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Operator index of a nonsingular algebraic curve

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Abstract: The present paper is devoted to a scheme-theoretic analog of the Fredholm theory. The continuity of the index function over the coordinate ring of an algebraic variety is investigated. It turns out that the index is closely related to the filtered topology given by finite products of maximal ideals. We prove that a variety over a field possesses the index function on nonzero elements of its coordinate ring iff it is an algebraic curve. In this case, the index is obtained by means of the multiplicity function from its normalization if the ground field is algebraically closed.

Key words: Algebraic variety, index of an operator tuple, integral extension, Dedekind extension, Taylor spectrum, Koszul homology groups of a variety

1. Introduction

The index $i(t)$ of a linear transformation t acting on a vector space X/k over a field k is the difference

$$i(t) = \dim_k(\ker(t)) - \dim_k(\operatorname{coker}(t))$$

of dimensions its kernel and cokernel whenever both dimensions are finite. The routine linear algebra result asserts that the index of every linear transformation acting on a finite dimensional vector space is vanishing. For an infinite dimensional complex Banach space X the set $\mathcal{F}(X)$ of all bounded linear operators having a finite index is exactly the class of Fredholm operators. It turns out that $\mathcal{F}(X)$ is an open subset of the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators acting on X equipped with the standard operator norm-topology, and $i : \mathcal{F}(X) \to \mathbb{Z}$ is a continuous function whenever $\mathbb{Z}$ is equipped with the discrete topology. A key fact in this direction is that the set of all invertible elements $\mathcal{B}(X)^{-1}$ of the algebra $\mathcal{B}(X)$ is an open subset, and $i^{-1}\{0\} = \mathcal{B}(X)^{-1} + \mathcal{K}(X)$ (Nikolskii criteria), where $\mathcal{K}(X)$ is the two-sided ideal of compact operators on X [13, 3.5], [11, 8.5]. That openness property fails to be true in the category of topological algebras [10, Part 1]. The index also pops up in the joint spectral theory within the concept of joint essential spectrum [5], [9].

To get a scheme-theoretic analog of the Fredholm theory is a challenging task after all works [6], [7] and [8] done toward algebraic spectral theory and multi-operator functional calculus for schemes. Many key results from the complex analytic geometry have their analogs in schemes such as the fundamental theorem of Serre on vanishing [12, 3.3.7]. Taylor's multivariable functional calculus [18], [19] (see also [15], [14]) has a scheme

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version suggested in [7], and the related algebraic spectral theory with all its key properties and their link to algebraic geometry were developed in [8]. It turns out that many key results of the commutative algebra are indeed reformulations of the joint spectral theory. The Taylor spectrum of the coordinate function tuple (acting as multiplication operators) on the coordinate ring of a variety is reduced to the variety itself, which in turn inherits the Koszul homology groups and the related index. The index in turn is reduced to the multiplicity from the local theory [16], and the multiplicity formula of Serre is used in [8] for the calculation of Koszul homology groups.

In the present paper we investigate the index function within the algebraic spectral theory proposed in [8] for elements taken from the coordinate ring R of an affine variety Y/k over k. It turns out that the index is closely related to the naturally filtered topology in R given by means of the family of maximal ideals introduced in [3, 3.2.13] by N. Bourbaki. Namely, the finite products $\mathfrak{a} = \mathfrak{m}_1 \cdots \mathfrak{m}_m$ of maximal ideals $\mathfrak{m}_j \subseteq R$ is a filter base of the filtered topology $\mathcal{F}$ in R. The length m of the product $\mathfrak{a}$ is called its multiplicity and it is denoted by $\rho(\mathfrak{a})$. If R/k is a Dedekind extension then every nonzero ideal $\mathfrak{a} \subseteq R$ has a unique primary factorization $\mathfrak{a} = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r}$ and $\rho(\mathfrak{a}) = m_1 + \cdots + m_r$. In particular, $\rho(z) = \rho(\langle z \rangle)$ defines *a multiplicity function* $\rho : R^* \to \mathbb{Z}$ whenever R is a Dedekind domain, where $R^* = R - \{0\}$. Our first result asserts that if Y/k is a variety with its coordinate ring R and $R^i = \{y \in R : i(y) < \infty\}$, then the index $i : \left(R^i, \mathcal{F}\right) \to \mathbb{Z}$ turns out to be a continuous homomorphism of abelian semigroups whenever $\mathbb{Z}$ is equipped with the left order topology. In particular, $\{i \geq n\}$ is a closed set in R^i for every $n \in \mathbb{Z}$. The index $i_M(y)$ of an r-tuple y of elements of R acting on a Noetherian R-module M can be calculated in terms of $i(y)$ in the following way

$$i_M(y) = i(y) \dim_K (K \otimes_R M)$$

whenever $i(y) < \infty$ and $\dim(Y) \leq 1$, where $K = \mathrm{Frac}(R)$ is the fraction field of R. If $R^i = R^*$ we say that Y/k is an i-variety. In this case, $i(y) < \infty$ for every r-tuple y of elements of R^*, and $i(y) = 0$ whenever $r > 1$. The main result of the paper asserts that an affine variety Y/k is an i-variety if and only if it is an irreducible curve. In this case, the equality

$$i = -\rho | R^*$$

holds if k is algebraically closed, where ρ is the multiplicity function of the normalization $\overline{R}$ of R in K, which is a Dedekind domain. Finally, the obtained result is illustrated in the case of an elliptic curve.

2. Preliminaries

In the present section, we provide the paper with the necessary basic material.

2.1. Discrete valuation rings

Let K be a field with a discrete valuation $v : K^* \to \mathbb{Z}$, which is a surjective group homomorphism such that $v(x+y) \geq \min\{v(x), v(y)\}$ whenever $x \neq -y$. The subring $A = \{v \geq 0\} \subseteq K$ turns out to be a local domain with its maximal ideal $\mathfrak{m} = \{v > 0\}$. Actually, it is a PID (in particular, UFD), and it is a regular local ring of $\dim(A) = 1$ with $K = \mathrm{Frac}(A)$. If $\zeta \in A$ with $v(\zeta) = 1$ then every $x \in K^*$ has a unique factorization $x = u\zeta^n$ with $n = v(x)$, and every ideal $\mathfrak{a} \subseteq A$ turns out to be principal $\mathfrak{a} = \left\langle \zeta^{v(\mathfrak{a})} \right\rangle$ with $v(\mathfrak{a}) = \min\{v(x) : x \in \mathfrak{a}\}$. In particular, $\mathfrak{m} = \langle \zeta \rangle$. A ring obtained by means of a valuation on a field is called a discrete valuation ring or DVR.

If k is a field and $K = k((\zeta))$ is the field of all formal Laurent series $x = \sum_{i\geq n} \lambda_n \zeta^n$, $n \in \mathbb{Z}$, $\lambda_i \in k$ over k, then $v : K^* \to \mathbb{Z}$, $v(x) = n$ (to be the order of vanishing of x, that is, $\lambda_n \neq 0$ and $\lambda_i = 0$ for all $i < n$) is a valuation on K. In this case, DVR is the k-algebra $A = k[[\zeta]]$ of all formal power series with its maximal ideal $\mathfrak{m} = \langle \zeta \rangle$.

Now let A be a DVR which contains the residue field $k = A/\mathfrak{m}$, so it is a k-algebra. Being PID, A is a Noetherian local k-algebra with $\mathfrak{m} = \langle \zeta \rangle = \mathrm{rad}(A)$. By Krull Intersection, $\cap_n \mathfrak{m}^n = \{0\}$ and A is embedded into its $\mathfrak{m}$-adic completion $\widehat{A}$, which is a Noetherian local k-algebra of $\dim\left(\widehat{A}\right) = 1$ too (see [1, 22.62]). By the Cohen Structure Theorem [1, 22.58], we conclude that $\widehat{A} = k[[\zeta]]$ up to a natural isomorphism. In particular, $\widehat{A}$ is DVR with its valuation $\widehat{v}$ induced from $k((\zeta))$. If $x = \sum_{i\geq n} \lambda_n \zeta^n \in A \subseteq \widehat{A}$ then $x = u\zeta^n$ with $u = \lambda_n + \lambda_{n+1}\zeta + \cdots \in A$. But u is invertible in A and $v(x) = v(u\zeta^n) = v(u) + v(\zeta^n) = n = \widehat{v}(x)$, that is, $v = \widehat{v}|A$.

