Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1733 – 1745
© TÜBİTAK
doi:10.55730/1300-0098.3459

Liouville-type theorem for one-dimensional porous medium systems with sources

Anh Tuan DUONG*
School of Applied Mathematics and Informatics, Hanoi University of Science and Technology
1 Dai Co Viet, Hai Ba Trung, Ha noi, Viet Nam

Received: 28.07.2022 • **Accepted/Published Online:** 19.07.2023 • **Final Version:** 25.09.2023

Abstract: In this paper, we are concerned with the one-dimensional porous medium system with sources

$$\begin{cases} u_t - (u^m)_{xx} = a_{11}u^p + a_{12}u^r v^{r+m}, (x,t) \in J \times I \subset \mathbb{R} \times \mathbb{R} \\ v_t - (v^m)_{xx} = a_{21}u^{r+m}v^r + a_{22}v^p,\ (x,t) \in J \times I \subset \mathbb{R} \times \mathbb{R}, \end{cases}$$

where $p = 2r + m,\ m > 1,\ r > 0$. Under the conditions $a_{12} \geq 0, a_{21} \geq 0,\ a_{11} > 0$, and $a_{22} > 0$, we prove that the system does not possess any nontrivial nonnegative weak solution.

Key words: Liouville-type theorem, Porous medium system with sources, Nonexistence result

1. Introduction

In recent years, the Liouville properties for elliptic and parabolic equations/systems have been much investigated and emerged as one of the most powerful tools in the study of initial and boundary value problems. From Liouville-type theorems, one can deduce a variety of results on qualitative properties of solutions such as: universal, pointwise, a priori estimates of local solutions; universal and singularity estimates; decay estimates; blow-up rate of solutions, see [25, 26, 29] and references therein.

Let us first go back to the pioneering work of Gidas and Spruck [14] where the existence and nonexistence of a positive solution to the Lane-Emden equation

$$-\Delta u = u^p \text{ in } \mathbb{R}^N$$

was completely established. The optimal range of the exponent p for the nonexistence of positive solutions is $1 < p < p_s(N) := \frac{N+2}{N-2}$. However, a similar question for the Lane-Emden system

$$\begin{cases} -\Delta u = v^p \text{ in } \mathbb{R}^N \\ -\Delta v = u^q \text{ in } \mathbb{R}^N \end{cases}$$

has not been completely solved. It is conjectured that the Lane-Emden system has no positive solution if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}.$$

*Correspondence: tuan.duonganh@hust.edu.vn

2010 *AMS Mathematics Subject Classification:* 35B53, 35K57

This conjecture has been confirmed in the low dimensions $N \le 4$, see [19, 31, 32, 34]. In higher dimensions $N \ge 5$, it has not been solved yet, see [34].

We next consider the parabolic model

$$u_t - \Delta u = u^p \text{ in } \mathbb{R}^N \times I \subset \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

which has been extensively studied by many mathematicians. The well-known Fujita result ensures the nonexistence of nontrivial nonnegative supersolutions in $\mathbb{R}^N \times (0, +\infty)$ of (1.1) provided that $1 < p \le \frac{N+2}{N}$, see [10], [20, Sec. 26], and [2, 15, 23] for generalized models. In the supercritical case $p > \frac{N+2}{N}$, problem (1.1) possesses, see [16, Example 1], a nonnegative supersolution in $\mathbb{R}^N \times \mathbb{R}$ of the form

$$u(x,t) = \begin{cases} kt^{-\frac{1}{p-1}} e^{-\gamma \frac{1+|x|^2}{t}} & \text{if } t > 0,\ x \in \mathbb{R}^N \\ 0 & \text{if } t \le 0,\ x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where k, γ are suitably chosen, see also [8] for the system. The complete classification of solutions to (1.1) is one of the most interesting and challenging problems, see [21, 27]. Very recently, the optimal Liouville type theorem for solutions of (1.1) in any dimension has been completely proved by Quittner [28]. We also refer to papers of Bidaut-Véron [4], Quittner [27] and the references [3, 9, 24, 26, 29] for related results.

We now consider the quasilinear parabolic equation

$$u_t - \Delta u^m = u^p \text{ in } \mathbb{R}^N \times I \subset \mathbb{R}^N \times \mathbb{R} \tag{1.3}$$

with $m > 1$. For this equation, some local solvability and general regularity results of solutions of (1.3) can be found in [1, 6, 11–13, 30, 33, 35]. In particular, the authors in [13, 30] proved that when $p \le m + \frac{2}{N}$, the solution u of (1.3) in $\mathbb{R}^N \times (0, +\infty)$ with bounded, continuous initial data $u_0 \not\equiv 0$ does not exist globally and blow up in a finite time, i.e. there is $T > 0$ such that

$$\sup_{x \in \mathbb{R}^N} u(x,t) \to +\infty \text{ as } t \to T.$$

Under the same condition $p \le m + \frac{2}{N}$, it was also established in [6] that any solution u of (1.3) in $\mathbb{R}^N \times (0, T)$ satisfies the blow up estimate

$$u(x,t) \le C(N,m,p) \left(t^{-\frac{1}{p-1}} + (T-t)^{-\frac{1}{p-1}} \right). \tag{1.4}$$

Furthermore, the range of the exponent p is extended in [1] such that the estimate (1.4) is still true. The extended range is $p < p_0(m,N)$ where $p_0(m,N)$ is explicitly given by

$$p_0(m,N) = \begin{cases} \frac{N(N+2)}{2(N-1)^2}(1 + \theta + \sqrt{1+2\theta}) & \text{if } N \ge 2, \\ \infty & \text{if } N = 1 \end{cases}$$

with $\theta = \frac{(N-1)(m-1)}{N}$. Remark that when $m = 1$,

$$p_0(1.N) = p_B(N) = \begin{cases} \frac{N(N+2)}{2(N-1)^2} & \text{if } N \ge 2, \\ \infty & \text{if } N = 1, \end{cases}$$

which is called the Bidaux-Véron exponent.

