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Free ordered products-ordered semigroup amalgams-ordered dominions

(*dedicated to the memory of Prof. Niovi Kehayopulu*)

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Abstract: Given an indexed family $\{(S_i, \cdot_i, \leq_i), i \in I\}$ of disjoint ordered semigroups, we construct an ordered semigroup having $(S_i, \cdot_i, \leq_i)$, $i \in I$ as subsemigroups (with respect to the operation and order relation of each $(S_i, \cdot_i, \leq_i)$, $i \in I$). This ordered semigroup is the free ordered product $\prod\limits_{i\in I}{}^{*}S_i$ of the family $\{S_i, i \in I\}$ and we give the crucial property which essentially characterizes the free products. Next we study the same problem in the case that the family $\{(S_i, \cdot_i, \leq_i), i \in I\}$ of ordered semigroups has as intersection the ordered semigroup $(U, \cdot_U, \leq_U)$ which is a subsemigroup of $(S_i, \cdot_i, \leq_i)$ for every $i \in I$ (with respect to the operation and order relation of each $(S_i, \cdot_i, \leq_i)$, $i \in I$). To do this, we first consider the ordered semigroup amalgam $\mathfrak{A} = [\{(S_i, \cdot_i, \leq_i), i \in I\}; (U, \cdot_U, \leq_U); \{\varphi_i : U \to S_i, i \in I\}]$ (where $\{\varphi_i : U \to S_i, i \in I\}$ is a family of monomorphisms) and then we construct the free ordered product $\prod\limits_{i\in I}{}^{*}_{U} S_i$ of the ordered semigroup amalgam $\mathfrak{A}$ considering the ordered quotient of the free ordered product $\prod\limits_{i\in I}{}^{*}S_i$ by an appropriate pseudoorder of $\prod\limits_{i\in I}{}^{*}S_i$ through which for each $i, j \in I$ and for each $u \in U$, $\varphi_i(u) \in S_i$ is identified (by means of monomorphisms) with $\varphi_j(u) \in S_j$. We give a sufficient and necessary condition so that an ordered semigroup amalgam is embedded in an ordered semigroup. At the end of the paper, we introduce the notion of ordered dominions. An element d of an ordered semigroup S is dominated by a subsemigroup U of S if for all ordered semigroups $(T, \cdot, \leq)$ and for all homomorphisms $\beta, \gamma : S \to T$ such that $\beta(u) = \gamma(u)$ for each $u \in U$, we have $[\beta(d))_{\leq}^{T} \cap [\gamma(d))_{\leq}^{T} \neq \varnothing$. In the last Theorem of the paper, we give an expression of the set of elements of S dominated by U based on ordered semigroup amalgams.

Key words: Ordered semigroup, pseudoorder on an ordered semigroup, ordered quotient of an ordered semigroup by a pseudoorder, free ordered product of ordered semigroups, ordered semigroup amalgam, free ordered product of an ordered semigroup amalgam, ordered dominions

1. Introduction

A semigroup amalgam $\mathfrak{A}$ may conveniently be thought of as a family $\{(S_i, \cdot_i), i \in I\}$ of semigroups intersecting pairwise in a common subsemigroup $(U, \cdot_U)$ (i.e. $S_i \cap S_j = U$ for $i, j \in I$ with $i \neq j$ and $\cdot_U = \cdot_i \cap (U \times U)$ for every $i \in I$). In general the set $\bigcup\limits_{i\in I} S_i$ is not a semigroup with respect to the operations of S_i $(i \in I)$. The

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2010 *AMS Mathematics Subject Classification:* 06F05 (20M10) main question concerning a semigroup amalgam is whether or not the set $\bigcup_{i\in I} S_i$ is embeddable in a semigroup $(T, \cdot_T)$ with respect to the operations of S_i, $i \in I$, that is, whether or not there exists an one-one mapping $f : \bigcup_{i\in I} S_i \to T$ with the property that for each $i \in I$ and for all $\alpha, b \in S_i$, $f(\alpha \cdot_i b) = f(\alpha) \cdot_T f(b)$. We can answer the above question considering the free product of the semigroup amalgam $\mathfrak{A}$ (see [2, §9.4] and [3, §VII.1]). The aim of the paper is to study "similar concepts" in the case of ordered semigroups, also taking into account the order relation of S_i, $i \in I$. We follow the same steps as in the case of semigroups, that is, if

$$\mathfrak{A} = [\{(S_i, \cdot_i, \leq_i), i \in I\}; (U, \cdot_U, \leq_U); \{\varphi_i : U \to S_i, i \in I\}]$$

(where $\{\varphi_i : U \to S_i, i \in I\}$ is a family of monomorphisms) is an ordered semigroup amalgam, we first construct the free ordered product $\prod_{i\in I}{}^* S_i$ of the family $\{S_i, i \in I\}$ and then the free ordered product $\prod_{U, i\in I}^* S_i$ of the ordered semigroup amalgam $\mathfrak{A}$ which is the ordered quotient of $\prod_{i\in I}{}^* S_i$ by the pseudoorder of $\prod_{i\in I}{}^* S_i$ generated by the set of those words of $\prod_{i\in I}{}^* S_i$ that need to be "identified". In particular, in $\prod_{U, i\in I}^* S_i$ for each $i, j \in I$ the image $\varphi_i(u)$ of an element $u \in U$ in S_i is identified (by means of monomorphisms) with its image $\varphi_j(u)$ in S_j. Then, based on the previous identification, we construct the new equality relation and order relation of $\prod_{i\in I}{}^* S_i$. We prove that an ordered semigroup amalgam is embeddable in an ordered semigroup if and only if it is naturally embedded in its free ordered product. The concepts of free ordered product and of free ordered product of an ordered semigroup amalgam was introduced by the author in his Doctoral Dissertation (cf. [9]). As an application of ordered semigroup amalgams, we introduce the notion of ordered dominion which is "analogous" to the notion of dominion of semigroups in the case of ordered semigroups. Following Isbell [4], if U is a subsemigroup of a semigroup S and $d \in S$, we say that U dominates d if for all semigroups T and for all homomorphisms $\beta, \gamma : S \to T$ such that $\beta(u) = \gamma(u)$ for each $u \in U$, we have $\beta(d) = \gamma(d)$. The set of elements of S dominated by U is called the dominion of U in S. According to Howie [3, §VII.2] there is a connection between dominions and semigroup amalgams. In the last chapter of the paper we study "similar concepts" in the case of ordered semigroups, first defining the notion of ordered dominion and then characterizing ordered dominions in some more accessible way (by means of ordered semigroup amalgams). The results provided by the paper generalize the analogous ones concerning semigroups without order because a semigroup without order can be considered as an ordered semigroup with order relation being its equality relation and so one can easily obtain the corresponding results on semigroups without order from those presented in the paper.

2. Prerequisites

Let X be a nonempty set and $\mathfrak{B}(X)$ be the set of all binary relations on X. If $\rho, \sigma \in \mathfrak{B}(X)$ we define

$$\rho \circ \sigma := \{(\alpha, b) \in X \times X : \ (\exists c \in X) \ \ (\alpha, c) \in \rho \text{ and } (c, b) \in \sigma\}$$

It is well known that $(\mathfrak{B}(X), \circ)$ is a semigroup (see [1, §1.4] and [3, §I.4]). If $R \in \mathfrak{B}(X)$ and $n \in \mathbb{N}$ we write R^n instead of $\underset{\leftarrow \text{ n times } \rightarrow}{R \circ R \circ ... \circ R}$ and we define $R^\infty := \bigcup_{n=1}^{\infty} R^n$. Then (see [1, §1.4] and [3, §I.4])) R^∞ is the smallest transitive relation on X containing R (called transitive closure). An equivalence relation ρ on a semigroup $(S, \cdot)$ is called congruence if for α, b, c$\in S$, $(\alpha, b) \in \rho$ implies $(\alpha \cdot c, b \cdot c) \in \rho$ and $(c \cdot \alpha, c \cdot b) \in \rho$ (see [1, §1.5] and [3, §1.5]). We denote by $(x)_\rho$, $x \in S$, the ρ-class containing x, i.e.