The well known [3, 6.3.6], [1, 23.6] characterization result of DVR's asserts that for a Noetherian local ring A with its maximal ideal $\mathfrak{m}$ the following statements are equivalent: (1) A is DVR; (2) A is a normal domain of $\dim(A) = 1$; (3) A is a normal domain of depth 1; (4) A is a regular local ring of $\dim(A) = 1$; (5) $\mathfrak{m}$ is principal of height at least 1.

2.2. Dedekind domains and multiplicity

A Noetherian normal domain R of $\dim(R) = 1$ is called a Dedekind domain. So are PID's and rings of algebraic integers. In particular, R is DVR iff it is a local Dedekind domain. Actually, a Noetherian domain R which is not a field is a Dedekind domain iff the localization $R_{\mathfrak{p}}$ is DVR at every nonzero prime $\mathfrak{p} \subseteq R$. The main theorem of classical ideal theory (see [3, 7.2.3], [1, 24.8]) asserts that every nonzero ideal $\mathfrak{a}$ of a Dedekind domain R has a unique factorization $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$, where $\mathfrak{p}$ is running over all associated primes $\mathrm{Ass}_R(R/\mathfrak{a})$ of the R-module $R/\mathfrak{a}$, and $v_{\mathfrak{p}}$ is a discrete valuation of $R_{\mathfrak{p}}$. In particular, a Noetherian domain R of $\dim(R) = 1$ is Dedekind iff every primary ideal of R is a power of its radical [1, 24.9].

Every Dedekind domain admits *a multiplicity function* $\rho : R^* \to \mathbb{Z}$ defined in the following way. Let $\mathcal{I}$ be the set of all nonzero ideals of R, which in turn defines a filtered topology in R. For every $\mathfrak{a} \in \mathcal{I}$ we put $\rho(\mathfrak{a}) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{a})$ based on the unique primary factorization of $\mathfrak{a}$ in R. Notice that $\rho(\mathfrak{a}\mathfrak{b}) = \rho(\mathfrak{a}) + \rho(\mathfrak{b})$ whenever $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$. The multiplicity function $\rho : R^* \to \mathbb{Z}$ is given by the rule $\rho(z) = \rho(\langle z \rangle)$, where $\langle z \rangle \in \mathcal{I}$ is the principal ideal in R generated by z. Thus $\rho(zw) = \rho(z) + \rho(w)$ for all $z, w \in R^*$.

Lemma 2.1 *The multiplicity function is a continuous homomorphism $\rho : (R^*, \mathcal{I}) \to \mathbb{Z}$ of abelian semigroups whenever $\mathbb{Z}$ is equipped with the right order topology. In particular, $\{\rho \leq n\}$ is a closed subset in R^* for every $n \in \mathbb{Z}$.*

Proof Take $z \in R^*$ with its neighborhood $z + \mathfrak{b}$, $\mathfrak{b} \in \mathcal{I}$. Put $\mathfrak{a} = \langle z \rangle$ and $l = \rho(z)$. Notice that $\mathfrak{a} \in \mathcal{I}$. By shrinking $\mathfrak{b}$, if that is necessary, we may assume that $\mathfrak{b} \subseteq \mathfrak{a}$. One can take $\mathfrak{b}$ to be $\langle z \rangle$ or $\langle z^2 \rangle$. Then $z + \mathfrak{b} \subseteq \mathfrak{a}$. For every $w \in (z + \mathfrak{b}) \cap R^*$ we obtain that $w \in \mathfrak{a} \cap R^*$ and $w = yz$. It follows that $\rho(w) = \rho(y) + \rho(z) \geq l$, which means that $\rho((z + \mathfrak{b}) \cap R^*) \subseteq [l, +\infty)$. Whence ρ is a continuous mapping. $\square$

3. The finite index function

In this section we investigate the index of tuples of elements of a ring R acting on an R-module and prove a key result (see below Lemma 3.4) on the index using a spectral decomposition of a Noetherian R-module.

Let k be a field, R/k a finitely generated k-algebra, which is not a field. If R is reduced then it is the coordinate ring of an algebraic set Y/k over k, whereas it is a variety if R is a domain. If $R=k[x]$ for an n-tuple $x=(x_1,\ldots,x_n)$ then we can assume that $Y\subseteq\mathbb{A}^n$ and $\operatorname{Spec}(R)$ is the related affine scheme. If R is a domain then $\operatorname{Spec}(R)$ has the generic point ξ corresponding to the prime ideal $\langle 0\rangle$. Let M be an R-module and y an r-tuple of elements of R, which in turn is identified with a family of mutually commuting multiplication operators on M. We have the Koszul complex $\operatorname{Kos}(y,M)$:

$$0\leftarrow M\xleftarrow{\partial_0}\cdots\xleftarrow{\partial_{p-2}}M\otimes_k\wedge^{p-1}k^r\xleftarrow{\partial_{p-1}}M\otimes_k\wedge^{p}k^r\xleftarrow{\partial_p}\cdots\xleftarrow{\partial_{r-1}}M\leftarrow 0$$

of R-modules with the differential

$$\partial_{p-1}\left(v\otimes e_{i_1}\wedge\ldots\wedge e_{i_p}\right)=\sum_{s=1}^{p}(-1)^{s+1}y_{i_s}v\otimes e_{i_1}\wedge\ldots\wedge\widehat{e_{i_s}}\wedge\ldots\wedge e_{i_p},$$

where $v\in M$, $(e_1,\ldots,e_r)$ is the standard basis for k^r, and the notation $\widehat{e_{i_s}}$ stands for skipping e_{i_s} from the p-vector (see [4, 9.1]). The homology groups of $\operatorname{Kos}(y,M)$ are denoted by $H_p(y,M)$, $p\geq 0$, which are R-modules.

The following assertion is well known (see [18] or to the proof of [8, Lemma 2.4]).

Lemma 3.1 *If R/k is an algebra and y is an r-tuple of elements of R, then $\langle y\rangle H_p(y,M)=\{0\}$ for every $p\geq 0$, where $\langle y\rangle\subseteq R$ is the ideal generated by the tuple y. In particular, $\operatorname{Kos}(y,M)$ is exact whenever $\langle y\rangle=R$.*

Proof Fix i and define the following R-linear map

$$\gamma_i: M\otimes_k\wedge^p k^m\to M\otimes_k\wedge^{p+1}k^m,\quad \gamma_i(u_p)=u\otimes(e_i\wedge v_p),$$

where $u_p=u\otimes v_p$, $e_i\wedge v_p=(-1)^{s-1}e_{i_1}\wedge\ldots\wedge e_i\wedge\ldots\wedge e_{i_p}=(-1)^{s-1}v_{p+1}$ whenever $v_p=e_{i_1}\wedge\ldots\wedge e_{i_p}$ and $i_1<\cdots<i_{s-1}<i<i_s<\cdots<i_p$ for some s. Let $v_{p,k}$ be the $p-1$-vector $e_{i_1}\wedge\ldots\wedge\widehat{e_{i_k}}\wedge\ldots\wedge e_{i_p}$, where $\widehat{e_{i_k}}$ stands for skipping e_{i_k} from the p-vector v_p. Then

$$\begin{aligned}\partial_p\gamma_i(u_p)&=(-1)^{s-1}\sum_{k<s}(-1)^{k+1}y_{i_k}u\otimes v_{p+1,k}+y_iu\otimes v_p+(-1)^{s-1}\sum_{k\geq s}(-1)^{k}y_{i_k}u\otimes v_{p+1,k}\\&=-\sum_{k<s}(-1)^{k+1}y_{i_k}u\otimes e_i\wedge v_{p,k}+\sum_{k\geq s}(-1)^{k}y_{i_k}u\otimes e_i\wedge v_{p,k}+y_iu\otimes v_p\\&=-\gamma_i\partial_{p-1}(u_p)+y_iu_p.\end{aligned}$$

If $u_p\in\ker(\partial_{p-1})$ then $y_iu_p=\partial_p\gamma_i(u_p)\in\operatorname{im}(\partial_p)$, which means that the action of y_i over $H_p(y,M)$ is trivial, that is, $yH_p(y,M)=\{0\}$. Taking into account that $H_p(y,M)$ is an R-module, we conclude that $\langle y\rangle H_p(y,M)=\{0\}$ too. □

In analysis the statement of Lemma 3.1 sounds like that the Taylor spectrum is smaller than the Harte spectrum of the operator tuple y.