The approach in [1] is based on establishing a Liouville type theorem on the whole space. In fact, the authors showed that the equation (1.3) has no nontrivial nonnegative weak solution in the whole space $\mathbb{R}^N \times \mathbb{R}$ when $m < p < p_0(m, N)$. This nonexistence results in the full range $m < p < mp_S$ is still left open. Among other things, it was shown in [1] that:

Theorem A. *Let $1 < m < p$. Then, the equation $u_t - (u^m)_{xx} = u^p$ has no nontrivial nonnegative weak solution in $\mathbb{R} \times \mathbb{R}$.*

Let us next consider the semilinear cooperative parabolic system

$$\begin{cases} u_t - \Delta u = a_{11}u^p + a_{12}u^r v^{r+1}, (x,t) \in \Omega \times I \subset \mathbb{R}^N \times \mathbb{R} \\ v_t - \Delta v = a_{21}u^{r+1}v^r + a_{22}v^p, \ (x,t) \in \Omega \times I \subset \mathbb{R}^N \times \mathbb{R}, \end{cases} \tag{1.5}$$

where $p = 2r+1$ and $r > 0$. This system has been studied in [7, 22, 27] in any dimension. Some Liouville type theorems were established in the case $a_{12} = a_{21}$ in [22] and in the general case $a_{12} \neq a_{21}$ in [7]. In particular, the following result was proved in [27], see also [28].

Theorem B. *Let $N = 1$ and $a_{12}, a_{21} \geq 0; a_{11}, a_{22} > 0$. Then, the system (1.5) has no nontrivial nonnegative solution in $\mathbb{R} \times \mathbb{R}$.*

The main tools in [27] are scaling argument and energy estimates. Noticing that, by a simple scaling, one can reduce the system (1.5) to a parabolic system with gradient structure as in [27].

Our purpose in this paper is to study the following porous medium system

$$\begin{cases} u_t - (u^m)_{xx} = a_{11}u^p + a_{12}u^r v^{r+m}, (x,t) \in J \times I \subset \mathbb{R} \times \mathbb{R} \\ v_t - (v^m)_{xx} = a_{21}u^{r+m}v^r + a_{22}v^p, \ (x,t) \in J \times I \subset \mathbb{R} \times \mathbb{R}, \end{cases} \tag{1.6}$$

where the exponents p, m, r and the coefficients a_{ij} satisfy

$$\begin{aligned} &m > 1, r > 0, p = 2r + m, \\ &a_{12}, a_{21} \geq 0; a_{11}, a_{22} > 0. \end{aligned} \tag{1.7}$$

As mentioned above, there are many contributions to the porous medium equations. Nevertheless, to the best of our knowledge, there are a few results on porous medium system with sources, see [5, 17, 18, 36] for the global existence and boundedness of solutions. In this paper, we propose to study the Liouville properties of the porous medium system with sources. The first attempt is to establish a nonexistence result for the one-dimensional porous medium system with sources (1.6). As in [1], by a nonnegative weak solution (u, v) of (1.6) we mean $u, v \in C(J \times I)$, $u, v \geq 0$, satisfying (1.6) in the distributional sense.

Our main result in this paper is the following.

Theorem 1.1 *Under the assumption (1.7), the system (1.6) does not possess any nontrivial nonnegative weak solution in $\mathbb{R} \times \mathbb{R}$.*

In order to prove Theorem 1.1, we shall develop the idea in [7, 22] where the main tool is a combination of the Bochner formula, nonlinear integral estimates, the scaling invariance argument, and some idea from [4, 14]. Remark that our proof is not straightforward in comparison with that for the case $m = 1$ in [7, 22]. The main difficulty arising in the proof is the presence of a quasilinear term ($m > 1$). Some key estimates for the semilinear case ($m = 1$) do not work for the case ($m > 1$) and the nonlinear integral estimates become more delicate.

Before closing the introduction, we present a consequence of Theorem 1.1 without proof since it is totally similar to [1, Theorem 2.1], see also [7, Proposition 1].

Proposition 1.2 *Let $p > m > 1$ and $J \subset \mathbb{R}$. Suppose that (1.7) holds. If (u, v) is a nonnegative weak solution of (1.6) in $J \times (0, T)$, then there holds*

$$u(x,t) + v(x,t) \le C(t^{-\frac{1}{p-1}} + (T-t)^{-\frac{1}{p-1}} + \mathrm{dist}^{-\frac{2}{p-m}}(x, J)),\ x \in J, t \in (0,T).$$

The rest of this paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

For the sake of simplicity, we denote by $\int$ the integral $\int_{(-1,1)\times(-1,1)} dxdt$. Denote by C a generic positive constant whose value may change from line to line.

In order to prove Theorem 1.1, it is sufficient to prove that the following system

$$\begin{cases} (u^{\frac{1}{m}})_t - u_{xx} = a_{11}u^{\frac{p}{m}} + a_{12}u^{\frac{r}{m}}v^{\frac{r+m}{m}}, (x,t) \in J \times I \subset \mathbb{R}\times\mathbb{R} \\ (v^{\frac{1}{m}})_t - v_{xx} = a_{21}u^{\frac{r+m}{m}}v^{\frac{r}{m}} + a_{22}v^{\frac{p}{m}},\ (x,t) \in J \times I \subset \mathbb{R}\times\mathbb{R}. \end{cases} \tag{2.1}$$

has no nontrivial nonnegative weak solution. In what follows, we shall prove this assertion.

An integral estimate, which plays a crucial role in the proof of Theorem 1.1, is given in the following lemma.