$$(x)_\rho := \{y \in S / (x, y) \in \rho\}$$

For a congruence ρ on S, the quotient set $S/_\rho$ is the set of all ρ-classes and is a semigroup endowed with the operation "$*$" [3, §I.5]

$$(\alpha)_\rho * (b)_\rho = (\alpha \cdot b)_\rho, \; \alpha, b \in S$$

If $(S, \cdot)$ is a semigroup and the "symbol" 1 does not belong to S, then $\left(S^1, \odot\right)$ is the semigroup obtained from S by adjoining an identity (see [1, §1.1], [3, §I.1] and [7]), that is $S^1 := S \cup \{1\}$ and

$$\alpha \odot b := \begin{cases} \alpha \cdot b, & \alpha, b \in S \\ \alpha, & b = 1 \\ b, & \alpha = 1 \end{cases}, \quad \alpha, b \in S^1$$

Clearly 1 is the identity of S^1. For any $\alpha, b \in S^1$ it is customary to write αb instead of $\alpha \odot b$ (just as we do for $\alpha \cdot b$, $\alpha, b \in S$). An ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation "$\leq$" which is compatible with the operation "$\cdot$" (i.e. for α, b, c $\in S$, $\alpha \leq b$ implies $\alpha \cdot c \leq b \cdot c$ and $c \cdot \alpha \leq c \cdot b$). For a nonempty subset A of S we define

$$[A)_\leq^S := \{x \in S / (\exists \alpha \in A)\, \alpha \leq x\} \text{ and } (A]_\leq^S := \{x \in S / (\exists \alpha \in A)\, x \leq \alpha\}$$

For $\alpha \in S$ we usually write $[\alpha)_\leq^S$ instead of $[\{\alpha\})_\leq^S$ and $(\alpha]_\leq^S$ instead of $(\{\alpha\}]_\leq^S$, that is

$$[\alpha)_\leq^S = \{x \in S / \alpha \leq x\} \text{ and } (\alpha]_\leq^S = \{x \in S / x \leq \alpha\}$$

A mapping $f : (S, \cdot, \leq) \to (T, \centerdot, \preceq)$ between two ordered semigroups is (see [6, 10])

- a <u>homomorphism</u> if
  - o $f(\alpha \cdot b) = f(\alpha) \centerdot f(b)$, α, b $\in S$
  - o $f(\alpha) \preceq f(b)$, α, b $\in S$ such that $\alpha \leq b$.
- <u>reverse isotone</u> (or <u>antitone</u>) if for α, b $\in S$, $f(\alpha) \preceq f(b)$ implies $\alpha \leq b$.
- a <u>monomorphism</u> if it is a reverse isotone homomorphism.
- an <u>isomorphism</u> if it is a monomorphism and onto (in this case we write $S \simeq T$).

It is easy to see that

- a reverse isotone mapping is one-one (in general the reverse statement doesn't hold).
- the composition of homomorphisms (resp. monomorphisms, isomorphisms) is a homomorphism (resp. monomorphism, isomorphism).

Proposition 2.1 *Let $(S, \cdot, \leq)$, $(T, \centerdot, \preceq)$ be ordered semigroups and $f : S \to T$, $g : T \to S$ be homomorphisms such that $f \circ g = 1_T$ and $g \circ f = 1_S$ (where 1_T, 1_S are the identity mappings on T, S respectively). Then $S \simeq T$.*

Proof We shall prove that f is an isomorphism. Since f is a homomorphism, it suffices to show that f is a reverse isotone and onto mapping.

- Let $\alpha, b \in S$ such that $f(\alpha) \preceq f(b)$. Since g is a homomorphism, then $g(f(\alpha)) \leq g(f(b))$. Thus (since $g \circ f = 1_S$) $\alpha \leq b$.
- Let $b \in T$. Then for $\alpha := g(b) \in S$ we have $f(\alpha) = f(g(b)) \underset{(f \circ g = 1_T)}{=} b$

$\square$

Definition 2.2 *([5, 6, 10]) Let $(S, \cdot, \leq)$ be an ordered semigroup and $\sigma \subseteq S \times S$ (: σ is a binary relation on S). The relation σ is called pseudoorder if*

i) $\leq \ \subseteq \sigma$

ii) σ *is transitive*

iii) for $x, y, z \in \sigma$ such that $(x, y) \in \sigma$ we have $(xz, yz) \in \sigma$ and $(zx, zy) \in \sigma$.

Now let $(S, \cdot, \leq)$ be an ordered semigroup and σ be a pseudoorder on S. Then we define (see [5, 6, 10])

$$\bar{\sigma} := \sigma \cap \sigma^{-1}$$

where $\sigma^{-1} := \{(\alpha, b) \in S \times S / (b, \alpha) \in \sigma\}$, that is,

$$\bar{\sigma} := \{(\alpha, b) \in S \times S / (\alpha, b) \in \sigma, (b, \alpha) \in \sigma\}$$

The relation $\bar{\sigma}$ is a congruence on S (see [5, 6, 10] – given a congruence ρ on an ordered semigroup S, it has been constructed [8] by the author and N. Kehayopulu a pseudoorder σ on S such that $\rho \subseteq \bar{\sigma}$ using the quasi-chains modulo ρ). We can define an order relation "$\leq_\sigma$" on the quotient semigroup ${}^{S}/_{\bar{\sigma}}$ as follows (see [5, 6, 10]):

$$(\alpha)_{\bar{\sigma}} \leq_\sigma (b)_{\bar{\sigma}} \Leftrightarrow (x, y) \in \sigma \text{ for some } x \in (\alpha)_{\bar{\sigma}},\ y \in (b)_{\bar{\sigma}}$$

It can be proved that

$$(\alpha)_{\bar{\sigma}} \leq_\sigma (b)_{\bar{\sigma}} \text{ if and only if } (x, y) \in \sigma \text{ for any } x \in (\alpha)_{\bar{\sigma}},\ y \in (b)_{\bar{\sigma}}$$

(see [5, 6, 10]). Therefore

$$(\alpha)_{\bar{\sigma}} \leq_\sigma (b)_{\bar{\sigma}} \text{ if and only if } (\alpha, b) \in \sigma$$

Then $\left({}^{S}/_{\bar{\sigma}}, *, \leq_\sigma\right)$ is an ordered semigroup (called ordered quotient of S by σ) and the mapping $\sigma^{\#} : (S, \cdot, \leq) \rightarrow \left({}^{S}/_{\bar{\sigma}}, *, \leq_\sigma\right)$ defined by

$$\sigma^{\#}(x) = (x)_{\bar{\sigma}},\ x \in S$$

is an onto homomorphism called natural homomorphism (see [5, 6, 10]). For any relation R on S we define (see [3, §I.5] and [10])

$$R^c := \left\{(x\alpha y, xby) \in S \times S : x, y \in S^1, (\alpha, b) \in R\right\}$$

Proposition 2.3 *Let $(S, \cdot, \leq)$ be an ordered semigroup and $R \subseteq S \times S$. Then*

i) $(R^c \cup \leq)^\infty$ *is a pseudoorder of S.*

ii) $R \subseteq (R^c \cup \leq)^\infty$

iii) If T is a pseudoorder of S containing R then $(R^c \cup \leq)^\infty \subseteq T$.

Proof

i) It is evident that $\leq \;\subseteq\; (R^c \cup \leq)^\infty$ and $(R^c \cup \leq)^\infty$ is transitive. Since for $\alpha, b, c \in S$ with $(\alpha, b) \in (R^c \cup \leq)$ we clearly have $(\alpha c, bc), (c\alpha, cb) \in (R^c \cup \leq)$ then for $\alpha, b, c \in S$, $(\alpha, b) \in (R^c \cup \leq)^\infty$ implies $(\alpha c, bc), (c\alpha, cb) \in (R^c \cup \leq)^\infty$. So $(R^c \cup \leq)^\infty$ is a pseudoorder of S.

ii) It follows immediately from $R \subseteq R^c \subseteq (R^c \cup \leq) \subseteq (R^c \cup \leq)^\infty$.

iii) First we prove that $(R^c \cup \leq) \subseteq T$. For $(\alpha, b) \in (R^c \cup \leq)$ we have $(\alpha, b) \in R^c$ or $\alpha \leq b$.