We define *the index*

$$i_M(y) = \sum_{p=0}^{r} (-1)^{p+1} \dim_k (H_p(y, M))$$

of the tuple y whenever $\dim_k (H_p(y, M)) < \infty$ for all p. In this case we write $i_M(y) < \infty$. If $M = R$ we use the notation $i(y)$ instead of $i_M(y)$. If $r = 1$ we come up with a singleton y and $i_M(y) = \dim_k \ker(y|M) - \dim_k \operatorname{coker}(y|M) = i(y|M)$ is the index of a k-linear transformation.

The set of those $y \in R$ with $i_M(y) < \infty$ is denoted by $R^{i,M}$, and we write R^i instead of $R^{i,R}$. Thus there is a well defined *index function*

$$i_M : R^{i,M} \to \mathbb{Z}, \quad z \mapsto i(z).$$

Notice that $k^* \subseteq R^{-1} \subseteq R^{i,M} \subseteq R^*$. If $\dim_k(M) < \infty$ then $R^* = R^{i,M}$ and $i_M(y) = 0$ for every $y \in R^*$.

One of the key results of [8] asserts that $i_M(y) < \infty$ for an r-tuple y whenever $k[y] \subseteq R$ is an integral extension and M is a finitely generated R-module. Moreover, $i_M(y) = 0$ if M has finite length $\ell_R(M)$. If $M = R = k[x]$, $x = (x_1, \ldots, x_n)$, $d = \dim(R) = \dim(Y)$ and $Y \subseteq \mathbb{A}^n$ is the related variety, then $i(x) = -\delta_{nd}$ (see [8, Corollary 2.5]). Recall that δ_{nd} stands for Kronecker delta-symbol. Finally, if R is a domain and $R^i = R^*$, then we say that Y/k is *an* i*-variety.*

Recall that a multiplicative subset $S \subseteq R^*$ is called a saturated one if $zw \in S$ iff $z, w \in S$.

Lemma 3.2 *Let R be a finitely generated k-algebra and let M be an R-module. The set $R^{i,M}$ is multiplicative and $i_M(zw) = i_M(z) + i_M(w)$ for all $z, w \in R^{i,M}$. If R is a domain then R^i is a saturated multiplicative set in R and $R^{-1} = i^{-1}\{0\}$.*

Proof First assume that R is a domain. If z invertible then $\operatorname{coker}(z|R) = \{0\}$ and $i(z) = 0$. Conversely, $i(z) = 0$ iff $\operatorname{coker}(z|R) = \{0\}$, for R is a domain and $\ker(z|R) = \{0\}$. But $\operatorname{coker}(z|R) = \{0\}$ means that $\langle z \rangle = R$ or z is invertible. Pick $z, w \in R^*$. There is a canonical exact sequence

$$0 \leftarrow \operatorname{coker}(w|R) \xleftarrow{\beta} \operatorname{coker}(wz|R) \xleftarrow{\alpha} \operatorname{coker}(z|R) \leftarrow 0$$

of k-vector spaces with $\alpha(y^\sim \bmod z) = (wy)^\sim \bmod wz$ and $\beta(y^\sim \bmod wz) = y^\sim \bmod w$ for all $y \in R$, where the tilde sign indicates to the related equivalence class of an element. In particular, $i(wz) < \infty$ iff $i(w) < \infty$ and $i(z) < \infty$, which means that R^i is a saturated multiplicative subset of R.

Now consider the general case of R and let M be an R-module. If $z, w \in R^{i,M}$ then using the Snowflake Lemma [11, 2.3.16], we obtain the following exact sequence

$$0 \leftarrow \operatorname{coker}(w|M) \xleftarrow{\beta} \operatorname{coker}(wz|M) \xleftarrow{\alpha} \operatorname{coker}(z|M) \leftarrow$$
$$\leftarrow \ker(w|M) \leftarrow \ker(wz|M) \leftarrow \ker(z|M) \leftarrow 0$$

of the finite dimensional k-vector spaces whose Euler characteristic is vanishing. Thus

$$i_M(z) - i_M(wz) + i_M(w) = 0$$

or $i_M(zw) = i_M(z) + i_M(w)$ for all $z, w \in R^{i,M}$. □

As above let R be a finitely generated k-algebra and y an r-tuple of elements of R. An exact sequence

$$0 \leftarrow N \longleftarrow M \longleftarrow K \leftarrow 0$$

of R-modules generates the following exact sequence

$$0 \leftarrow \mathrm{Kos}(y, N) \longleftarrow \mathrm{Kos}(y, M) \longleftarrow \mathrm{Kos}(y, K) \leftarrow 0$$

of Koszul complexes, which in turn associates the long exact sequence

$$\cdots \longleftarrow H_{i-1}(y, K) \longleftarrow H_i(y, N) \longleftarrow H_i(y, M) \longleftarrow H_i(y, K) \longleftarrow \cdots$$

of homology groups, where $i \geq 1$ (see [16, 4.A.2]). In particular, finiteness of any two integers from $i_N(y)$, $i_M(y)$ and $i_K(y)$ imply the third one left. In this case,

$$i_M(y) = i_N(y) + i_K(y). \tag{3.1}$$

In particular, the formula works for the direct sums of R-modules with their finite index.

Lemma 3.3 *Let R/k be an algebra and let M be an R-module. Then for every r-tuple y of elements of $R^{i,M}$ the equality $i_M(y) = 0$ holds whenever $r > 1$. In particular, $i_M(y) < \infty$ for all finite tuples from $R^{i.M}$.*

Proof Pick an r-tuple y of elements of $R^{i,M}$ and let us prove by induction on r that $i_M(y) < \infty$. Put $y' = (y_1, \ldots, y_{r-1})$. By induction hypothesis, we have $i_M(y') < \infty$. Using Lemma Bourbaki [8] (see also [4, 9.5, Lemma 3]), we derive an exact sequence

$$0 \leftarrow \ker(y_r | H_{p-1}(y', M)) \longleftarrow H_p(y, M) \longleftarrow \mathrm{coker}(y_r | H_p(y', M)) \leftarrow 0 \tag{3.2}$$

of R-modules for every $p \geq 0$. But $\dim_k(H_p(y', M)) < \infty$ for all p, that is, both corners of (3.2) are finite dimensional k-vector spaces. It follows that $\dim_k(H_p(y, M)) < \infty$ for all p, that is, $i_M(y) < \infty$. Finally, taking into account that $i(y_r | H_p(y', M)) = 0$, $p \geq 0$, we deduce that

$$\begin{aligned} i_M(y) &= \sum_{p=0}^{r} (-1)^{p+1} \left(\dim_k \ker(y_r | H_{p-1}(y', M)) + \dim_k \mathrm{coker}(y_r | H_p(y', M))\right) \\ &= \sum_{p=0}^{r} (-1)^{p+1} \dim_k \mathrm{coker}(y_r | H_p(y', M)) + (-1)^p \dim_k \ker(y_r | H_p(y', M)) \\ &= \sum_{p=0}^{r} (-1)^p \, i(y_r | H_p(y', M)) = 0, \end{aligned}$$

that is, $i_M(y) = 0$ whenever $r > 1$. □

In the case of a Noetherian module the index can be calculated by means of the spectral decomposition (see [7, 4.4]). If $M = R/\mathfrak{a}$ for an ideal $\mathfrak{a} \subseteq R$ we write $i_{\mathfrak{a}}(y)$ instead of $i_M(y)$. The set of all minimal elements from $\mathrm{Supp}(M)$ is denoted by $\mathrm{Min}(M)$. The nilradical $\mathrm{nil}(M)$ of an R-module M is defined as the radical of the ideal $\mathrm{Ann}(M)$. If M is a finitely generated R-module then $\mathrm{nil}(M) = \cap \mathrm{Supp}(M)$. An R-module M is said to be reduced if $\mathrm{nil}(M) = \{0\}$.