Lemma 2.1 *Assume that (1.7) holds. Let (u, v) be a positive regular solution of (2.1) on $(-1, 1) \times (-1, 1)$. Fix $\chi \in C_0^\infty((-1,1)\times(-1,1))$ and put*

$$I := a_{21}\int \chi u^{-1-\frac{1}{m}}|u_x|^4 + a_{12}\int \chi v^{-1-\frac{1}{m}}|v_x|^4$$

and

$$L := a_{21}\int \chi u^{1-\frac{1}{m}}\left(a_{11}u^{\frac{p}{m}} + a_{12}u^{\frac{r}{m}}v^{\frac{r}{m}+1}\right)^2 + a_{12}\int \chi v^{1-\frac{1}{m}}\left(a_{21}u^{\frac{r}{m}+1}v^{\frac{r}{m}} + a_{22}v^{\frac{p}{m}}\right)^2.$$

Then, there exists $C > 0$ independent of u, v and χ such that

$$\begin{aligned} I + L \le & C\int \chi\left(|(u^{\frac{1}{m}})_t|u^{-\frac{1}{m}}|u_x|^2 + |(v^{\frac{1}{m}})_t|v^{-\frac{1}{m}}|v_x|^2\right) \\ & + C\int u^{1-\frac{1}{m}}|\chi_x u_x|\left(u^{\frac{p}{m}} + u^{\frac{r}{m}}v^{\frac{r}{m}+1} + |(u^{\frac{1}{m}})_t| + u^{-1}|u_x|^2\right) \\ & + C\int v^{1-\frac{1}{m}}|\chi_x v_x|\left(v^{\frac{p}{m}} + v^{\frac{r}{m}}u^{\frac{r}{m}+1} + |(v^{\frac{1}{m}})_t| + v^{-1}|v_x|^2\right) \\ & + C\int |\chi_{xx}|\left(u^{1-\frac{1}{m}}|u_x|^2 + v^{1-\frac{1}{m}}|v_x|^2\right) + C\int |\chi_t|\left(u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}\right) \\ & + C\int \chi\left(u^{1-\frac{1}{m}}|(u^{\frac{1}{m}})_t|^2 + v^{1-\frac{1}{m}}|(v^{\frac{1}{m}})_t|^2\right). \end{aligned} \tag{2.2}$$

Proof Define

$$I_1 = \int \chi u^{-1-\frac{1}{m}} |u_x|^4, I_2 = \int \chi v^{-1-\frac{1}{m}} |v_x|^4$$

and $J = a_{21}J_1 + a_{12}J_2$ with

$$J_1 = \int \chi u^{-\frac{1}{m}} |u_x|^2 u_{xx}, J_2 = \int \chi v^{-\frac{1}{m}} |v_x|^2 v_{xx}.$$

Then, we have

$$I = a_{21}I_1 + a_{12}I_2.$$

Applying [1, Lemma 4.1] with $q = 1 - \frac{1}{m}$, $N = 1$ and $k \in \mathbb{R}$, we obtain

$$\begin{aligned}\left(-\frac{k}{m} + \frac{m-1}{2m^2}\right) I_1 &+ 3\left(k - \frac{m-1}{2m}\right) J_1 \\ &\leq \frac{1}{2}\int u^{1-\frac{1}{m}} |u_x|^2 \chi_{xx} + \int u^{1-\frac{1}{m}} \left(u_{xx} + \left(\frac{m-1}{m} - k\right) u^{-1}|u_x|^2\right) u_x \chi_x\end{aligned}$$

and

$$\begin{aligned}\left(-\frac{k}{m} + \frac{m-1}{2m^2}\right) I_2 &+ 3\left(k - \frac{m-1}{2m}\right) J_2 \\ &\leq \frac{1}{2}\int v^{1-\frac{1}{m}} |v_x|^2 \chi_{xx} + \int v^{1-\frac{1}{m}} \left(v_{xx} + \left(\frac{m-1}{m} - k\right) v^{-1}|v_x|^2\right) v_x \chi_x.\end{aligned}$$

Multiplying the first inequality by a_{21}, the second one by a_{12}, we deduce that

$$\begin{aligned}\left(-\frac{k}{m} + \frac{m-1}{2m^2}\right) I + 3\left(k - \frac{m-1}{2m}\right) J &\leq C\int \left(u^{1-\frac{1}{m}}|u_x|^2 + v^{1-\frac{1}{m}}|v_x|^2\right)|\chi_{xx}| \\ &+ C\int u^{1-\frac{1}{m}} \left|\left(u_{xx} + \left(\frac{m-1}{m} - k\right) u^{-1}|u_x|^2\right) u_x\chi_x\right| \\ &+ C\int v^{1-\frac{1}{m}} \left|\left(v_{xx} + \left(\frac{m-1}{m} - k\right) v^{-1}|v_x|^2\right) v_x\chi_x\right|.\end{aligned} \tag{2.3}$$

Since (u, v) is a positive regular solution of (2.1), we have

$$\begin{aligned}-J =& a_{21}\int \chi u^{-\frac{1}{m}} |u_x|^2(-u_{xx}) + a_{12}\int \chi v^{-\frac{1}{m}} |v_x|^2(-v_{xx}) \\ =& a_{21}\int \chi a_{11} u^{\frac{p-1}{m}} |u_x|^2 + a_{12}\int \chi a_{22} v^{\frac{p-1}{m}} |v_x|^2 \\ &+ a_{12}a_{21}\left(\int \chi|u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi|v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1}\right) \\ &- a_{21}\int \chi u^{-\frac{1}{m}} |u_x|^2 (u^{\frac{1}{m}})_t - a_{12}\int \chi v^{-\frac{1}{m}} |v_x|^2 (v^{\frac{1}{m}})_t.\end{aligned} \tag{2.4}$$

It follows from an integration by parts that

$$\begin{aligned}\int \chi|u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} &= \frac{m}{r+m-1}\int \chi v^{\frac{r}{m}+1} u_x \left(u^{\frac{r+m-1}{m}}\right)_x \\ &= -\frac{m}{r+m-1}\int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} - \frac{r+m}{r+m-1}\int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}} u_x v_x \\ &- \frac{m}{r+m-1}\int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} \chi_x u_x\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\int \chi|v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} &= \frac{m}{r+m-1}\int \chi u^{\frac{r}{m}+1} v_x \left(v^{\frac{r+m-1}{m}}\right)_x \\ &= -\frac{m}{r+m-1}\int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx} - \frac{r+m}{r+m-1}\int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}} v_x u_x \\ &- \frac{m}{r+m-1}\int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} \chi_x v_x.\end{aligned} \tag{2.6}$$