α) If $(\alpha, b) \in R^c$ then there exist $x, y \in S^1$ and $(c, d) \in R$ such that $\alpha = xcy$ and $b = xdy$. Since $R \subseteq T$ we have $(c, d) \in T$ which implies that $(xcy, xdy) \in T$ (T is a pseudoorder on S). Therefore $(\alpha, b) \in T$.

β) If $\alpha \leq b$ then, since T is a pseudoorder on S, $(\alpha, b) \in T$.

Consequently $(R^c \cup \leq) \subseteq T$ and so T is a transitive relation on S containing $(R^c \cup \leq)$ and hence, since $(R^c \cup \leq)^\infty$ is the smallest transitive relation on S containing $(R^c \cup \leq)$, it follows that $(R^c \cup \leq)^\infty \subseteq T$.

□

From the above Proposition, $(R^c \cup \leq)^\infty$ is the smallest pseudoorder on S containing R (called pseudoorder on S generated by R) denoted by ${\sigma_R}^S$ (if no confusion arises, we usually simplify the notation to σ_R).
Let now S, T be ordered semigroups and $f : S \to T$ be a homomorphism. We define (see [5, 6, 10])

$$\underset{\sim}{f} := \{(\alpha, b) \in S \times S : f(\alpha) \leq f(b)\}$$

and

$$\ker f := \{(\alpha, b) \in S \times S : f(\alpha) = f(b)\}$$

Then (see [5, 6, 10])

- $\underset{\sim}{f}$ is a pseudoorder on S
- $\ker f = \overline{\underset{\sim}{f}}$ $\left(= \underset{\sim}{f} \cap \underset{\sim}{f}^{-1}\right)$

Theorem 2.4 *Let S, T be ordered semigroups, $f : S \to T$ be a homomorphism and σ be a pseudoorder on S such that $\sigma \subseteq \underset{\sim}{f}$. Then there exists a homomorphism $\eta : {}^{S}/_{\bar{\sigma}} \to T$ such that $\eta \circ \sigma^{\#} = f$ and $f(S) = \eta\left({}^{S}/_{\bar{\sigma}}\right)$.*

Proof Define $\eta : {}^{S}/_{\bar{\sigma}} \to T$ by

$$\eta\left((\alpha)_{\bar{\sigma}}\right) := f(\alpha),\ \alpha \in S$$

Since $\sigma \subseteq \underset{\sim}{f}$ we immediately have $\bar{\sigma} \subseteq \ker f$ and hence η is well-defined. Also for all $\alpha, b \in S$

$$\eta\left((\alpha)_{\bar{\sigma}} * (b)_{\bar{\sigma}}\right) = \eta\left((\alpha b)_{\bar{\sigma}}\right) = f(\alpha b) = f(\alpha) f(b) = \eta\left((\alpha)_{\bar{\sigma}}\right) \eta\left((b)_{\bar{\sigma}}\right)$$

So in order η to be a homomorphism, it suffices to show that $\eta\left((\alpha)_{\bar{\sigma}}\right) \leq \eta\left((b)_{\bar{\sigma}}\right)$ for $(\alpha)_{\bar{\sigma}} \leq_{\sigma} (b)_{\bar{\sigma}}$:
Let $\alpha, b \in S$ such that $(\alpha)_{\bar{\sigma}} \leq_{\sigma} (b)_{\bar{\sigma}}$. Then $(\alpha, b) \in \sigma \subseteq \underset{\sim}{f}$. Thus $f(\alpha) \leq f(b)$ and hence $\eta\left((\alpha)_{\bar{\sigma}}\right) \leq \eta\left((b)_{\bar{\sigma}}\right)$.
Since for any $\alpha \in S$ we have

$$\left(\eta \circ \sigma^{\#}\right)(\alpha) = \eta\left(\sigma^{\#}(\alpha)\right) = \eta\left((\alpha)_{\bar{\sigma}}\right) = f(\alpha)$$

it is clear that $\eta \circ \sigma^{\#} = f$ and $f(S) = \eta\left({}^{S}/_{\bar{\sigma}}\right)$. □

Remark If $\gamma : {}^{S}/_{\bar{\sigma}} \to T$ is a homomorphism such that $\gamma \circ \sigma^{\#} = f$ then for every $\alpha \in S$ we have

$$\gamma\left((\alpha)_{\bar{\sigma}}\right) = \gamma\left(\sigma^{\#}(\alpha)\right) = f(\alpha) = \eta\left((\alpha)_{\bar{\sigma}}\right)$$

and hence $\gamma = \eta$. So there is a unique homomorphism $\gamma : {}^{S}/_{\bar{\sigma}} \to T$ such that $\gamma \circ \sigma^{\#} = f$ (this is η).

3. Free ordered products

Let $\{(S_i, \cdot_i, \leq_i), i \in I\}$ be an indexed family of disjoint ordered semigroups. Let $S := \bigcup\limits_{i \in I} S_i$. Then for any $\alpha \in S$ we denote $\sigma(\alpha)$ the unique $\kappa \in I$ with the property that $\alpha \in S_\kappa$. If no confusion arises, we shall not use index for the order relation on S_i, $i \in I$ (i.e. we shall just write $\leq$ instead of $\leq_i$) and also we shall not use any symbol for the operation of S_i, $i \in I$ (i.e. we shall just write $\alpha\mathrm{b}$ instead of $\alpha \cdot_i b$ for all $\alpha, b \in S_i$).
Now let $n \in \mathbb{N}$.

- If $n = 1$ we denote $F_1 := \{(\alpha) : \alpha \in S\}$
- If $n \geq 2$ we denote

$$F_n := \left\{(\alpha_1, \alpha_2, ..., \alpha_n) \in \underset{\leftarrow\, n \text{ times} \,\rightarrow}{S \times S \times ... \times S} : \sigma(\alpha_\lambda) \neq \sigma(\alpha_{\lambda+1}),\ \lambda = 1, ..., \mathrm{n} - 1\right\}$$

We define $\prod\limits_{i \in I} {}^{*}S_i := \bigcup\limits_{n \in \mathbb{N}} F_n$. The set $\prod\limits_{i \in I} {}^{*}S_i$

- consists of all finite strings

  $$(\alpha_1, \alpha_2, ..., \alpha_n) \in \underbrace{S \times S \times ... \times S}_{n}$$

  where if $n \geq 2$ then $\sigma(\alpha_\lambda) \neq \sigma(\alpha_{\lambda+1})$, $\lambda = 1, ..., n-1$, and
- is a semigroup with operation "$\bullet$" defined by the rule that

$$(\alpha_1,...,\alpha_m)\bullet(b_1,...,b_r):=\begin{cases}(\alpha_1,...,\alpha_m,b_1,...,b_r)\,, & if\ \sigma(\alpha_m)\neq\sigma(b_1)\\ (\alpha_1,...,\alpha_m b_1,...,b_r)\ \ , & if\ \sigma(\alpha_m)=\sigma(b_1)\end{cases}$$

(see [2, §9.4] and [3, §VII.1])

It is evident that $\prod\limits_{i\in I}{}^*S_i$ is generated by the strings of length one (i.e. by (α), $\alpha\in S_i$, $i\in I$). On $\prod\limits_{i\in I}{}^*S_i$ we define a binary relation "$\preceq$" as follows

$$(\alpha_1,...,\alpha_m)\preceq(b_1,...,b_r)\Leftrightarrow\left\{\begin{matrix}m=r=q\,(say)\\ \sigma(\alpha_j)=\sigma(b_j)\,,\ j=1,...,q\\ \alpha_j\leq b_j,\ j=1,...,q\end{matrix}\right\}$$