Lemma 3.4 *Let k be a field, R a finitely generated k-algebra, M a Noetherian R-module, and let y be an r-tuple of elements of R such that $k[y] \subseteq R$ is integral. Then*

$$i_M(y) = \sum_{\mathfrak{p} \in \operatorname{Min}(M)} i_{\mathfrak{p}}(y)\, \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \sum_{\mathfrak{q} \in \mathfrak{S}} i_{\mathfrak{q}}(y)$$

for some multisubset $\mathfrak{S} \subseteq \operatorname{Supp}(M) \setminus \operatorname{Min}(M)$ (each its element can be included into $\mathfrak{S}$ several times). If $\mathfrak{p} \in \operatorname{Ass}(M)$ with $\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$, then $\mathfrak{p} \in \operatorname{Min}(M)$, $M_{\mathfrak{p}} = (R/\mathfrak{p}^n)_{\mathfrak{p}} \otimes_R M$ turns out to be a finite-length module over the Artinian ring $R_{\mathfrak{p}}/\mathfrak{p}^n R_{\mathfrak{p}}$ for some $n = n(\mathfrak{p})$, and

$$\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \ell_{(R/\mathfrak{p}^n)_{\mathfrak{p}}}\left((R/\mathfrak{p}^n)_{\mathfrak{p}} \otimes_R M\right).$$

If $M_{\mathfrak{p}}$ is reduced then $n = 1$ and

$$\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim_{K(\mathfrak{p})}(K(\mathfrak{p}) \otimes_R M),$$

where $K(\mathfrak{p}) = \operatorname{Frac}(R/\mathfrak{p})$.

Proof Put $R' = k[y]$. Since R/R' is a finitely generated algebra and integral, it follows that R is a finitely generated module over R' [1, 10.28]. But M is a finitely generated module over R, therefore M is a finitely generated module over R' (see [1, 10.26]). Since R'/k is Noetherian (Hilbert Basis), it follows that M is a Noetherian R'-module too, and so are all homology groups $H_p(y, M)$. By Lemma 3.1, the R'-module structure of $H_p(y, M)$ is diagonalizable, and it is just the k-vector space structure one. In particular, every ascending chain of vector subspaces in $H_p(y, M)$ turns out to be a chain of R'-submodules which has to stabilize being a Noetherian R'-module. Hence $H_p(y, M)$ is a finite dimensional k-vector space, and $i_M(y) < \infty$. In particular, $i_{\mathfrak{a}}(y) < \infty$ for all ideals $\mathfrak{a} \subseteq R$ (see also [8, Theorem 2.1]).

It is well known [1, 17.16], [3, 4.1.4] that M has a chain

$$\langle 0 \rangle = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{l-1} \subsetneq M_l = M$$

of its submodules with $Q_j = M_j/M_{j-1} = R/\mathfrak{p}_j$ for some primes $\mathfrak{p}_j$ such that

$$\operatorname{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} \subseteq \operatorname{Supp}(M),$$

where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ is a multiset. Notice that every $\mathfrak{p}$ in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ could occur in the set several times called the multiplicity of $\mathfrak{p}$.

Fix $\mathfrak{p} \in \operatorname{Min}(M)$. Then $\mathfrak{p} \in \operatorname{Ass}(M)$ and $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. In this case, $(Q_j)_{\mathfrak{p}} = (R/\mathfrak{p}_j)_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}_j R_{\mathfrak{p}} = \langle 0 \rangle$ unless $\mathfrak{p}_j = \mathfrak{p}$. If $\mathfrak{p} = \mathfrak{p}_j$ then $(Q_j)_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}$. Thus $M_{\mathfrak{p}}$ has the ascending chain of $R_{\mathfrak{p}}$-submodules $\left\{(M_j)_{\mathfrak{p}}\right\}$ whose quotients either the zero module or the simple module $R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}$. We come up with a Jordan-Hölder series for the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ and $\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ coincides with the multiplicity of $\mathfrak{p}$ in the multiset $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Put $\mathfrak{S} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} \setminus \operatorname{Min}(M)$ to be another multiset. Using (3.1), we deduce that

$$i_M(y) = \sum_j i_{Q_j}(y) = \sum_{\mathfrak{p} \in \operatorname{Min}(M)} i_{\mathfrak{p}}(y) + \sum_{\mathfrak{q} \in \mathfrak{S}} i_{\mathfrak{q}}(y) = \sum_{\mathfrak{p} \in \operatorname{Min}(M)} i_{\mathfrak{p}}(y)\, \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \sum_{\mathfrak{q} \in \mathfrak{S}} i_{\mathfrak{q}}(y),$$

that is, the equality for $i_M(y)$ holds.

Now assume that $\mathfrak{p} \in \operatorname{Ass}(M)$ with $\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. Then $\mathfrak{p} \in \operatorname{Min}(M)$ (see [3, 4.2.5, Corollary 2]). Namely, $\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ and the fact that $R_\mathfrak{p}$ is local imply that $\operatorname{Supp}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$ [1, 19.4]. For every prime ideal $\mathfrak{q} \subseteq R$ with $\mathfrak{q} \subsetneqq \mathfrak{p}$ we have $M_\mathfrak{q} = (M_\mathfrak{p})_\mathfrak{q} = (M_\mathfrak{p})_{\mathfrak{q}R_\mathfrak{p}} = 0$, which means that $\mathfrak{q} \notin \operatorname{Supp}(M)$. Hence $\mathfrak{p} \in \operatorname{Min}(M)$ (the argument belongs to the referee), and $\mathfrak{p}R_\mathfrak{p} = \operatorname{nil}(M_\mathfrak{p})$. It follows that $\mathfrak{p}R_\mathfrak{p}$ has a nilpotent action on the module $M_\mathfrak{p}$, that is, $\mathfrak{p}^n M_\mathfrak{p} = \{0\}$ for some positive integer $n = n(\mathfrak{p})$. Thus

$$M_\mathfrak{p} = M_\mathfrak{p}/\mathfrak{p}^n M_\mathfrak{p} = (R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} M_\mathfrak{p} = (R/\mathfrak{p}^n)_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p} \otimes_R M = (R/\mathfrak{p}^n)_\mathfrak{p} \otimes_R M$$

turns out to be a finitely generated $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-module, and $\operatorname{Spec}(R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}\}$, which means that $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$ is an Artinian ring. Moreover,

$$(R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}) / (\mathfrak{p}R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = (R/\mathfrak{p})_\mathfrak{p} = \operatorname{Frac}(R/\mathfrak{p}) = K(\mathfrak{p})$$

is the fraction field of $R/\mathfrak{p}$. If $N \subseteq M$ is a submodule then $N_\mathfrak{p} = R_\mathfrak{p} \otimes_R N = (R/\mathfrak{p}^n)_\mathfrak{p} \otimes_R N$ holds too, and $N_\mathfrak{p}$ turns out to be an $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-module. Conversely, every $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-submodule of $M_\mathfrak{p}$ is in turn an $R_\mathfrak{p}$-submodule, and it has the form $N_\mathfrak{p}$ for the uniquely given $\mathfrak{p}$-saturated submodule $N \subseteq M$. Thus all $R_\mathfrak{p}$-submodules of $M_\mathfrak{p}$ are in turn $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-submodules. Since M is a Noetherian R-module, we deduce that $\ell_{(R/\mathfrak{p}^n)_\mathfrak{p}}\left((R/\mathfrak{p}^n)_\mathfrak{p} \otimes_R N\right) < \infty$ for every submodule $N \subseteq M$. As above using a chain of submodules for M, we obtain that $M_\mathfrak{p}$ has the ascending chain of $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-submodules $\left\{(M_j)_\mathfrak{p}\right\}$ whose quotients either the zero module or $K(\mathfrak{p})$. Hence $\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) = \ell_{(R/\mathfrak{p}^n)_\mathfrak{p}}\left((R/\mathfrak{p}^n)_\mathfrak{p} \otimes_R M\right)$. Finally, if $M_\mathfrak{p}$ is reduced then $\mathfrak{p}R_\mathfrak{p} = \{0\}$ and $R_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = K(\mathfrak{p})$, $M_\mathfrak{p} = K(\mathfrak{p}) \otimes_R M$ and $\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim_{K(\mathfrak{p})}(K(\mathfrak{p}) \otimes_R M)$. □

If M is a module over a domain R and S is the multiplicative set $R - \{0\}$, then we define the torsion submodule $T(M)$ of M to be the S-saturation of $\langle 0 \rangle$, that is, $m \in T(M)$ iff $sm = 0$ for some $s \in S$. Recall that M is said to be torsion-free if $T(M) = 0$. The quotient module $M/T(M)$ is torsion-free.