By combining (2.5) and (2.6), we arrive at

$$\begin{aligned}&\int \chi|u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi|v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} \\ &= -\frac{m}{r+m-1}\int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} - \frac{m}{r+m-1}\int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx} \\ &- \frac{r+m}{r+m-1}\int \chi \left(u^{\frac{r+m-1}{m}} v^{\frac{r}{m}} + v^{\frac{r+m-1}{m}} u^{\frac{r}{m}}\right) u_x v_x \\ &- \frac{m}{r+m-1}\int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1}\chi_x u_x - \frac{m}{r+m-1}\int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1}\chi_x v_x.\end{aligned} \tag{2.7}$$

On the other hand, by using the Young inequality and some elementary computations, it holds

$$2(u^{\frac{r+m-1}{m}} v^{\frac{r}{m}} + v^{\frac{r+m-1}{m}} u^{\frac{r}{m}}) u_x v_x \leq |u_x|^2 (u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + u^{\frac{p-1}{m}}) + |v_x|^2 (v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} + v^{\frac{p-1}{m}}).$$

This together with (2.7) implies that

$$\begin{aligned}&\left(1+\frac{r+m}{2(r+m-1)}\right)\left(\int \chi|u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi|v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1}\right) \\ &+ \frac{r+m}{2(r+m-1)}\left(\int \chi(|u_x|^2 u^{\frac{p-1}{m}} + |v_x|^2 v^{\frac{p-1}{m}}\right) \\ &\geq -\frac{m}{r+m-1}\int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} - \frac{m}{(r+m-1)}\int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx} \\ &- \frac{m}{r+m-1}\int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1}\chi_x u_x - \frac{m}{r+m-1}\int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1}\chi_x v_x.\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int \chi|u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi|v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} \\
&+ \frac{r+m}{3(r+m)-2}\left(\int \chi(|u_x|^2 u^{\frac{p-1}{m}} + |v_x|^2 v^{\frac{p-1}{m}}\right) \\
&\geq -\frac{2m}{3(r+m)-2}\left(\int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} + \int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx}\right) \\
&- \frac{2m}{3(r+m)-2}\left(\int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} \chi_x u_x + \int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} \chi_x v_x\right).
\end{aligned} \tag{2.8}$$

We next use an integration by parts to get

$$\begin{aligned}
\int \chi u^{\frac{p-1}{m}} |u_x|^2 &= \frac{m}{p+m-1} \int \chi u_x \left(u^{\frac{p-1}{m}+1}\right)_x \\
&= -\frac{m}{p+m-1} \int \chi u^{\frac{p-1}{m}+1} u_{xx} - \frac{m}{p+m-1} \int u^{\frac{p-1}{m}+1} \chi_x u_x
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\int \chi v^{\frac{p-1}{m}} |v_x|^2 &= \frac{m}{p+m-1} \int \chi v_x \left(v^{\frac{p-1}{m}+1}\right)_x \\
&= -\frac{m}{p+m-1} \int \chi v^{\frac{p-1}{m}+1} v_{xx} - \frac{m}{p+m-1} \int v^{\frac{p-1}{m}+1} \chi_x v_x.
\end{aligned} \tag{2.10}$$

By taking into account (2.4), (2.8), (2.9), (2.10) and the fact that $\frac{m}{p+m-1} < \frac{r+m}{3(r+m)-2}$, we have

$$\begin{aligned}
&-\left(1 + \left(\frac{a_{12}a_{21}}{a_{11}} + \frac{a_{12}a_{21}}{a_{22}}\right)\frac{r+m}{3(r+m)-2}\right) J \\
&\geq \frac{r+m}{3(m+r)-2} a_{21} \int \chi(a_{11} u^{\frac{p-1}{m}+1} + a_{12} u^{1+\frac{r-1}{m}} v^{\frac{r}{m}+1})(-u_{xx}) \\
&+ \frac{r+m}{3(m+r)-2} a_{12} \int \chi(a_{21} v^{1+\frac{r-1}{m}} u^{\frac{r}{m}+1} + a_{22} v^{\frac{p-1}{m}+1})(-v_{xx}) \\
&- C \int \left(\chi|(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 + \chi|(v^{\frac{1}{m}})_t| v^{-\frac{1}{m}} |v_x|^2\right) \\
&- C \int \left((u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1} v^{\frac{r}{m}+1})|\chi_x u_x| + (v^{\frac{p-1}{m}+1} + v^{\frac{r-1}{m}+1} u^{\frac{r}{m}+1})|\chi_x v_x|\right).
\end{aligned} \tag{2.11}$$

In (2.11) we use

$$\begin{cases} -u_{xx} = a_{11} u^{\frac{p}{m}} + a_{12} u^{\frac{r}{m}} v^{\frac{r}{m}+1} - (u^{\frac{1}{m}})_t \\ -v_{xx} = a_{21} v^{\frac{r}{m}} u^{\frac{r}{m}+1} + a_{22} v^{\frac{p}{m}} - (v^{\frac{1}{m}})_t \end{cases}$$

and an integrating by parts in t to obtain

$$\begin{aligned}
-J \geq\ & \varepsilon \left(a_{21} \int \chi u^{1-\frac{1}{m}} (a_{11} u^{\frac{p}{m}} + a_{12} u^{\frac{r}{m}} v^{\frac{r}{m}+1})^2 + a_{12} \int \chi v^{1-\frac{1}{m}} (a_{21} v^{\frac{r}{m}} u^{\frac{r}{m}+1} + a_{22} v^{\frac{p}{m}})^2 \right) \\
& - C \int \left(\chi |(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 + \chi |(v^{\frac{1}{m}})_t| v^{-\frac{1}{m}} |v_x|^2 \right) \\
& - C \int \left((u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1} v^{\frac{r}{m}+1}) |\chi_x u_x| + (v^{\frac{p-1}{m}+1} + v^{\frac{r-1}{m}+1} u^{\frac{r}{m}+1}) |\chi_x v_x| \right) \\
& - C \int |\chi_t| (u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}) \qquad\qquad (2.12) \\
& - C \int \chi (u^{1+\frac{r-1}{m}} v^{\frac{r}{m}+1} (u^{\frac{1}{m}})_t + v^{1+\frac{r-1}{m}} u^{\frac{r}{m}+1} (v^{\frac{1}{m}})_t).
\end{aligned}$$