Theorem 3.1 $\left(\prod\limits_{i\in I}{}^*S_i,\bullet,\preceq\right)$ *is an ordered semigroup.*

Proof It suffices to show that "$\preceq$" is an order relation on S with respect to the operation $''\bullet''$. It is clear that "$\preceq$" is an order relation on S. Now let $\mathbf{a}=(\alpha_1,...,\alpha_m)$, $\mathbf{b}=(b_1,...,b_r)$ and $\mathbf{c}=(c_1,...,c_n)$ be elements of $\prod\limits_{i\in I}{}^*S_i$ with $\mathbf{a}\preceq\mathbf{b}$. We shall show that $\mathbf{a}\bullet\mathbf{c}\preceq\mathbf{b}\bullet\mathbf{c}$ and $\mathbf{c}\bullet\mathbf{a}\preceq\mathbf{c}\bullet\mathbf{b}$. Since $a\preceq b$ then

$$\left\{\begin{matrix}m=r=q\,(say)\\ \sigma(\alpha_j)=\sigma(b_j)\,,\ j=1,...,q\\ \alpha_j\leq b_j,\ j=1,...,q\end{matrix}\right\}$$

Hence $\mathbf{a}=(\alpha_1,...,\alpha_q)$ and $\mathbf{b}=(b_1,...,b_q)$. We distinguish two cases:

i) $\underline{\sigma(\alpha_q)\neq\sigma(c_1)}$

Then $\mathbf{a}\bullet\mathbf{c}=(\alpha_1,...,\alpha_q)\bullet(c_1,...,c_n)=(\alpha_1,...,\alpha_q,c_1,...,c_n)$. Since $\sigma(\alpha_q)=\sigma(b_q)$ we have $\sigma(b_q)\neq\sigma(c_1)$ and hence $\mathbf{b}\bullet\mathbf{c}=(b_1,...,b_q)\bullet(c_1,...,c_n)=(b_1,...,b_q,c_1,...,c_n)$. Since $\sigma(\alpha_j)=\sigma(b_j)$, $\alpha_j\leq b_j$, $j=1,...,q$ we immediately have $\mathbf{a}\bullet\mathbf{c}\preceq\mathbf{b}\bullet\mathbf{c}$.

ii) $\underline{\sigma(\alpha_q)=\sigma(c_1)}$

Then $\mathbf{a}\bullet\mathbf{c}=(\alpha_1,...,\alpha_q)\bullet(c_1,...,c_n)=(\alpha_1,...,\alpha_q c_1,...,c_n)$. Since $\sigma(\alpha_q)=\sigma(b_q)$ we have $\sigma(b_q)=\sigma(c_1)$ and hence $b\bullet c=(b_1,...,b_q)\bullet(c_1,...,c_n)=(b_1,...,b_q c_1,...,c_n)$. Since $\sigma(\alpha_q c_1)=\sigma(\alpha_q)=\sigma(b_q)=\sigma(b_q c_1)$ we have $\sigma(\alpha_q c_1)=\sigma(b_q c_1)$. Also since $\sigma(\alpha_q)=\sigma(b_q)=\sigma(c_1)=k$ (say) we have $\alpha_q,b_q,c_1\in S_k$ and so $\alpha_q\leq b_q$ implies $\alpha_q c_1\leq b_q c_1$. Thus, since $\sigma(\alpha_j)=\sigma(b_j)$, $\alpha_j\leq b_j$, $j=1,...,q-1$, we immediately have $\mathbf{a}\bullet\mathbf{c}\preceq\mathbf{b}\bullet\mathbf{c}$.

Similarly we show that $\mathbf{c}\bullet\mathbf{a}\preceq\mathbf{c}\bullet\mathbf{b}$. □

The ordered semigroup $\left(\prod\limits_{i\in I}{}^*S_i,\bullet,\preceq\right)$ is called *free ordered product* of the family $\{(S_i,\cdot_i,\leq_i)\,,i\in I\}$. For every $j\in I$ we consider the mapping

$$\theta_j:S_j\to\prod_{i\in I}{}^*S_i\,/\ \alpha\to(\alpha)$$

It is evident that θ_j is a monomorphism for any $j\in I$.

Proposition 3.2 *Let* $(T, \circ, \leq_T)$ *be an ordered semigroup and* $\{\psi_i : S_i \to T, i \in I\}$ *be a family of homomorphisms. Then there exists a unique homomorphism* $\gamma : \prod_{i \in I}{}^{*} S_i \to T$ *such that* $\gamma \circ \theta_j = \psi_j$ *for each* $j \in I$*.*

Proof We define $\gamma : \prod_{i \in I}{}^{*} S_i \to T$ by

$$\gamma\left((\alpha_1, \alpha_2, ..., \alpha_m)\right) := \left(\psi_{\sigma(\alpha_1)}(\alpha_1)\right) \circ \left(\psi_{\sigma(\alpha_2)}(\alpha_2)\right) \circ ... \circ \left(\psi_{\sigma(\alpha_m)}(\alpha_m)\right)$$

i) Let $(\alpha_1, ..., \alpha_m), (b_1, ..., b_r) \in \prod_{i \in I}{}^{*} S_i$. Then

$$\gamma\left((\alpha_1, ..., \alpha_m) \bullet (b_1, ..., b_r)\right) = \gamma\left((\alpha_1, ..., \alpha_m)\right) \circ \gamma\left((b_1, ..., b_r)\right)$$

(see [2, §9.4] and [3, §VII.1]).

ii) Let $(\alpha_1, ..., \alpha_m), (b_1, ..., b_r) \in \prod_{i \in I}{}^{*} S_i$ with $(\alpha_1, ..., \alpha_m) \preceq (b_1, ..., b_r)$. Then

$$\left\{ \begin{array}{c} m = r = q \ (say) \\ \sigma(\alpha_p) = \sigma(b_p), \ p = 1, ..., q \\ \alpha_p \leq b_p, \ p = 1, ..., q \end{array} \right\}$$

Since ψ_i is a homomorphism for every $i \in I$ and $\sigma(\alpha_p) = \sigma(b_p)$, $p = 1, ..., q$, then $\alpha_p \leq b_p$ implies $\psi_{\sigma(\alpha_p)}(\alpha_p) \leq_T \psi_{\sigma(b_p)}(b_p)$, $p = 1, ..., q$. Therefore ($(T, \circ, \leq_T)$ is an ordered semigroup)

$$\left(\psi_{\sigma(\alpha_1)}(\alpha_1)\right) \circ \left(\psi_{\sigma(\alpha_2)}(\alpha_2)\right) \circ ... \circ \left(\psi_{\sigma(\alpha_q)}(\alpha_q)\right) \leq_T \left(\psi_{\sigma(b_1)}(b_1)\right) \circ \left(\psi_{\sigma(b_2)}(b_2)\right) \circ ... \circ \left(\psi_{\sigma(b_q)}(b_q)\right)$$

and hence $(m = r = q)$ we have

$$\gamma\left((\alpha_1, \alpha_2, ..., \alpha_m)\right) \leq_T \gamma\left((b_1, b_2, ..., b_r)\right)$$

Consequently, γ is a homomorphism.
Now for $j \in I$ and $\alpha \in S_j$ we have

$$\gamma\left(\theta_j(\alpha)\right) = \gamma\left((\alpha)\right) = \psi_{\sigma(\alpha)}(\alpha)$$

But $\sigma(\alpha) = j$ and so $\psi_{\sigma(\alpha)}(\alpha) = \psi_j(\alpha)$. Thus $\gamma(\theta_j(\alpha)) = \psi_j(\alpha)$. Therefore $\gamma \circ \theta_j = \psi_j$ for every $j \in I$.
The uniqueness of γ follows immediately from the fact that $\prod_{i \in I}{}^{*} S_i$ is generated by strings of length one (see [2, §9.4] and [3, §VII.1]). □

Proposition 3.3 *Let* H *be an ordered semigroup and* $\{\varphi_i : S_i \to H, i \in I\}$ *be a family of monomorphisms having the following property (say (+)):*