Proposition 3.5 *Let Y/k be a variety in $\mathbb{A}^n$ over a field k with its coordinate ring R and $K = \operatorname{Frac}(R)$, y an r-tuple of elements of R such that $i(y) < \infty$, and let M be a Noetherian R-module. Then*

$$i_M(y) = i(y) \dim_K (K \otimes_R M)$$

whenever $\dim(Y) \leq 1$. *In this case, $R^i \subseteq R^{i,M}$ and $R^{-1} \subseteq i_M^{-1}\{0\}$.*

Proof Since R is a domain, we derive that $K = S^{-1}R$ for $S = R - \{0\}$. For a while assume that M is a torsion-free module. Note that $M \otimes_R K = S^{-1}M$ and it is a finitely generated K-module, that is, there is a finite K-basis $\{m_i/1 : 1 \leq i \leq d\}$ for $S^{-1}M$. There is a well defined R-linear map $\alpha : R^d \to M$, $\alpha(e_i) = m_i$, $1 \leq i \leq d$ whose localization $S^{-1}\alpha : S^{-1}R^d \to S^{-1}M$ is nothing but the K-isomorphism $\lambda : K^d \to M \otimes_R K$, $\lambda(e_i) = m_i \otimes 1$, where $\{e_i\}$ is the standard basis in the related free modules. It follows that $S^{-1}\ker(\alpha) = \ker\left(S^{-1}\alpha\right) = \ker(\lambda) = \{0\}$. But $\ker(\alpha) \subseteq R^d$ is a torsion-free submodule. Hence $\ker(\alpha) = \{0\}$.

Now put $N = M/\operatorname{im}(\alpha)$ to be a finitely generated R-module with

$$S^{-1}N = S^{-1}M/S^{-1}\operatorname{im}(\alpha) = S^{-1}M/\operatorname{im}(\lambda) = \{0\}.$$

If $N = \sum_{i=1}^{l} Rn_i$ then $s_i n_i = 0$ for some $s_i \in S$, which in turn implies that $sn_i = 0$, $1 \le i \le l$ for some $s \in S$. In particular, $s \in \mathrm{Ann}(N)$, hence $\mathrm{Ann}(N) \ne \langle 0 \rangle$. But $\mathrm{Supp}(N) = V(\mathrm{Ann}(N))$, therefore $\xi = (0) \notin \mathrm{Supp}(N)$. Taking into account that $\dim(R) \le 1$, we conclude that $\mathrm{Supp}(N) \subseteq Y$ (the maximal ideals). Then $\ell_R(R/\mathfrak{q}) = 1 < \infty$ (see below Proposition 4.1, or [1, 19.4]), and $i_{\mathfrak{q}}(y) = 0$ due to [8, Corollary 3.4] for every $\mathfrak{q} \in \mathrm{Supp}(N)$. As in the proof of Lemma 3.4, one can use a chain of submodules for N to conclude that $i_N(y) = 0$. Using the exact sequence

$$0 \leftarrow N \longleftarrow M \overset{\alpha}{\longleftarrow} R^d \leftarrow 0$$

and (3.1), we deduce that $i_M(y) = i_N(y) + i_{R^d}(y) = i_{R^d}(y) = i(y)\, d$, where $d = \dim_K(K \otimes_R M)$.

If M is not a torsion-free module then $T(M)$ is a nonzero Noetherian module. Hence $\mathrm{Ass}(T(M)) \ne \varnothing$ and it does not contain ξ. As above in the case of the module N, we deduce that $i_{T(M)}(y) = 0$. But $M/T(M)$ is a torsion-free R-module and (3.1) applied to the exact sequence

$$0 \leftarrow M/T(M) \longleftarrow M \longleftarrow T(M) \leftarrow 0$$

results in $i_M(y) = i_{M/T(M)}(y) + i_{T(M)}(y) = i_{M/T(M)}(y)$. Notice also that the R-modules $K \otimes_R M$ and $K \otimes_R M/T(M)$ are canonically isomorphic. Thus the formula holds for $i_M(y)$, and $R^i \subseteq R^{i,M}$, $i_M^{-1}\{0\} \supseteq i^{-1}\{0\}$ (see Lemma 3.2). □

Thus $i(y) \,|\, i_M(y)$ whenever Y/k is an irreducible algebraic curve in $\mathbb{A}^n$ (see also [1, 26.12], [3, 7.2.5]). Note also that the formula from Proposition 3.5 works whenever y is an r-tuple of elements of R with the integral extension $k[y] \subseteq R$.

Corollary 3.6 *Let Y/k be a variety in $\mathbb{A}^n$ over a field k with its coordinate ring R of $\dim(Y) \le 1$, y an r-tuple of elements of R such that $k[y] \subseteq R$ is integral and K'/K a finite field extension. If R' is the normalization of R in K' then $i_{R'}(y) = i(y)\dim_K(K')$.*

Proof First notice that R' is a finite R-module and a finitely generated k-algebra [3, 5.3.2]. In particular, R' is Noetherian (Hilbert Basis) and $k[y] \subseteq R'$ is integral. By Lemma 3.4, we conclude that $i_{R'}(y) < \infty$. Moreover, R' is a torsion-free R-module being a domain. Using Proposition 3.5, we conclude that $i_{R'}(y) = i(y)\dim_K(K \otimes_R R')$. It remains to notice that $K \otimes_R R' = K'$. Namely, every $z' \in K'$ being integral over K turns out to be a root of a monic polynomial $p(X)$ over $K = \mathrm{Frac}(R)$ with the coefficients $c_j = a_j/s$, $a_j, s \in R$ and $n = \deg(p(X))$. Since $s^n p(z') = 0$, we obtain that sz' is integral over R, which means that $sz' = x' \in R'$. Hence $z' = (1/s) \otimes x' \in K \otimes_R R'$. The rest is clear. □

4. The filtered topology

In this section, we introduce a suitable filtered topology in a finitely generated k-algebra R that would make the index function continuous. We assume that $k \subsetneqq R$.

Let $\mathrm{Max}(R)$ be the set of all maximal ideals of R. By *a support on* R we mean a nonnegative function $m : \mathrm{Max}(R) \to \mathbb{Z}$ whose support (as a function) is finite. Every support m on R defines its ideal $\mathfrak{b}_m = \cap \{\mathfrak{p}^{m(\mathfrak{p})} : \mathfrak{p} \in \mathrm{Max}(R)\} \subseteq R$. If $m = 0$ (the trivial support) then we assume that $\mathfrak{b}_m = R$. If R is a domain then $\mathfrak{b}_m \ne \langle 0 \rangle$ (Prime Avoidance). Let $\mathcal{F}$ be the set of these ideals $\{\mathfrak{b}_m\}$ of the ring R, where m

is running over all supports on R. Then $\mathcal{F}$ is a filter base, which defines a filtered topology in R called the $\mathcal{F}$-topology. If R is reduced then the $\mathcal{F}$-topology is Hausdorff due to Hilbert Nullstellensatz (we do not need k to be algebraically closed). Notice that $(R,\mathcal{F})$ is a topological ring. The related Hausdorff completion $\widehat{R}$ of R is topologically isomorphic to the direct product $\prod_{\mathfrak{p}\in \operatorname{Max}(R)} \widehat{R_{\mathfrak{p}}}$, where $\widehat{R_{\mathfrak{p}}}$ is the Hausdorff completion of R with respect to the $\mathfrak{p}$-adic topology (see [3, 3.2.13]). Note that $R/\mathfrak{p}^l$ is a local k-algebra, for no prime could contain $\mathfrak{p}^l$ apart from $\mathfrak{p}$ thanks to the fact $\mathfrak{p}$ is maximal (Prime Avoidance). It follows that

$$R/\mathfrak{p}^l = \left(R/\mathfrak{p}^l\right)_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}^l R_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{m}^l, \tag{4.1}$$

where $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ is the maximal ideal of the localization $R_{\mathfrak{p}}$ at $\mathfrak{p}$. Using (4.1), we derive that $\widehat{R_{\mathfrak{p}}} = \varprojlim \left\{R/\mathfrak{p}^l\right\} = \varprojlim \left\{R_{\mathfrak{p}}/\mathfrak{m}^l\right\}$, which is the $\mathfrak{m}$-adic completion of the localization $R_{\mathfrak{p}}$ at $\mathfrak{p}$.