Here $\varepsilon > 0$ is some small positive constant and is independent of u, v and χ. Applying an integration by parts, the last term in (2.12) becomes

$$\int \chi (u^{1+\frac{r-1}{m}} v^{\frac{r}{m}+1} (u^{\frac{1}{m}})_t + v^{1+\frac{r-1}{m}} u^{\frac{r}{m}+1} (v^{\frac{1}{m}})_t) = -\frac{1}{r+1} \int \chi_t u^{\frac{r}{m}+1} v^{\frac{r}{m}+1}.$$

This equality combined with (2.12) and the Young inequality $u^{\frac{r}{m}+1} v^{\frac{r}{m}+1} \leq C(u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1})$ yield

$$\begin{aligned}
-J \geq\ & \varepsilon L - C \int \left(\chi |(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 + \chi |(v^{\frac{1}{m}})_t| v^{-\frac{1}{m}} |v_x|^2 \right) \\
& - C \int \left((u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1} v^{\frac{r}{m}+1}) |\chi_x . u_x| + (v^{\frac{p-1}{m}+1} + v^{\frac{r-1}{m}+1} u^{\frac{r}{m}+1}) |\chi_x v_x| \right) \\
& - C \int |\chi_t| (u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}). \qquad\qquad (2.13)
\end{aligned}$$

By plugging (2.13) into (2.3) and choosing k such that $k - \frac{m-1}{2m} < 0$, we obtain (2.2). Lemma 2.1 is proved. □

Lemma 2.2 *In addition to* (1.7), *assume that* $a_{12}, a_{21} > 0$. *Let* (u, v) *be a nonnegative weak solution of* (2.1) *in* $(-1,1) \times (-1,1)$. *Then, there exists* $C > 0$ *independent of* u, v *such that*

$$\int_{(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{2},\frac{1}{2})} \left(u^{\frac{2p+m-1}{m}} + v^{\frac{2p+m-1}{m}} \right) dxdt \leq C. \qquad (2.14)$$

Proof The proof is based on the idea in [1]. We first assume that (u, v) is a positive regular solution of the system (2.1) in $(-1,1) \times (-1,1)$.

Let $\xi \in C_0^\infty((-1,1) \times (-1,1))$ be a test function such that $\xi = 1$ in $(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ and $0 \leq \xi \leq 1$. Put $\chi = \xi^{\frac{2(2p+m-1)}{p-m}}$. Then, it is easy to see that

$$\begin{aligned}
|\chi_x| &\leq C \chi^{\frac{1}{2}}, \\
|\chi_{xx}| &\leq C \chi^{\frac{p+2m-1}{2p+m-1}}, \\
|\chi_t| &\leq C \chi^{\frac{p+2m-1}{2p+m-1}}.
\end{aligned}$$

We use again the notation $\int$ which stands for $\int_{(-1,1)\times(-1,1)} dxdt$ for simplicity. From [1, Formula (4.11)], given any constant $\eta > 0$ and any positive function $w \in C^{2,1}((-1,1)\times(-1,1))$, we have

$$\int w^{1-\frac{1}{m}}|w_x|^2\left(|\chi_{xx}| + \chi^{-1}|\chi_x|^2 + |\chi_t|\right) \le \eta \int \chi\left(w^{-1-\frac{1}{m}}|w_x|^4 + w^{\frac{2p-1}{m}+1}\right) + C(\eta). \tag{2.15}$$

Here, $C(\eta)$ is a positive constant depending on η. Using (2.15) and the Young inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}$, $\varepsilon > 0$, we control the terms corresponding to u on the right hand side of (2.2) as follows

$$\int \chi|(u^{\frac{1}{m}})_t|u^{-\frac{1}{m}}|u_x|^2 \le \varepsilon \int \chi u^{-1-\frac{1}{m}}|u_x|^4 + \frac{1}{4\varepsilon}\int \chi u^{1-\frac{1}{m}}|(u^{\frac{1}{m}})_t|^2,$$

$$\begin{aligned}\int |\chi_x u_x|(u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1}v^{\frac{r}{m}+1}) &\le \varepsilon \int \chi u^{1-\frac{1}{m}}(u^{\frac{p}{m}} + u^{\frac{r}{m}}v^{\frac{r}{m}+1})^2 + \frac{1}{4\varepsilon}\int u^{1-\frac{1}{m}}\chi^{-1}|\chi_x u_x|^2 \\ &\le \varepsilon \int \chi u^{1-\frac{1}{m}}(u^{\frac{p}{m}} + u^{\frac{r}{m}}v^{\frac{r}{m}+1})^2 + \varepsilon \int \chi\big(u^{-1-\frac{1}{m}}|u_x|^4 + u^{\frac{2p-1}{m}+1}\big) + C(\varepsilon),\end{aligned}$$

$$\begin{aligned}\int |\chi_x u_x|u^{-\frac{1}{m}}|u_x|^2 &\le \varepsilon \int \chi u^{-1-\frac{1}{m}}|u_x|^4 + \frac{1}{4\varepsilon}\int u^{1-\frac{1}{m}}\chi^{-1}|\chi_x u_x|^2 \\ &\le \varepsilon \int \chi u^{-1-\frac{1}{m}}|u_x|^4 + \varepsilon \int \chi\big(u^{-1-\frac{1}{m}}|u_x|^4 + u^{\frac{2p-1}{m}+1}\big) + C(\varepsilon),\end{aligned}$$