«For any ordered semigroup T *and for any family of homomorphisms* $\{\beta_i : S_i \to T, i \in I\}$ *there exists a unique homomorphism* $\delta : H \to T$ *such that* $\delta \circ \varphi_j = \beta_j$ *for each* $j \in I$*»*

Then $\prod_{i \in I}{}^{*} S_i \simeq H$*.*

Proof

From the given property $(+)$ for $T = \prod_{i\in I}{}^*S_i$ and the family of homomorphisms $\left\{\theta_j : S_j \to \prod_{i\in I}{}^*S_i, j \in I\right\}$ it follows that there exists a unique homomorphism $\delta : H \to \prod_{i\in I}{}^*S_i$ such that $\delta \circ \varphi_i = \theta_i$ for every $i \in I$. From Proposition 3.2 for $T = H$ and the family of homomorphisms $\{\varphi_i : S_i \to H, i \in I\}$ it follows that there exists a unique homomorphism $\gamma : \prod_{i\in I}{}^*S_i \to H$ such that $\gamma \circ \theta_i = \varphi_i$ for every $i \in I$. Then $\gamma \circ \delta : H \to H$ is a homomorphism such that

$$\varphi_i = \gamma \circ \theta_i = \gamma \circ (\delta \circ \varphi_i) = (\gamma \circ \delta) \circ \varphi_i$$

for any $i \in I$.

Now from the given property $(+)$ for $T = H$ and the family of homomorphisms $\{\varphi_i : S_i \to H, i \in I\}$ it follows that there exists a unique homomorphism $\delta' : H \to H$ such that $\delta' \circ \varphi_i = \varphi_i$ for every $i \in I$. Since clearly the identity homomorphism $1_H : H \to H$ has this property, we immediately have $\gamma \circ \delta = \delta' = 1_H$. So $\gamma \circ \delta = 1_H$. Also $\delta \circ \gamma : \prod_{i\in I}{}^*S_i \to \prod_{i\in I}{}^*S_i$ is a homomorphism such that

$$\theta_j = \delta \circ \varphi_j = \delta \circ (\gamma \circ \theta_j) = (\delta \circ \gamma) \circ \theta_j$$

for any $j \in I$.

From Proposition 3.2 for $T = \prod_{i\in I}{}^*S_i$ and the family of homomorphisms $\left\{\theta_j : S_j \to \prod_{i\in I}{}^*S_i, j \in I\right\}$ it follows that there exists a unique homomorphism $\gamma' : \prod_{i\in I}{}^*S_i \to \prod_{i\in I}{}^*S_i$ such that $\gamma' \circ \theta_j = \theta_j$ for every $j \in I$. Since clearly the identity homomorphism $1_{\prod_{i\in I}{}^*S_i} : \prod_{i\in I}{}^*S_i \to \prod_{i\in I}{}^*S_i$ has this property, we immediately have $\delta \circ \gamma = \gamma' = 1_{\prod_{i\in I}{}^*S_i}$. So $\delta \circ \gamma = 1_{\prod_{i\in I}{}^*S_i}$. From this and the fact that $\gamma \circ \delta = 1_H$ (shown above), we now immediately deduce that $\prod_{i\in I}{}^*S_i \simeq H$ as required. $\square$

4. Ordered semigroup amalgams

An *ordered semigroup amalgam*

$$\mathfrak{A} = [\{(S_i, \cdot_i, \leq_i), i \in I\}; (U, \cdot_U, \leq_U); \{\varphi_i : U \to S_i, i \in I\}]$$

consists of

- a family $\{(S_i, \cdot_i, \leq_i), i \in I\}$ of disjoint ordered semigroups
- an ordered semigroup $(U, \cdot_U, \leq_U)$ (called the *core* of the ordered semigroup amalgam) such that $S_i \cap U = \varnothing$ for each $i \in I$
- a family $\{\varphi_i : U \to S_i, i \in I\}$ of monomorphisms.

We simplify the notation $\mathfrak{A} = [\{(S_i, \cdot_i, \leq_i), i \in I\}; (U, \cdot_U, \leq_U); \{\varphi_i : U \to S_i, i \in I\}]$ to $\mathfrak{A} = [S_i; U; \varphi_i, i \in I]$ when the context allows. We shall say that the ordered semigroup amalgam $\mathfrak{A}$ is *embedded* in the ordered semigroup T if there exists a monomorphism $\lambda : U \to T$ and a family $\{\lambda_i : S_i \to T, i \in I\}$ of monomorphisms with the following properties:

- $\lambda_i \circ \varphi_i = \lambda$ for any $i \in I$
- $\lambda_i(S_i) \cap \lambda_j(S_j) = \lambda(U),\ i,j \in I$ such that $i \neq j$.

Let now

$$R = \{(a,b) \in F_1 \times F_1 : (\exists (u,i,j) \in U \times I \times I)\ \ a = \theta_i(\varphi_i(u)), b = \theta_j(\varphi_j(u))\}$$

It is evident that R is a binary relation on $\prod\limits_{i\in I}{}^*S_i$ and hence we can consider the pseudoorder on $\prod\limits_{i\in I}{}^*S_i$ generated by R, $\sigma_R = (R^c \cup \preceq)^\infty$ where

$$R^c := \left\{(xay, xby) \in \prod_{i\in I}{}^*S_i \times \prod_{i\in I}{}^*S_i : x, y \in \left(\prod_{i\in I}{}^*S_i\right)^1,\ \ (a,b) \in R\right\}$$

Then the ordered quotient of $\prod\limits_{i\in I}{}^*S_i$ by σ_R is called *free ordered product of the ordered semigroup amalgam* $\mathfrak{A}$ and denoted $\prod\limits_{i\in I}{}^*_U S_i$ (i.e. $\prod\limits_{i\in I}{}^*_U S_i = {}^{\prod_{i\in I}{}^*S_i}/_{\overline{\sigma_R}}$ where $\overline{\sigma_R} = \sigma_R \cap {\sigma_R}^{-1}$).

For each $i \in I$ we define $\mu_i = {\sigma_R}^\# \circ \theta_i$, where ${\sigma_R}^\#$ is the natural homomorphism

$${\sigma_R}^\# : \prod_{i\in I}{}^*S_i \to \prod_{i\in I}{}^*_U S_i,\ {\sigma_R}^\#((\alpha_1, \alpha_2, ..., \alpha_n)) = ((\alpha_1, \alpha_2, ..., \alpha_n))_{\overline{\sigma_R}}$$

It is clear that for every $i \in I$ the mapping μ_i is a homomorphism ($\mu_i : S_i \to \prod\limits_{i\in I}{}^*_U S_i,\ \mu_i(\alpha) = ((\alpha))_{\overline{\sigma_R}}$).

Proposition 4.1 *For any $i, j \in I$ we have $\mu_i \circ \varphi_i = \mu_j \circ \varphi_j$.*

Proof Let $u \in U$. Then

$$\mu_i(\varphi_i(u)) = {\sigma_R}^\#(\theta_i(\varphi_i(u))) = (\theta_i(\varphi_i(u)))_{\overline{\sigma_R}}$$

and similarly $\mu_j(\varphi_j(u)) = (\theta_j(\varphi_j(u)))_{\overline{\sigma_R}}$. Since

- $(\theta_i(\varphi_i(u)), \theta_j(\varphi_j(u))) \in R \subseteq \sigma_R$
- $(\theta_j(\varphi_j(u)), \theta_i(\varphi_i(u))) \in R \subseteq \sigma_R$

it follows immediately that $(\theta_i(\varphi_i(u)), \theta_j(\varphi_j(u))) \in \sigma_R \cap {\sigma_R}^{-1} = \overline{\sigma_R}$ and thus $(\theta_i(\varphi_i(u)))_{\overline{\sigma_R}} = (\theta_j(\varphi_j(u)))_{\overline{\sigma_R}}$. Consequently $\mu_i(\varphi_i(u)) = \mu_j(\varphi_j(u))$. □

We define $\mu = \mu_i \circ \varphi_i$, $i \in I$ (from the above Proposition, the definition of μ does not depend on the choice of $i \in I$). Clearly

- $\mu : U \to \prod\limits_{i\in I}{}^*_U S_i,\ \mu(u) = ((\varphi_i(u)))_{\overline{\sigma_R}}$
- μ is a homomorphism.