Proposition 4.1 *Let R be a finitely generated k-algebra, and $\mathfrak{a} \subseteq R$ an ideal. The following assertions are equivalent:*

(i) $\dim_k(R/\mathfrak{a}) < \infty$*;*

(ii) $\ell_R(R/\mathfrak{a}) < \infty$*;*

(iii) *The ideal $\mathfrak{a}$ has a primary expansion $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s$ with $\operatorname{Max}(R/\mathfrak{a}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary.*

(iv) *The ideal $\mathfrak{a}$ contains $\mathfrak{b}_m$ and $\mathfrak{b}_m = \prod \left\{\mathfrak{p}^{m(\mathfrak{p})} : \mathfrak{p} \in \operatorname{Max}(R)\right\}$ for a certain support m on R.*

In this case the primary expansion of $\mathfrak{a}$ is unique and $\dim_k(R/\mathfrak{a}) = \sum_{j=1}^{s} \dim_k(R/\mathfrak{q}_j)$.

Proof Put $A = R/\mathfrak{a}$. First notice that if $A = \{0\}$ then the assertion is obvious, for $\operatorname{Max}(R/\mathfrak{a}) = \varnothing$ and the related support m is trivial. Therefore we can assume that $A \neq \{0\}$.

If $\dim_k(A) < \infty$ then obviously $\ell_R(A) < \infty$, that is, $(i) \Rightarrow (ii)$.

Now assume that $\ell_R(A) < \infty$. Then A is an Artinian ring. By Akizuki-Hopkins Theorem [1, 19.8], we conclude that $\operatorname{Spec}(A) = \operatorname{Supp}_R(A) = \operatorname{Max}(A) = \operatorname{Ass}_R(A)$ and it is a finite (discrete) set. By Lasker-Noether Theorem, $\mathfrak{a}$ has a unique irredundant primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$. If $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary then $\operatorname{Ass}_R(A) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ [1, 18.17], which in turn are identified with the maximal ideals of R. Hence $(ii) \Rightarrow (iii)$.

Suppose that $\mathfrak{a}$ has a primary expansion $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s$ with $\operatorname{Max}(A) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Then $\mathfrak{p}_j^{m_j} \subseteq \mathfrak{q}_j \subseteq \mathfrak{p}_j$ for some $m_j \geq 1$. As above $\mathfrak{p}_i^{m_i}$ and $\mathfrak{p}_j^{m_j}$ are comaximal for all $i \neq j$, therefore $\mathfrak{b}_m = \mathfrak{p}_1^{m_1} \cap \cdots \cap \mathfrak{p}_s^{m_s} = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_s^{m_s} \subseteq \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{a}$, that is, $(iii) \Rightarrow (iv)$.

Now assume that $\mathfrak{a} \supseteq \mathfrak{b}_m = \mathfrak{p}_1^{m_1} \cap \cdots \cap \mathfrak{p}_s^{m_s}$ for some m. If $\mathfrak{p}$ is a prime containing $\mathfrak{a}$ then $\mathfrak{p}_j^{m_j} \subseteq \mathfrak{p}$, which in turn implies that $\mathfrak{p}_j \subseteq \mathfrak{p}$ for some j (Prime Avoidance). Taking into account that $\mathfrak{p}_j$ is maximal, we conclude that $\mathfrak{p}_j = \mathfrak{p}$. Hence $\operatorname{Spec}(A) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} = \operatorname{Max}(A)$. Using again [1, 19.4], we obtain that A has the finite length $\ell_R(A)$. If $\{A_i : 0 \leq i \leq l\}$ is a Jordan-Hölder series of R-submodules in A, then $A_i/A_{i-1} = R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i \subseteq R$, $i \geq 1$. But $k \subseteq R/\mathfrak{m}_i$ is an algebra finite extension, which is finite by Zariski Nullstellensatz [1, 15.4]. It follows that $\dim_k(A_i/A_{i-1}) < \infty$ for every i. Hence $\dim_k(A) < \infty$, that is, $(iv) \Rightarrow (i)$.

Finally, note that $\dim(A)=0$, that is, all $\mathfrak{p}_i$ are minimal in $\operatorname{Spec}(A)$ as well. By Second Uniqueness [1, 18.22], $\mathfrak{q}_i$ is the saturation of $\mathfrak{a}$ with respect the set $R-\mathfrak{p}_i$, that is, $\mathfrak{q}_i=(\mathfrak{a}R_{\mathfrak{p}_i})\cap R$. Thus every $\mathfrak{q}_i$ is uniquely defined by means of the ideal $\mathfrak{a}$ and $\mathfrak{p}_i=\sqrt{\mathfrak{q}_i}$. Since R is Noetherian, $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are comaximal for $i\neq j$, it follows that so are ideals $\mathfrak{q}_i$ and $\mathfrak{q}_j$ (see [1, 18.11]). By Chinese Remainder Theorem, we have $\mathfrak{a}=\mathfrak{q}_1\cdots\mathfrak{q}_s$ and $R/\mathfrak{a}=\bigoplus_j R/\mathfrak{q}_j$ up to a natural isomorphism. In particular, $\dim_k(R/\mathfrak{a})=\sum_{j=1}^s\dim_k(R/\mathfrak{q}_j)$. Notice that $\dim_k(A)=\ell_R(A)$ whenever k is an algebraically closed field. $\square$

Based on Proposition 4.1, we conclude that the $\mathcal{F}$-topology in R is given by means of all ideals $\mathfrak{a}\subseteq R$ with $\dim_k(R/\mathfrak{a})<\infty$. As above in Section 2.2 we put $|m|(\mathfrak{a})=m_1+\cdots+m_s$. If $\mathfrak{a}=\langle z\rangle$ is principal then we write $|m|(z)$ instead of $|m|(\mathfrak{a})$.

Corollary 4.2 *Let R be a finitely generated k-algebra, y an r-tuple of elements of R with $r\geq 1$, and let M be a Noetherian R-module. If $H_p(y,M)=0$ for some p, then $H_j(y,M)=0$ for all $j\geq p$. If y is an r-tuple of elements of R^i with $r>1$ and $H_2(y,R)=0$, then $\dim_k(H_p(y,R))=\sum_{j=1}^s\dim_k(R/\mathfrak{q}_j)$, $p=0,1$ and $\dim_k(H_p(y,R))=0$, $p\geq 2$, where $\langle y\rangle=\mathfrak{q}_1\cap\ldots\cap\mathfrak{q}_s$ is a primary decomposition. For every $z\in R^i$ with $\langle z\rangle=\mathfrak{q}_1\cap\ldots\cap\mathfrak{q}_s$, we have*

$$i(z)=\dim(\operatorname{Ann}(z))-\sum_{j=1}^s\dim_k(R/\mathfrak{q}_j).$$

Proof Take an r-tuple y of elements of R with $r\geq 1$. Let us prove that if $H_p(y,M)=0$ for some p then $H_j(y,M)=0$ for all $j\geq p$. First assume that $\langle y\rangle=R$. By Lemma 3.1, we have $H_j(y,M)=0$ for all $j\geq 0$, and the result follows.

