\alpha}(\overline{\Omega})}<\infty$. So by Step 2 and Ascoli-Arzelà theorem, we get $\lim\limits_{k\to\infty}|u_k|_{L^\infty(\Omega)}=0$. □

Lemma 4.2 *There exists a $\mu_0>0$ such that $A_{\mu_0,\epsilon}$ consists of constants only.*

Proof Using the fractional Poincaré's inequality, there exists a constant $C_1>0$ such that

$$C_1\|\psi-\psi_\Omega\|^2_{L^2(\Omega)}\leq[\psi]^2_{H^{1/2}(\Omega)},\ \forall\psi\in H^{1/2}(\Omega), \tag{32}$$

where $\psi_\Omega:=\frac{1}{|\Omega|}\int_\Omega\psi(x)dx$. By [31, Lemma 2.4 and Theorem 2.5], there exists a constant $C_2>0$ such that

$$C_2[\phi]^2_{H^{1/2}(\Omega)}\leq\iint_{\mathcal{C}}|\nabla v^\phi|^2dxdy. \tag{33}$$

If we denote $M_\mu=\sup\{|u|_{L^\infty(\Omega)}:\ (u,\lambda)\in A_{\mu,\epsilon}\}$, then by Lemma 4.1 and Heine theorem, we have $\lim\limits_{\mu\to 0}M_\mu=0$. Let $f(t)=t^{2^*-1}$ $(t\geq 0)$, so there exists a $\mu_0>0$ such that $f'(M_{\mu_0})\leq\frac{C_1C_2}{2}$. We choose $(u,\lambda)\in A_{\mu_0,\epsilon}$ and write $u=u_\Omega+\phi$. Then $\int_\Omega\phi(x)dx=0$, and ϕ verifies

$$\begin{cases}(-\Delta)^{1/2}\phi+\lambda\phi-\big(\int_0^1 f'(u_\Omega+\theta\phi)dt\big)\phi=f(u_\Omega)-\lambda u_\Omega & \text{in } \Omega,\\ \partial_\nu\phi=0, & \text{on } \partial\Omega.\end{cases}$$

Let v^ϕ be the Neumann extension of ϕ, which satisfies the extension problem

$$\begin{cases}\Delta v^\phi=0 & \text{in } \mathcal{C},\\ \partial_\nu v^\phi=0 & \text{on } \partial_L\mathcal{C},\\ -(v^\phi)_y=\big(\int_0^1 f'(u_\Omega+\theta\phi)dt\big)\phi-\lambda\phi+f(u_\Omega)-\lambda u_\Omega & \text{on } \Omega\times\{0\}.\end{cases} \tag{34}$$

Taking v^ϕ as a test function in (34), thus,

$$\iint_{\mathcal{C}}|\nabla v^\phi|^2dxdy+\lambda\int_\Omega\phi^2dx=\int_\Omega\Big(\int_0^1 f'(u_\Omega+\theta\phi)dt\Big)\phi^2dx. \tag{35}$$

Since $0\leq u_\Omega+\theta\phi\leq M_{\mu_0}$, we get

$$\Big|\int_0^1 f'(u_\Omega+\theta\phi)dt\Big|\leq f'(M_{\mu_0})\leq\frac{C_1C_2}{2}. \tag{36}$$

Combining (32), (33), (35) with (36), we have

$$(C_1C_2+\lambda)\int_\Omega \phi^2 dx \le \iint_{\mathcal{C}} |\nabla v^\phi|^2 dxdy + \lambda \int_\Omega \phi^2 dx \le \frac{C_1C_2}{2}\int_\Omega \phi^2 dx,$$

which implies that $\phi \equiv 0$. Hence, u is a constant. □

Proof of Theorem 1.1 (ii). For $\lambda > 0$, the constant solution w_λ of (7) satisfies

$$w_\lambda = \lambda^{\frac{n-1}{2}} \to 0 \quad \text{as } \lambda \to 0.$$

Thus, there exists a $\mu_1 > 0$ such that

$$J_\lambda(w_\lambda) = \frac{1}{2n}\lambda^n |\Omega| \le \frac{S_0^n}{8n} \quad \text{for } \lambda \le \mu_1.$$

Let $\epsilon = \frac{1}{2}$ and $\tilde{\lambda} := \min\{\mu_0, \mu_1\}$, where μ_0 is given in Lemma 4.2. Let $\lambda < \tilde{\lambda}$ and u_λ is a positive minimal energy solution of (4), then $J_\lambda(v_\lambda) \le J_\lambda(w_\lambda)$ and $(u_\lambda, \lambda) \in A_{\mu_0,\epsilon}$. By Lemma 4.2, we get that u_λ is constant.

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