Proposition 4.2

i) Let $j \in I$ such that μ_j is reverse isotone. Then μ is reverse isotone.

ii) $\mu(U) \subseteq \mu_i(S_i)$ for all $i \in I$.

Proof

i) Since $\mu = \mu_j \circ \varphi_j$, μ_j is reverse isotone and φ_j is a monomorphism then clearly μ is reverse isotone.

ii) For every $i \in I$ we have $\mu(U) = \mu_i(\varphi_i(U)) \subseteq \mu_i(S_i)$.

□

Definition 4.3 *The ordered semigroup amalgam $\mathfrak{A}$ is naturally embedded in its free ordered product if*

- *μ_i is reverse isotone for all $i \in I$*
- *$\mu_i(S_i) \cap \mu_j(S_j) \subseteq \mu(U)$ for all $i, j \in I$ such that $i \neq j$.*

Proposition 4.4 *If the ordered semigroup amalgam $\mathfrak{A}$ is naturally embedded in its free ordered product then the ordered amalgam $\mathfrak{A}$ is embeddable in an ordered semigroup.*

Proof Since μ_i is reverse isotone for all $i \in I$ then clearly $\left\{\mu_i : S_i \to \Pi^*_{U\, i \in I} S_i, i \in I\right\}$ is a family of monomorphisms and also μ is a monomorphism. Moreover $\mu = \mu_i \circ \varphi_i$ for every $i \in I$. Now let $i, j \in I$ such that $i \neq j$. We shall prove that $\mu_i(S_i) \cap \mu_j(S_j) = \mu(U)$. Indeed:

- By Definition 4.3 we have $\mu_i(S_i) \cap \mu_j(S_j) \subseteq \mu(U)$.
- By Proposition 4.2 ii) it follows immediately that $\mu(U) \subseteq \mu_i(S_i) \cap \mu_j(S_j)$.

Thus $\mu_i(S_i) \cap \mu_j(S_j) = \mu(U)$ and hence the ordered amalgam $\mathfrak{A}$ is embedded in the ordered semigroup $\Pi^*_{U\, i \in I} S_i$.

□

Proposition 4.5 *Let Q be an ordered semigroup and $\{\nu_i : S_i \to Q, i \in I\}$ be a family of homomorphisms such that $\nu_i \circ \varphi_i = \nu_j \circ \varphi_j$ for all $i, j \in I$. Then there exists a unique homomorphism $\delta : \Pi^*_{U\, i \in I} S_i \to Q$ such that $\delta \circ \mu_i = \nu_i$ for each $i \in I$.*

Proof By Proposition 3.2 we obtain a unique homomorphism $\gamma : \prod_{i \in I} {}^* S_i \to Q$ such that $\gamma \circ \theta_i = \nu_i$ for every $i \in I$. Then for $i, j \in I$ and $u \in U$ we have

$$\gamma(\theta_i(\varphi_i(u))) = \nu_i(\varphi_i(u)) = \nu_j(\varphi_j(u)) = \gamma(\theta_j(\varphi_j(u)))$$

and hence $(\theta_i(\varphi_i(u)), \theta_j(\varphi_j(u))) \in \ker\gamma$ which clearly means that $R \subseteq \ker\gamma$. Since $\ker\gamma = \underset{\sim}{\gamma} \cap \underset{\sim}{\gamma}^{-1}$ where $\underset{\sim}{\gamma} := \left\{(a,b) \in \left(\prod_{i \in I} {}^* S_i\right) \times \left(\prod_{i \in I} {}^* S_i\right) : \gamma(a) \preceq \gamma(b)\right\}$, it follows immediately that $R \subseteq \underset{\sim}{\gamma}$. Thus $\underset{\sim}{\gamma}$ is a pseudoorder on $\prod_{i \in I} {}^* S_i$ containing R and so $\sigma_R \subseteq \underset{\sim}{\gamma}$. Then by Theorem 2.4 there exists a (unique) homomorphism $\delta : \Pi^*_{U\, i \in I} S_i \to Q$ such that $\delta \circ {\sigma_R}^\# = \gamma$. Consequently for each $i \in I$ and $\alpha \in S_i$ it follows that

$$\delta\left(\mu_i\left(\alpha\right)\right)=\delta\left(\left(\left(\alpha\right)\right)_{\overline{\sigma_R}}\right)=\delta\left(\pi_{\overline{\sigma_R}}\left(\left(\alpha\right)\right)\right)=\gamma\left(\left(\alpha\right)\right)=\gamma\left(\theta_i\left(\alpha\right)\right)=\nu_i\left(\alpha\right)$$

and hence $\delta\circ\mu_i=\nu_i$ for all $i\in I$.

Now let $\zeta:\prod\limits_{i\in I}{}_U^{*}S_i\rightarrow Q$ be a homomorphism such that $\zeta\circ\mu_i=\nu_i$ for each $i\in I$. We shall prove that $\zeta=\delta$.

Indeed:

For $(\alpha_1,\alpha_2,...,\alpha_n)\in\prod\limits_{i\in I}{}^{*}S_i$ $(n\in\mathbb{N})$ we have

$$\begin{aligned}\zeta\left(\left(\left(\alpha_1,\alpha_2,...,\alpha_n\right)\right)_{\overline{\sigma_R}}\right)&=\zeta\left(\left(\left(\alpha_1\right)\right)_{\overline{\sigma_R}}\left(\left(\alpha_2\right)\right)_{\overline{\sigma_R}}...\left(\left(\alpha_n\right)\right)_{\overline{\sigma_R}}\right)=\\&=\zeta\left(\left(\left(\alpha_1\right)\right)_{\overline{\sigma_R}}\right)\zeta\left(\left(\left(\alpha_2\right)\right)_{\overline{\sigma_R}}\right)...\zeta\left(\left(\left(\alpha_n\right)\right)_{\overline{\sigma_R}}\right)=\\&=\zeta\left(\mu_{\sigma(\alpha_1)}\left(\alpha_1\right)\right)\zeta\left(\mu_{\sigma(\alpha_2)}\left(\alpha_2\right)\right)...\zeta\left(\mu_{\sigma(\alpha_n)}\left(\alpha_n\right)\right)=\\&=\nu_{\sigma(\alpha_1)}\left(\alpha_1\right)\nu_{\sigma(\alpha_2)}\left(\alpha_2\right)...\nu_{\sigma(\alpha_n)}\left(\alpha_n\right)=\\&=\delta\left(\mu_{\sigma(\alpha_1)}\left(\alpha_1\right)\right)\delta\left(\mu_{\sigma(\alpha_2)}\left(\alpha_2\right)\right)...\delta\left(\mu_{\sigma(\alpha_n)}\left(\alpha_n\right)\right)=\\&=\delta\left(\left(\left(\alpha_1\right)\right)_{\overline{\sigma_R}}\right)\delta\left(\left(\left(\alpha_2\right)\right)_{\overline{\sigma_R}}\right)...\delta\left(\left(\left(\alpha_n\right)\right)_{\overline{\sigma_R}}\right)=\\&=\delta\left(\left(\left(\alpha_1\right)\right)_{\overline{\sigma_R}}\left(\left(\alpha_2\right)\right)_{\overline{\sigma_R}}...\left(\left(\alpha_n\right)\right)_{\overline{\sigma_R}}\right)=\delta\left(\left(\left(\alpha_1,\alpha_2,...,\alpha_n\right)\right)_{\overline{\sigma_R}}\right)\end{aligned}$$

and so the proof is complete. □

Theorem 4.6 *Let* $\mathfrak{A}=\left[\left\{\left(S_i,\cdot_i,\leq_i\right),i\in I\right\};\left(U,\cdot_U,\leq_U\right);\left\{\varphi_i:U\rightarrow S_i,i\in I\right\}\right]$ *be an ordered semigroup amalgam. The following are equivalent:*