Now assume that $\langle y\rangle\neq R$ and $M\neq 0$. Since $H_j(y,M)$ are the same homology groups of the $k[y]$-module M, one can assume that $R=k[y]$ and $\langle y\rangle=\mathfrak{m}$ is a maximal ideal of R. Using Lemma 3.1, we obtain that the R-module structure of every $H_j(y,M)$ is reduced to its k-vector space one. Since the localization is an exact functor, we conclude that $H_j(y,M)=H_j(y,M)_{\mathfrak{m}}=H_j(y/1,M_{\mathfrak{m}})$ for all $j\geq 0$, where $y/1$ is the tuple representing $(y_1/1,\ldots,y_r/1)\subseteq\mathfrak{m}R_{\mathfrak{m}}=\operatorname{rad}R_{\mathfrak{m}}$. Therefore we can assume that R/k is a local Noetherian k-algebra with $y\subseteq\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. Since $M/y_1M\neq 0$ (Nakayama lemma), the first statement on the Corollary follows in the case of $r=1$. We proceed by induction on r. Put $y'=(y_1,\ldots,y_{r-1})$ and suppose $H_p(y,M)=0$ for some $p>0$ (recall that $H_0(y,M)=M/\langle y\rangle M\neq 0$). Let us prove that $H_j(y,M)=0$ for all $j\geq p$. Using the exact sequence (3.2), we obtain that $\operatorname{coker}(y_r|H_p(y',M))=0$. Since $H_p(y',M)$ is a Noetherian R-module and $y_r\in\operatorname{rad}R$, it follows that $H_p(y',M)/y_rH_p(y',M)\neq 0$ whenever $H_p(y',M)\neq 0$ (Nakayama lemma). Hence $H_p(y',M)=0$, and $H_j(y',M)=0$, $j\geq p$ by induction hypothesis. Using again (3.2), we obtain the exact sequence

$$0\leftarrow\ker(y_r|H_j(y',M))\longleftarrow H_{j+1}(y,M)\longleftarrow\operatorname{coker}(y_r|H_{j+1}(y',M))\leftarrow 0$$

for every $j\geq p$. It follows that $H_{j+1}(y,M)=0$ for all $j\geq p$.

Finally, assume that $r>1$ and $H_2(y,R)=0$. Then $H_j(y,R)=0$ for all $j\geq 2$, and

$$i(y)=-\dim_k(H_0(y,R))+\dim_k(H_1(y,R)).$$

By Lemma 3.3, we have $i(y) = 0$. Then $\dim_k(H_1(y,R)) = \dim_k(H_0(y,R)) = \dim_k(R/\langle y\rangle) < \infty$, and it remains to use Proposition 4.1. $\square$

Now we can prove the continuity property of the index function.

Theorem 4.3 *Let Y/k be a variety with its coordinate ring R. Then $i : (R^i, \mathcal{F}) \to \mathbb{Z}$ is a continuous homomorphism of abelian semigroups whenever $\mathbb{Z}$ is equipped with the left order topology. In particular, $\{i \geq n\}$ is a closed set in R^i for every $n \in \mathbb{Z}$.*

Proof We use very similar arguments from Lemma 2.1. Pick $z \in R^i$ and its neighborhood $z + \mathfrak{b}_m$ with $\mathfrak{b}_m \in \mathcal{F}$. Put $\mathfrak{a} = \langle z\rangle$ and $l = i(z)$. Then $\dim_k(R/\mathfrak{a}) = -i(z) < \infty$. By Proposition 4.1, we conclude that $\mathfrak{a} \in \mathcal{F}$. Thus we can assume that $\mathfrak{b}_m \subseteq \mathfrak{a}$, that is, $z + \mathfrak{b}_m \subseteq \mathfrak{a}$. For every nonzero $w \in (z + \mathfrak{b}_m) \cap R^i$ we have $w \in \mathfrak{a} \cap R^i$. It follows that $w = yz$ and $i(w) = i(y) + i(z) \leq l$ thanks to Lemma 3.2. Hence $i((z + \mathfrak{b}_m) \cap R^i) \subseteq (-\infty, l]$, which means that i is a continuous mapping. $\square$

The property of being an open subset for R^i (and also for R^{-1}) fails to be true in general (see Section 1). If $\dim(Y) \leq 1$ the continuity property from Theorem 4.3 holds also for Noetherian modules thanks to Proposition 3.5.

5. Dedekind extensions

Let $R = k[x]$ be a finitely generated k-algebra with an n-tuple $x = (x_1, \ldots, x_n)$. If R is a domain of $\dim(R) = 1$, then $\operatorname{Spec}(R) = Y \cup \{\xi\}$ with $Y \subseteq \mathbb{A}^n$ an algebraic curve that corresponds to the set of all nonzero prime ideals of R (in fact they are maximal ideals), and the generic point ξ. If R is a Dedekind domain we say that R/k is a Dedekind extension. Notice that Dedekind domains stand for nonsingular irreducible affine curves [12, 1.5].

Lemma 5.1 *Let R/k be a Dedekind extension. Then $\dim_k(R/\mathfrak{a}) < \infty$ whenever $\mathfrak{a} \subseteq R$ is a nonzero ideal. In this case, $\mathfrak{a} = \mathfrak{b}_m$ and $|m|(\mathfrak{a}) = \rho(\mathfrak{a})$, $R^* = R^i$, and for every $z \in R^*$ there exists $w \in \langle z\rangle$ such that $k[w] \subseteq R$ is an integral extension.*

Proof By the main theorem of classical ideal theory (see Section 2.2) and Proposition 4.1, we deduce that $\dim_k(R/\mathfrak{a}) < \infty$, $\mathfrak{a} = \mathfrak{b}_m$ with $m_j = v_{\mathfrak{p}_j}(\mathfrak{a})$ for all j. Hence $|m|(\mathfrak{a}) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{a}) = \rho(\mathfrak{a})$ (see Section 2.2). In particular, $R^* = R^i$.

Take $z \in R^*$ and put $\mathfrak{a} = \langle z\rangle$. By assumption, $R = k[x]$ for an n-tuple x, and every x_i is acting on the finite-dimensional vector space $R/\mathfrak{a}$. By Cayley-Hamilton Theorem, $p_i(x_i) = 0$ on $R/\mathfrak{a}$ for a monic polynomial $p_i(\zeta) \in k[\zeta]$. It follows that $p_i(x_i) - y_i = 0$ for some $y_i \in \mathfrak{a}$, that is, $k[y] \subseteq R$ is integral with $y = (y_1, \ldots, y_n)$. In particular, $\dim(k[y]) = \dim(R) = 1$ [1, 15.25]. By Noether Nullstellensatz [1, 15.1] applied to the maximal ideal $\langle y\rangle$ of $k[y]$ results in $w \in \langle y\rangle$ such that $k[w] \subseteq k[y]$ is integral. Notice that k is any field and the element w obtained by Noether Nullstellensatz can be constructed by finitely many transformations $y_i \to y_i + y_j^d$ for a suitable choice of integers d. It remains to note that $\langle y\rangle \subseteq \mathfrak{a}$. $\square$

Thus $\mathcal{F} = \mathcal{I}$ and $\rho(\mathfrak{a}) = |m|(\mathfrak{a})$ for every $\mathfrak{a} \in \mathcal{I}$ whenever Y/k is a nonsingular algebraic curve. Now we calculate $\dim_k(R/\mathfrak{a})$ in terms of $\rho(\mathfrak{a})$.

Lemma 5.2 *Let R/k be a Dedekind extension of an algebraically closed field k. If $\mathfrak{a} \in \mathcal{I}$ then* $\dim_k(R/\mathfrak{a}) = \rho(\mathfrak{a})$.