$$\begin{aligned}\int u^{1-\frac{1}{m}}|\chi_x u_x||(u^{\frac{1}{m}})_t| &\le \int u^{1-\frac{1}{m}}\chi|(u^{\frac{1}{m}})_t|^2 + \frac{1}{4}\int u^{1-\frac{1}{m}}\chi^{-1}|\chi_x u_x|^2 \\ &\le \int u^{1-\frac{1}{m}}\chi|(u^{\frac{1}{m}})_t|^2 + \varepsilon \int \chi\big(u^{-1-\frac{1}{m}}|u_x|^4 + u^{\frac{2p-1}{m}+1}\big) + C(\varepsilon), \\ \int u^{1-\frac{1}{m}}|\chi_{xx}||u_x|^2 &\le \varepsilon \int \chi\big(u^{-1-\frac{1}{m}}|u_x|^4 + u^{\frac{2p-1}{m}+1}\big) + C(\varepsilon), \\ \int |\chi_t|u^{\frac{p}{m}+1} &\le \varepsilon \int \chi u^{\frac{2p-1}{m}+1} + C(\varepsilon)\int \chi^{-\frac{p+m}{p-1}}|\chi_t|^{\frac{2p+m-1}{p-1}} \\ &\le \varepsilon \int \chi u^{\frac{2p-1}{m}+1} + C(\varepsilon).\end{aligned} \tag{2.16}$$

Similarly, we obtain the estimates corresponding to v. Combining these estimates and Lemma 2.1, we arrive at

$$\begin{aligned}I + L \le& \varepsilon C\int \chi u^{1-\frac{1}{m}}\left((u^{\frac{p}{m}} + u^{\frac{r}{m}}v^{\frac{r}{m}+1})^2 + (v^{\frac{p}{m}} + v^{\frac{r}{m}}u^{\frac{r}{m}+1})^2 + u^{-2}|u_x|^4 + v^{-2}|v_x|^4\right) \\ &+ C\left(1+\frac{1}{\varepsilon}\right)\int \chi\left(u^{1-\frac{1}{m}}|(u^{\frac{1}{m}})_t|^2 + v^{1-\frac{1}{m}}|(v^{\frac{1}{m}})_t|^2\right) + C(\varepsilon).\end{aligned}$$

This inequality and the assumption that $a_{ij} > 0$ for all $i,j = 1,2$ give

$$I + L \le \varepsilon C(I+L) + C\left(1+\frac{1}{\varepsilon}\right)\int \chi\left(u^{1-\frac{1}{m}}|(u^{\frac{1}{m}})_t|^2 + v^{1-\frac{1}{m}}|(v^{\frac{1}{m}})_t|^2\right) + C(\varepsilon). \tag{2.17}$$

We next have

$$\begin{aligned}
&\int \chi\left(a_{21}u^{1-\frac{1}{m}}|(u^{\frac{1}{m}})_t|^2 + a_{12}v^{1-\frac{1}{m}}|(v^{\frac{1}{m}})_t|^2\right)\\
=&\int \chi a_{21}u^{1-\frac{1}{m}}(u^{\frac{1}{m}})_t\left(u_{xx} + a_{11}u^{\frac{p}{m}} + a_{12}u^{\frac{r}{m}}v^{\frac{r}{m}+1}\right)\\
&+\int \chi a_{12}v^{1-\frac{1}{m}}(v^{\frac{1}{m}})_t(v_{xx} + a_{22}v^{\frac{p}{m}} + a_{21}u^{\frac{r}{m}+1}v^{\frac{r}{m}})\\
=&\frac{1}{m}\int \chi a_{21}u_t(u_{xx} + a_{11}u^{\frac{p}{m}} + a_{12}u^{\frac{r}{m}}v^{\frac{r}{m}+1}) + \frac{1}{m}\int \chi a_{12}v_t(v_{xx} + a_{22}v^{\frac{p}{m}} + a_{21}u^{\frac{r}{m}+1}v^{\frac{r}{m}})\\
=&-\frac{1}{2m}\int \chi\partial_t\left(a_{21}|u_x|^2 + a_{12}|v_x|^2\right) + \frac{1}{p+m}\int \chi\partial_t\left(a_{21}a_{11}u^{\frac{p}{m}+1} + a_{12}a_{22}v^{\frac{p}{m}+1}\right)\\
&+\frac{a_{21}a_{12}}{r+m}\int \chi\partial_t(u^{\frac{r}{m}+1}v^{\frac{r}{m}+1}) - \frac{1}{m}\int (a_{21}u_t\chi_x u_x + a_{12}v_t\chi_x v_x)\,.
\end{aligned}$$

Integrating by parts in t and using again the Young inequality, we arrive at

$$\begin{aligned}
&\int \chi\left(a_{21}u^{1-\frac{1}{m}}|(u^{\frac{1}{m}})_t|^2 + a_{12}v^{1-\frac{1}{m}}|(v^{\frac{1}{m}})_t|^2\right)\\
=&\frac{1}{2m}\int \chi_t\left(a_{21}|u_x|^2 + a_{12}|v_x|^2\right) - \frac{1}{p+m}\int \chi_t\left(a_{21}a_{11}u^{\frac{p}{m}+1} + a_{12}a_{22}v^{\frac{p}{m}+1}\right)\\
&-\frac{a_{12}a_{21}}{r+m}\int \chi_t u^{\frac{r}{m}+1}v^{\frac{r}{m}+1} - \frac{1}{m}\int (a_{21}u_t\chi_x u_x + a_{12}v_t\chi_x v_x)\\
\le& C\int |\chi_t|\left(|u_x|^2 + |v_x|^2 + u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}\right) + \frac{1}{2}\int \chi\left(a_{21}u^{1-\frac{1}{m}}(u^{\frac{1}{m}})_t^2 + a_{12}v^{1-\frac{1}{m}}(v^{\frac{1}{m}})_t^2\right)\\
&+C\int \chi^{-1}|\chi_x|^2(u^{1-\frac{1}{m}}|u_x|^2 + v^{1-\frac{1}{m}}|v_x|^2).
\end{aligned}$$