i) $\mathfrak{A}$ *is embeddable in an ordered semigroup*

ii) $\mathfrak{A}$ *is naturally embedded in its free ordered product.*

Proof Since ii)⇒i) is obvious, it suffices to show only i)⇒ii). So suppose that $\mathfrak{A}$ is embeddable in an ordered semigroup $(T,\cdot_T,\leq_T)$, i.e. that there exists a monomorphism $\lambda:U\rightarrow T$ and a family $\{\lambda_i:S_i\rightarrow T,i\in I\}$ of monomorphisms such that $\lambda_i\circ\varphi_i=\lambda$ for every $i\in I$ and $\lambda_i\left(S_i\right)\cap\lambda_j\left(S_j\right)=\lambda\left(U\right)$ for all $i,j\in I$ with $i\neq j$. By Proposition 4.5 there exists a unique homomorphism $\delta:\prod\limits_{i\in I}{}_U^{*}S_i\rightarrow T$ such that $\delta\circ\mu_i=\lambda_i$ for each $i\in I$. Then for every $i\in I$, μ_i is reverse isotone since for $x,y\in S_i$ we have

$$\mu_i\left(x\right)\preceq_{\sigma_R}\mu_i\left(y\right)\Rightarrow\delta\left(\mu_i\left(x\right)\right)\leq_T\delta\left(\mu_i\left(y\right)\right)\Rightarrow\lambda_i\left(x\right)\leq_T\lambda_i\left(y\right)\Rightarrow x\preceq y$$

So the first condition of Definition 4.3 holds. For the second condition of Definition 4.3, suppose $i,j\in I$ such that $i\neq j$. We shall prove that $\mu_i\left(S_i\right)\cap\mu_j\left(S_j\right)\subseteq\mu\left(U\right)$. Indeed:

Let $z\in\mu_i\left(S_i\right)\cap\mu_j\left(S_j\right)$. Then there exist $\alpha\in S_i$ and $b\in S_j$ such that $z=\mu_i\left(\alpha\right)=\mu_j\left(b\right)$ and hence $\delta\left(z\right)=\delta\left(\mu_i\left(\alpha\right)\right)=\lambda_i\left(\alpha\right)\in\lambda_i\left(S_i\right)$. Similarly $\delta\left(z\right)\in\lambda_j\left(S_j\right)$ and so $\delta\left(z\right)\in\lambda_i\left(S_i\right)\cap\lambda_j\left(S_j\right)=\lambda\left(U\right)$. Therefore there exists $u\in U$ such that $\delta\left(z\right)=\lambda\left(u\right)$, i.e. $\lambda_i\left(\alpha\right)=\lambda_i\left(\varphi_i\left(u\right)\right)$. Since λ_i is a monomorphism (and hence one-one), it follows that $\alpha=\varphi_i\left(u\right)$. Thus

$$z=\mu_i\left(\alpha\right)=\mu_i\left(\varphi_i\left(u\right)\right)=\mu\left(u\right)\in\mu\left(U\right)$$

Consequently, we have shown that both the conditions of Definition 4.3 hold, and hence $\mathfrak{A}$ is naturally embedded in its free ordered product as required. □

5. Ordered dominions

Let S be an ordered semigroup and let U be a subsemigroup of S. We say that U *dominates* an element $d \in S$ if for all ordered semigroups $(T, \cdot, \leq)$ and for all homomorphisms $\beta, \gamma : S \to T$ such that $\beta(u) = \gamma(u)$ for each $u \in U$, we have

$$[\beta(d))^T_{\leq} \cap [\gamma(d))^T_{\leq} \neq \varnothing$$

The set of elements of S dominated by U is called the *ordered dominion* of U in S and is written $Dom_S(U)$.

Proposition 5.1 *Let U be a subsemigroup of an ordered semigroup S. Then*

i) $U \subseteq Dom_S(U) \subseteq S$

ii) $Dom_S(U)$ *is a subsemigroup of* S

Proof

i) Let $w \in U \subseteq S$, $(T, \cdot, \leq)$ be an ordered semigroup and $\beta, \gamma : S \to T$ be homomorphisms such that $\beta(u) = \gamma(u)$ for each $u \in U$. We will prove that $w \in Dom_S(U)$. For this it suffices to show that $[\beta(w))^T_{\leq} \cap [\gamma(w))^T_{\leq} \neq \varnothing$. Since $w \in U$ we have $\beta(w) = \gamma(w)$. Thus $\beta(w) \in [\beta(w))^T_{\leq} \cap [\gamma(w))^T_{\leq}$ and so $[\beta(w))^T_{\leq} \cap [\gamma(w))^T_{\leq} \neq \varnothing$.

ii) From i) it immediately follows that $\varnothing \neq Dom_S(U) \subseteq S$. Let now $d_1, d_2 \in Dom_S(U)$. We will prove that $d_1 d_2 \in Dom_S(U)$. To do this, take an ordered semigroup $(T, \cdot, \leq)$ and homomorphisms $\beta, \gamma : S \to T$ such that $\beta(u) = \gamma(u)$ for each $u \in U$. We need to show that $[\beta(d_1 d_2))^T_{\leq} \cap [\gamma(d_1 d_2))^T_{\leq} \neq \varnothing$. Since $d_1, d_2 \in Dom_S(U)$ then $[\beta(d_1))^T_{\leq} \cap [\gamma(d_1))^T_{\leq} \neq \varnothing$ and $[\beta(d_2))^T_{\leq} \cap [\gamma(d_2))^T_{\leq} \neq \varnothing$. Thus there exist $t_1, t_2 \in T$ such that $\beta(d_1) \leq t_1$, $\gamma(d_1) \leq t_1$, $\beta(d_2) \leq t_2$ and $\gamma(d_2) \leq t_2$. Since β, γ are homomorphisms and T is an ordered semigroup, it is clear that $\beta(d_1 d_2) \leq t_1 t_2$ and $\gamma(d_1 d_2) \leq t_1 t_2$. Hence

$$t_1 t_2 \in [\beta(d_1 d_2))^T_{\leq} \cap [\gamma(d_1 d_2))^T_{\leq}$$

and so $[\beta(d_1 d_2))^T_{\leq} \cap [\gamma(d_1 d_2))^T_{\leq} \neq \varnothing$.

$\square$

Let $(S, \cdot, \leq)$, $(S_i, \cdot_i, \leq_i)$, $i \in I$, U be ordered semigroups, $\alpha_i : (S, \cdot, \leq) \to (S_i, \cdot_i, \leq_i)$ be isomorphisms and $f : U \to S$ be a monomorphism. We consider the ordered semigroup amalgam

$$\mathfrak{A} = [\{S, S_i, i \in I\}; U; \{f, \alpha_i \circ f, i \in I\}]$$

and the free ordered product of $\mathfrak{A}$, say P. Denote "$\leq_P$" the order relation of P. Let also the associated (with the free ordered product P of $\mathfrak{A}$) homomorphisms

$$\mu : S \to P,\ \mu(s) = ((s))_{\overline{\sigma_R}}$$
$$\mu_i : S_i \to P,\ \mu_i(x) = ((x))_{\overline{\sigma_R}}$$

Proposition 5.2 *The homomorphisms μ, μ_i, $i \in I$ are reverse isotone.*

Proof For any $i, j \in I$ we define

$$\xi_{i,j} = \alpha_i \circ {\alpha_j}^{-1} : S_j \to S_i$$

Clearly $\xi_{i,j}$ is an isomorphism for all $i, j \in I$ and also $\xi_{i,i} = 1_{S_i}$ for every $i \in I$.
Let now $\kappa \in I$.

- $\alpha_\kappa \circ f = \alpha_\kappa \circ {\alpha_\lambda}^{-1} \circ \alpha_\lambda \circ f = \xi_{\kappa,\lambda} \circ (\alpha_\lambda \circ f)$, $\lambda \in I$
- $\xi_{\kappa,\lambda} \circ (\alpha_\lambda \circ f) = \alpha_\kappa \circ {\alpha_\lambda}^{-1} \circ \alpha_\lambda \circ f = \alpha_\kappa \circ f = \alpha_\kappa \circ {\alpha_\nu}^{-1} \circ \alpha_\nu \circ f = \xi_{\kappa,\nu} \circ (\alpha_\nu \circ f)$, $\nu, \lambda \in I$.