Proof By Lemma 5.1, $\dim_k(R/\mathfrak{a}) < \infty$, $\mathfrak{a} = \mathfrak{b}_m = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_s^{m_s}$ and $|m|(\mathfrak{a}) = \rho(\mathfrak{a})$ (see Proposition 4.1). Take $\mathfrak{p} \in \mathrm{Spec}(R/\mathfrak{a})$, that is, $\mathfrak{p} = \mathfrak{p}_j$ for some j. Note that $R_\mathfrak{p}$ is DVR with its residue field $R_\mathfrak{p}/\mathfrak{m} = R/\mathfrak{p}$ (see (4.1)), which is a finitely generated k-algebra. By Zariski Nullstellensatz, we have $R/\mathfrak{p} = k$, that is, $R_\mathfrak{p}$ contains its residue field. Then $\dim_k(R_\mathfrak{p}/\mathfrak{m}^l) = l$ for every positive integer l. One needs to use the fact that $R_\mathfrak{p}$ is DVR. Optionally, one can use the Cohen Structure Theorem [1, 22.58] (see Section 2.1). Namely, we have $R_\mathfrak{p} \subseteq \widehat{R_\mathfrak{p}} = k[[\zeta]]$ and $v = \widehat{v}|R_\mathfrak{p}$, where $\widehat{R_\mathfrak{p}}$ is the $\mathfrak{m}$-adic completion of $R_\mathfrak{p}$. Moreover, $R_\mathfrak{p}/\mathfrak{m}^l$ is complete (the filtration is vanishing) and $R_\mathfrak{p}/\mathfrak{m}^l = \widehat{R_\mathfrak{p}}/\langle \zeta^l \rangle$. Using (4.1), we deduce that

$$R/\mathfrak{p}^l = R_\mathfrak{p}/\mathfrak{m}^l = \widehat{R_\mathfrak{p}}/\langle \zeta^l \rangle = k[[\zeta]]/\langle \zeta^l \rangle = \bigoplus_{i=0}^{l-1} k\zeta^i,$$

that is, $\dim_k(R_\mathfrak{p}/\mathfrak{m}^l) = l$. Finally,

$$\dim_k(R/\mathfrak{a}) = \sum_{j=1}^{s} \dim_k\left(R/\mathfrak{p}_j^{m_j}\right) = \sum_{j=1}^{s} m_j = |m|(\mathfrak{a}) = \rho(\mathfrak{a})$$

thanks to Proposition 4.1 and Lemma 5.1. □

Now we can prove the main result of the paper.

Theorem 5.3 *Let Y/k be an affine variety over a field k with its coordinate ring R. Then Y/k is an i-variety iff Y is an irreducible curve over k. Moreover, if k is algebraically closed, then for the index function $i : R^* \to \mathbb{Z}$ we have*

$$i = -\rho|R^*,$$

where ρ is the multiplicity on the normalization $\overline{R}$ of R in the fraction field K of R. If M is a Noetherian R-module and $d = \dim_K(K \otimes_R M)$, then $i_M = di$ and $i_M : R^* \to \mathbb{Z}$ is a continuos map whenever $\mathbb{Z}$ is equipped with the left order topology.

Proof First assume that Y/k is an i-variety with its finite index function $i : R^* \to \mathbb{Z}$. Then $\dim_k(R/\mathfrak{a}) < \infty$ for every nonzero ideal $\mathfrak{a} \subseteq R$. By Proposition 4.1, we have $\mathfrak{a} \supseteq \mathfrak{b}_m$ for a support m on R. In particular, every nonzero prime $\mathfrak{p}$ turns out to be maximal, that is, R is a Noetherian domain of $\dim(R) = 1$.

Conversely, if R/k is a Noetherian domain of $\dim(R) = 1$, and $\mathfrak{a} \subseteq R$ is a nonzero ideal, then $\mathrm{Spec}(R/\mathfrak{a}) = \mathrm{Max}(R/\mathfrak{a})$, which means that $\ell_R(R/\mathfrak{a}) < \infty$. By Proposition 4.1, we deduce that $\dim_k(R/\mathfrak{a}) < \infty$. In particular, $i(z) = -\dim_k(R/\langle z \rangle) < \infty$ for every $z \in R^*$. If y is an r-tuple of elements of R^* with $r > 1$, then $i(y) = 0$ by Lemma 3.3. Using Theorem 4.3 and Proposition 3.5, we derive the formula for i_M whenever M is a Noetherian R-module.

Finally, put $M = \overline{R}$ to be the normal closure of R in K. Then M is a torsion-free R-module being a domain, and

$$K \otimes_R M = K \otimes_R \overline{R} = S^{-1}R \otimes_R \overline{R} = S^{-1}\overline{R} = \overline{S^{-1}R} = K$$

[1, 11.32], where $S = R - \{0\}$. Using Corollary 3.6, we obtain that $i_M = i$. But M is a Dedekind domain thanks to Krull-Akizuki Theorem [1, 26.13]. By Lemma 5.2, the Dedekind extension M/k possesses the multiplicity function ρ, and we have $i_M = -\rho$. The rest is clear. □

Remark 5.4 *The formula $i(z) = -\rho(z)$ fails to be true when k is not an algebraically closed field. The equality $\widehat{R_{\mathfrak{p}}} = k[[\zeta]]$ used in the proof of Lemma 5.2 does not hold. The related problems were investigated in [17].*

Corollary 5.5 *Let M/k be a vector space over an algebraically closed field k, t a k-linear transformation on M which defines a finitely generated $k[X]$-module structure on M, and let $f(X) \in k[X]$ be a nonzero polynomial. Then $\deg(f(X))$ divides the index $i_M(f(t))$ of the transformation $f(t)$ acting on M.*

Proof Put $R = k[X]$, which is a Dedekind domain. By assumption M is a Noetherian R-module. By Theorem 5.3, $i(f(t)) = -\rho(f(X)) = -\deg(f(X))$ (see Lemma 2.1) and $i_M(f(t)) = di(f(t))$ with $d = \dim_K(K \otimes_R M/T(M))$. □

Remark 5.6 *Actually, the assertion of Corollary 5.5 can be derived from the theory of elementary divisors in the classification of finitely generated modules over PID [1, 5.38], [2, 7.4]. Namely, based on [2, 7.4.3 Theorem 2] one can reduce the computation of $i_M(f(t))$ to the case of a free R-module M of finite rank, thereafter it turns out to be a multiple of $i(f(t))$.*

Remark 5.7 *All key results of the paper can easily be proved for the graded Noetherian modules just using homogeneous elements of the related graded ring R. If k is an algebraically closed field, Y/k is a projective nonsingular curve in $\mathbb{P}^n$ and $f \in R^*$ is a homogenous element representing a homogenous polynomial $f(X) \in k[X]$, then*

$$i(f) = -\deg(Y)\deg(f(X)).$$

Indeed, by scheme-theoretical version of Bezout's Theorem [12, 1.7.8], we have $\deg(Y)\deg(H) = \sum_{j=1}^{s} i(Y, H : p_j)$ is the sum of multiplicities of the intersection $Y \cap H = \{p_1, \ldots, p_s\}$, where $H = Z(f(X))$ is the hypersurface in $\mathbb{P}^n$ given by $f(X)$. In this case $\deg(H) = \deg(f(X))$. But $-i(f) = \rho(f) = \sum_{j=1}^{s} i(Y, H : p_j)$ thanks to Theorem 5.3.

For illustration of Theorem 5.3, consider an example of an elliptic curve $Y = \{y^2 = x^3 - x\}$ in $\mathbb{A}^2$ ($\operatorname{char}(k) \neq 2$), which is a nonsingular variety. In this case, $R = k[x, y]$ with its polynomial subalgebra $k[x]$. Notice that y is integral over the subalgebra $k[x]$, and x, y are irreducible elements of R (see [12, 1.6, Exer 6.2]) If $\lambda + \mu y \in \langle x \rangle$ in R then $\lambda = \mu = 0$. Taking into account that $y^2 \in \langle x \rangle$, we deduce that $i(x) = -2$. Further, $R/\langle y \rangle = \operatorname{span}_k\{1, x, x^2\}$. If $x^2 = \lambda + \mu x$ in $R/\langle y \rangle$ for some $\lambda, \mu \in k$, then $\lambda \in \langle x, y \rangle$ in R, which means that $\lambda = 0$ (note that $0 \in Y$). Moreover, $x^2 - \mu x = fy$ implies that $fy \in \langle x \rangle$, which is a prime ideal. Hence $f \in \langle x \rangle$ or $x - \mu \in \langle y \rangle$. As above $\mu = 0$. Thus $i(y) = -3$.

Based on Remark 5.7, we can see that $-i(y)$ is the number of intersection points of the x-axis with Y whereas $-i(x)$ represents the intersection of the y-axis with Y counted with multiplicities. In particular, $i(x^s y^t) = -2s - 3t$ for all $s, t \geq 0$ (see Lemma 3.2). Finally, $i(x, y) = 0$, $H_0(x, y) = H_1(x, y) = k$ and $H_2(x, y) = 0$ for the homology groups of $\operatorname{Kos}((x, y), R)$.

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