Consequently, this estimate yields

$$\begin{aligned}
\int \chi\left(a_{21}u^{1-\frac{1}{m}}(u^{\frac{1}{m}})_t^2 + a_{12}v^{1-\frac{1}{m}}(v^{\frac{1}{m}})_t^2\right) \le& C\int |\chi_t|\left(|u_x|^2 + |v_x|^2 + u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}\right)\\
&+C\int \chi^{-1}|\chi_x|^2(u^{1-\frac{1}{m}}|u_x|^2 + v^{1-\frac{1}{m}}|v_x|^2). \qquad (2.18)
\end{aligned}$$

Using $a_{12}, a_{21} > 0$, for any $\eta > 0$, we deduce from (2.15), (2.16), (2.18) and the Young inequality that

$$\int \chi\left(a_{21}u^{1-\frac{1}{m}}(u^{\frac{1}{m}})_t^2 + a_{12}v^{1-\frac{1}{m}}(v^{\frac{1}{m}})_t^2\right) \le C\eta(I+L) + C(\eta). \qquad (2.19)$$

Therefore, it follows from (2.17) and (2.19) that

$$I + L \le C\varepsilon(I+L) + C(\varepsilon) + C\Big(1 + \frac{1}{\varepsilon}\Big)\Big(C\eta(I+L) + C(\eta)\Big).$$

By taking $\eta = \varepsilon^2$ and choosing ε sufficiently small, we obtain $I + L \le C$ and this ends proof of Lemma for positive regular solutions.

We now consider the case where (u, v) is a nonnegative weak solutions of the system in $(-1,1)\times(-1,1)$. By using the similar argument as in [33, Proposition 2.4], there is a sequence $(u_k, v_k)_k$ of positive regular solutions of the system in $(-1,1)\times(-1,1)$ such that $(u_k, v_k) \to (u, v)$ locally uniformly in $(-1,1)\times(-1,1)$. By combining the above argument and the Fatou lemma, we get

$$\int_{(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{2},\frac{1}{2})} (u^{\frac{2p+m-1}{m}} + v^{\frac{2p+m-1}{m}})dxdt \leq \lim_{k\to+\infty}\inf \int_{(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{2},\frac{1}{2})} (u_k^{\frac{2p+m-1}{m}} + v_k^{\frac{2p+m-1}{m}})dxdt \leq C.$$

Lemma 2.2 is proved. □

With the above preparation at hand, we are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 is followed from Theorem A if $a_{12} = 0$ or $a_{21} = 0$ since the system is reduced to scalar equation. We then assume that $a_{12} > 0$, $a_{21} > 0$. Suppose that (u, v) is a nonnegative weak solution of (2.1) in $\mathbb{R}\times\mathbb{R}$. We are going to prove that $u \equiv v \equiv 0$.

For any $R > 0$, by the scaling invariance of the system (2.1), we rescale

$$u_R(x,t) = R^{\frac{2m}{p-m}} u\left(Rx, R^{\frac{2(p-1)}{p-m}} t\right), \quad v_R(x,t) = R^{\frac{2m}{p-m}} v\left(Rx, R^{\frac{2(p-1)}{p-m}} t\right).$$

Then, it is not difficult to check that (u_R, v_R) is also a nonnegative weak solution to (2.1). By Lemma 2.2, we have

$$\begin{aligned}
&\int_{|y|<\frac{1}{2}R}\int_{|s|<\frac{1}{2}R^{\frac{2(p-1)}{p-m}}} \left(u^{\frac{2p+m-1}{m}} + v^{\frac{2p+m-1}{m}}\right)(y,s)dyds \\
&\qquad = R^{1+\frac{2(p-1)}{p-m}-\frac{2(2p+m-1)}{p-m}} \int_{|x|<\frac{1}{2}}\int_{|t|<\frac{1}{2}} \left(u_R^{\frac{2p+m-1}{m}} + v_R^{\frac{2p+m-1}{m}}\right)(x,t)dxdt \\
&\qquad \leq CR^{1+\frac{2(p-1)}{p-m}-\frac{2(2p+m-1)}{p-m}} = CR^{-\frac{p+3m}{p-m}}.
\end{aligned}$$

Letting $R \to \infty$ and noting that the exponent on the right hand side is negative, we deduce that $u \equiv v \equiv 0$. The proof is complete.

Acknowledgment

This research was supported by Vietnam National Foundation for Science and Technology Development NAFOSTED (101.02-2021.33).

References

[1] Ammar K, Souplet P. Liouville-type theorems and universal bounds for nonnegative solutions of the porous medium equation with source. Discrete and Continuous Dynamical Systems. Series A 2010; 2: 665-689. https://doi.org/10.3934/dcds.2010.26.665

[2] Baras P, Pierre, M. Critère d'existence de solutions positives pour des équations semi-linéaires non monotones. Annales de l'Institut Henri Poincaré. Analyse Non Linéaire 1985; 2 (3): 185-212.

[3] Bartsch T, Poláčik P, Quittner P. Liouville-type theorems and asymptotic behavior of nodal radial solutions of semilinear heat equations. Journal of the European Mathematical Society 2011; 13(1): 219-247. https://doi.org/10.4171/JEMS/250

[4] Bidaut-Véron MF. Initial blow-up for the solutions of a semilinear parabolic equation with source term. In Équations aux dérivées partielles et applications Gauthier-Villars 1998; 189-198.