Then, by Proposition 4.5 (taking S_κ as Q with homomorphisms $\alpha_\kappa : S \to S_\kappa$, $\xi_{\kappa,\lambda} : S_\lambda \to S_\kappa$, $\lambda \in I$), there exists a homomorphism $\delta : P \to S_\kappa$ with the property

$$\delta \circ \mu = \alpha_\kappa, \ \delta \circ \mu_\lambda = \xi_{\kappa,\lambda}, \ \lambda \in I$$

A) $\underline{\mu \text{ is reverse isotone}}$

Let $s_1, s_2 \in S$ such that $\mu(s_1) \leq_P \mu(s_2)$. Then

$$\begin{aligned}
&\mu(s_1) \leq_P \mu(s_2) \Rightarrow && (\delta \ \textit{homomorphism})\\
&\delta(\mu(s_1)) \leq_\kappa \delta(\mu(s_2)) \Rightarrow && (\delta \circ \mu = \alpha_\kappa)\\
&\alpha_\kappa(s_1) \leq_\kappa \alpha_\kappa(s_2) \Rightarrow && (\alpha_\kappa \ \textit{reverse isotone})\\
&\alpha_\kappa(s_1) \leq_\kappa \alpha_\kappa(s_2) \Rightarrow && (\alpha_\kappa \ \textit{reverse isotone})\\
&s_1 \leq s_2
\end{aligned}$$

B) $\underline{\mu_\kappa \text{ is reverse isotone}}$

Let $x_1, x_2 \in S_\kappa$ such that $\mu_\kappa(x_1) \leq_P \mu_\kappa(x_2)$. Then

$$\begin{aligned}
&\mu_\kappa(x_1) \leq_P \mu_\kappa(x_2) \Rightarrow && (\delta \ \textit{homomorphism})\\
&\delta(\mu_\kappa(x_1)) \leq_\kappa \delta(\mu_\kappa(x_2)) \Rightarrow && (\delta \circ \mu_\kappa = \xi_{\kappa,\kappa} = 1_{S_\kappa})\\
&x_1 \leq_\kappa x_2
\end{aligned}$$

□

Since $f : U \to S$ is a homomorphism, then obviously $f(U)$ is a subsemigroup of S.

Theorem 5.3 $Dom_S(f(U)) = \bigcup_{w \in P} \left(\mu^{-1}\left((w]^P_{\leq_P} \right) \bigcap (\mu_i \circ \alpha_i)^{-1} \left((w]^P_{\leq_P} \right) \right)$, $i \in I$.

Proof Let $i \in I$.

A) $\underline{Dom_S(f(U)) \subseteq \bigcup_{w \in P} \left(\mu^{-1}\left((w]^P_{\leq_P} \right) \bigcap (\mu_i \circ \alpha_i)^{-1} \left((w]^P_{\leq_P} \right) \right)}$

Let $d \in Dom_S(f(U))$. By Proposition 4.1 we have $\mu \circ f = \mu_i \circ (\alpha_i \circ f)$. Therefore $\mu(f(u)) = (\mu_i \circ \alpha)(f(u))$ for each $u \in U$ and thus $\mu(\upsilon) = (\mu_i \circ \alpha)(\upsilon)$ for any $\upsilon \in f(U)$. Since $d \in Dom_S(f(U))$ and $\mu, \mu_i \circ \alpha_i : S \to P$ are homomorphisms, then

$$[\mu(d))_{\leq_P}^P \cap [(\mu_i \circ \alpha_i)(d))_{\leq_P}^P \neq \varnothing$$

Hence there exists $w \in P$ such that $\mu(d) \leq_P w$ and $(\mu_i \circ \alpha_i)(d) \leq_P w$. Therefore

$$\mu(d) \in (w]_{\leq_P}^P, \ (\mu_i \circ \alpha_i)(d) \in (w]_{\leq_P}^P$$

that is, $d \in \mu^{-1}\left((w]_{\leq_P}^P\right) \bigcap (\mu_i \circ \alpha_i)^{-1}\left((w]_{\leq_P}^P\right)$.

B) $\underline{\bigcup_{w \in P}\left(\mu^{-1}\left((w]_{\leq_P}^P\right) \bigcap (\mu_i \circ \alpha_i)^{-1}\left((w]_{\leq_P}^P\right)\right) \subseteq Dom_S(f(U))}$

Let $w \in P$ and $d \in \mu^{-1}\left((w]_{\leq_P}^P\right) \bigcap (\mu_i \circ \alpha_i)^{-1}\left((w]_{\leq_P}^P\right)$. Thus

$$d \in \mu^{-1}\left((w]_{\leq_P}^P\right), \ d \in (\mu_i \circ \alpha_i)^{-1}\left((w]_{\leq_P}^P\right)$$

and hence

$$\mu(d) \in (w]_{\leq_P}^P, \ (\mu_i \circ \alpha_i)(d) \in (w]_{\leq_P}^P$$

that is, $\mu(d) \leq_P w$ and $(\mu_i \circ \alpha_i)(d) \leq_P w$.

Let now $(T, \cdot_T, \leq_T)$ be an ordered semigroup and $\beta, \gamma : S \to T$ homomorphisms such that $\beta(\upsilon) = \gamma(\upsilon)$ for any $\upsilon \in f(U)$ which means that $\beta \circ f = \gamma \circ f$. For any $j \in I$ we define

$$\zeta_j = \gamma \circ {\alpha_j}^{-1} : S_j \to T$$

Clearly ζ_j is a homomorphism for all $j \in I$.

- $\beta \circ f = \gamma \circ f = \gamma \circ {\alpha_\kappa}^{-1} \circ = \zeta_\kappa \circ (\alpha_\kappa \circ f), \ \kappa \in I$
- $\zeta_\kappa \circ (\alpha_\kappa \circ f) = \gamma \circ {\alpha_\kappa}^{-1} \circ \alpha_\kappa \circ f = \gamma \circ {\alpha_\lambda}^{-1} \circ \alpha_\lambda \circ f = \zeta_\lambda \circ (\alpha_\lambda \circ f), \ \kappa, \lambda \in I$.

Then, by Proposition 4.5 (taking T as Q with homomorphisms $\beta : S \to T$, $\zeta_\kappa : S_\kappa \to T$, $\kappa \in I$), there exists a homomorphism $\eta : P \to T$ with the property

$$\eta \circ \mu = \beta, \ \eta \circ \mu_\kappa = \zeta_\kappa, \ \kappa \in I$$

■ Since η is a homomorphism and $\mu(d) \leq_P w$, we have

$$\beta(d) = \eta(\mu(d)) \leq_T \eta(w)$$

that is, $\eta(w) \in [\beta(d))_{\leq_T}^T$.

■ Since η is a homomorphism and $(\mu_i \circ \alpha_i)(d) \leq_P w$, we have

$$\gamma(d) = \left(\gamma \circ {\alpha_i}^{-1}\right)(\alpha_i(d)) = \zeta_i(\alpha_i(d)) = (\eta \circ \mu_i)(\alpha_i(d)) = \eta((\mu_i \circ \alpha_i)(d)) \leq_T \eta(w)$$

that is, $\eta(w) \in [\gamma(d))_{\leq_T}^T$.

Consequently $\eta(w) \in [\beta(d))^{T}_{\leq_T} \cap [\gamma(d))^{T}_{\leq_T}$ and thus $[\beta(d))^{T}_{\leq_T} \cap [\gamma(d))^{T}_{\leq_T} \neq \varnothing$. Therefore $d \in Dom_S(f(U))$.

□

Since every semigroup without order can be considered as an ordered semigroup with its equality relation being its order relation, then we immediately have that the notions and results presented in the paper generalize the analogous ones of semigroup without order.

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