[5] Cui Z, Yang Z. Boundedness of global solutions for a nonlinear degenerate parabolic (porous medium) system with localized sources. Applied Mathematics and Computation 2008; 198 (2): 882-895. https://doi.org/10.1016/j.amc.2007.09.037

[6] DiBenedetto E. Continuity of weak solutions to a general porous medium equation. Indiana University Mathematics Journal 1983; 32 (1): 83-118. doi: 10.1512/iumj.1983.32.32008

[7] Duong AT, Phan QH. A Liouville-type theorem for cooperative parabolic systems. Discrete and Continuous Dynamical Systems. Series A 2018; 38 (2): 823-833. https://doi.org/10.3934/dcds.2018035

[8] Duong AT, Phan QH. Optimal Liouville-type theorems for a system of parabolic inequalities. Communications in Contemporary Mathematics 2020; 26 (2): 1950043. https://doi.org/10.1142/S0219199719500433

[9] Földes J. Liouville theorems, a priori estimates, and blow-up rates for solutions of indefinite superlinear parabolic problems. Czechoslovak Mathematical Journal 2011; 61 (136): 169-198. https://doi.org/10.1007/s10587-011-0005-2

[10] Fujita H. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. Journal of the Faculty of Science. University of Tokyo. Section I. Mathematics 1966; 13: 109-124.

[11] Galaktionov VA. A boundary value problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta}$. Differentsialńye Uravneniya 1981; 17 (5): 836-842.

[12] Galaktionov VA. Best possible upper bound for blowup solutions of the quasilinear heat conduction equation with source. SIAM Journal on Mathematical Analysis 1991; 22 (5): 1293-1302. https://doi.org/10.1137/0522083

[13] Galaktionov VA. Blow-up for quasilinear heat equations with critical Fujita's exponents. Proceedings of the Royal Society of Edinburgh. Section A. Mathematics 1994; 124 (3): 517-525. doi: 10.1017/S0308210500028766

[14] Gidas B, Spruck J. Global and local behavior of positive solutions of nonlinear elliptic equations. Communications on Pure and Applied Mathematics 1981; 34 (4): 525-598. https://doi.org/10.1002/cpa.3160340406

[15] Kurta VV. A Liouville comparison principle for solutions of semilinear parabolic inequalities in the whole space. Advances in Nonlinear Analysis 2014; 3 (2): 125-131. https://doi.org/10.1515/anona-2014-0011

[16] Kurta VV. A Liouville comparison principle for solutions of quasilinear singular parabolic inequalities. Advances in Nonlinear Analysis 2015; 4 (1): 1-11. https://doi.org/10.1515/anona-2014-0026

[17] Li Y, Gao W, Han Y. Boundedness of global solutions for a porous medium system with moving localized sources. Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal 2010; 72 (6): 3080-3090. doi: 10.1016/j.na.2009.11.047

[18] Lu H. Global solutions and blow-up problems to a porous medium system with nonlocal sources and nonlocal boundary conditions. Mathematical Methods in the Applied Sciences 2011; 34 (15): 1933-1944. https://doi.org/10.1002/mma.1504

[19] Mitidieri E. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$. Differential and Integral Equations. An International Journal for Theory & Applications 1996; 9 (3): 465-479.

[20] Mitidieri E, Pohozaev SI. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. Trudy Matematicheskogo Instituta Imeni V. A. Steklova. Rossiĭskaya Akademiya Nauk 2001; 234: 1-384.

[21] Phan QH. Optimal Liouville-type theorems for a parabolic system. Discrete and Continuous Dynamical Systems. Series A 2015; 35 (1): 399-409.

[22] Phan QH, Souplet P. A Liouville-type theorem for the 3-dimensional parabolic Gross–Pitaevskii and related systems. Mathematische Annalen 2016; 366 (3-4): 1561-1585. https://doi.org/10.1007/s00208-016-1368-3

[23] Pinsky RG. Existence and nonexistence of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbf{R}^d$. Journal of Differential Equations 1997; 113 (1): 152-177. https://doi.org/10.1006/jdeq.1996.3196

[24] Poláčik P, Quittner P. A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation. Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods 2006; 64 (8): 1679-1689. https://doi.org/10.1016/j.na.2005.07.016

[25] Poláčik P, Quittner P, Souplet P. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. Duke Mathematical Journal 2007; 139 (3): 555-579. https://doi.org/10.1215/S0012-7094-07-13935-8

[26] Poláčik P, Quittner P, Souplet P. Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations. Indiana University Mathematics Journal 2007; 56 (2): 879-908. https://doi.org/10.1512/iumj.2007.56.2911

[27] Quittner P. Liouville theorems for scaling invariant superlinear parabolic problems with gradient structure. Mathematische Annalen 2016; 364 (1-2): 269-292. https://doi.org/10.1007/s00208-015-1219-7

[28] Quittner P. Optimal Liouville theorems for superlinear parabolic problems. Duke Mathematical Journal 2021; 170 (6): 1113-1136. https://doi.org/10.1215/00127094-2020-0096

[29] Quittner P, Souplet P. Superlinear parabolic problems: Blow-up, global existence and steady states. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007.

[30] Samarskii AA, Galaktionov VA, Kurdyumov SP, Mikhailov AP. Blow-up in quasilinear parabolic equations, vol. 19 of De Gruyter Expositions in Mathematics. Walter de Gruyter & Co., Berlin, 1995.

[31] Serrin J, Zou H. Non-existence of positive solutions of Lane-Emden systems. Differential Integral Equations 1996; 9 (4): 635-653.

[32] Serrin J, Zou H. Existence of positive solutions of the Lane-Emden system. Atti del Seminario Matematico e Fisico dell'Università di Modena 1998; 46: 369-380.

[33] Souplet P. An optimal Liouville-type theorem for radial entire solutions of the porous medium equation with source. Journal of Differential Equations 2009; 246 (10): 3980-4005. https://doi.org/10.1016/j.jde.2008.10.018

[34] Souplet P. The proof of the Lane-Emden conjecture in four space dimensions. Advances in Mathematics 2009; 221 (5): 1409-1427. https://doi.org/10.1016/j.aim.2009.02.014

[35] Vázquez JL. The porous medium equation: Mathematical theory. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

[36] Ye Z, Xu X. Global existence and blow-up for a porous medium system with nonlocal boundary conditions and nonlocal sources. Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal 2013; 82: 115-126. https://doi.org/10.1016/j.na.2013.01.004