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Some congruences with $q-$binomial sums

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Abstract: In this paper, using some combinatorial identities and congruences involving $q-$harmonic numbers, we establish congruences that for any odd prime p and any positive integer α,

$$\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} (-1)^{nk} \frac{q^{-\alpha npk+n\binom{k+1}{2}+2k}}{[k]_q} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \pmod{[p]_q^2},$$

and

$$\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} (-1)^{nk} q^{-\alpha npk+n\binom{k+1}{2}+k} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \widetilde{H}_k(q) \pmod{[p]_q^2},$$

where n is any integer.

Key words: Congruence, $q-$analog, $q-$harmonic number

1. Introduction

The harmonic numbers are given by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for} \quad n \in \mathbb{Z}^+.$$

In [22], Wolstenholme discovered that for any prime number $p \geq 5$,

$$H_{p-1} \equiv 0 \pmod{p^2}.$$

The $q-$harmonic numbers and the $q-$alternating harmonic number are given by

$$H_n(q) = \sum_{k=1}^{n} \frac{1}{[k]_q}, \quad \widetilde{H}_n(q) = \sum_{k=1}^{n} \frac{q^k}{[k]_q}, \quad I_n(q) = \sum_{k=1}^{n} \frac{(-1)^k}{[k]_q},$$

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and the $q-$harmonic numbers of order 2 are given by

$$H_{n,2}(q)=\sum_{k=1}^{n}\frac{1}{[k]_q^2},\quad \widetilde{H}_{n,2}(q)=\sum_{k=1}^{n}\frac{q^k}{[k]_q^2},$$

where for nonnegative integer n, $[n]_q=(1-q^n)/(1-q)=1+q+q^2+...+q^{n-1}$. It is seen that for any prime p such that $0<m<p$,

$$\frac{1}{[p-m]_q}\equiv-\frac{q^m}{[m]_q}\pmod{[p]_q}. \tag{1.1}$$

The $q-$Pochhammer symbol is given by

$$(x;q)_0=1 \text{ and } (x;q)_n=\prod_{k=0}^{n-1}\left(1-xq^k\right).$$

For any $m,n\in\mathbb{N}$, the $q-$binomial coefficients are defined by

$$\begin{bmatrix} n\\ m\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}},$$

if $n\geq m$, and if $n<m$, then $\begin{bmatrix} n\\ m\end{bmatrix}_q=0$. It is clear that

$$\lim_{q\to1}\begin{bmatrix} n\\ m\end{bmatrix}_q=\binom{n}{m},$$

where $\binom{n}{m}$ is the usual binomial coefficient. The $q-$binomial coefficients satisfy the recurrence relation

$$\begin{bmatrix} n+1\\ m\end{bmatrix}_q=q^m\begin{bmatrix} n\\ m\end{bmatrix}_q+\begin{bmatrix} n\\ m-1\end{bmatrix}_q.$$

In [16], Pan and Cao defined the $q-$Fermat quotient by for an odd prime p,

$$Q_p(m,q)=\frac{(q^m;q^m)_{p-1}/(q;q)_{p-1}-1}{[p]_q},$$

where m is nonnegative integer such that $p\nmid m$.
In [21], Tauraso gave that for any prime p and any positive integer α,

$$\begin{bmatrix} \alpha p-1\\ k\end{bmatrix}_q\equiv(-1)^k q^{-\binom{k+1}{2}}\left(1-\alpha[p]_q H_k(q)\right)\pmod{[p]_q^2},$$

where k is integer such that $0\leq k<p$. Thus, it is clearly seen that

$$\begin{bmatrix} \alpha p-1\\ k\end{bmatrix}_q\equiv(-1)^k q^{\alpha pk-\binom{k+1}{2}}\left(1-\alpha[p]_q \widetilde{H}_k(q)\right)\pmod{[p]_q^2}. \tag{1.2}$$

In [7], Elkhiri et al. gave that for an odd prime p,

$$\sum_{k=1}^{p-1} \widetilde{H}_k(q) \equiv \frac{p-\left(q^{1-p};q\right)_{p-1}}{[p]_q} \pmod{[p]_q}. \tag{1.3}$$

In [2, 19], Shi et al. showed that for any prime $p \geq 5$,

$$H_{p-1}(q) \equiv \frac{p-1}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2 \ [p]_q \pmod{[p]_q^2}, \tag{1.4}$$

and

$$\widetilde{H}_{p-1}(q) \equiv \frac{1-p}{2}(1-q) \ + \frac{p^2-1}{24}(1-q)^2 \ [p]_q \pmod{[p]_q^2}. \tag{1.5}$$

In [14], Ömür et al. investigated that for any positive integer n,

$$\sum_{k=1}^{n} \frac{q^{2k}}{[k]_q} = \widetilde{H}_n(q) - q(1-q)\,[n]_q\,, \tag{1.6}$$

and

$$\sum_{k=1}^{n} \frac{(-q)^k}{[k]_q} = I_n(q) + (1-q)\frac{(-1)^{n+1}+1}{2}. \tag{1.7}$$

In [10], Kızılateş and Tuğlu gave that for any positive integer n,

$$\sum_{0\leq i\leq k\leq n-1} \frac{q^{i+k}}{[i]_q} = [n]_q(\widetilde{H}_n(q)-q). \tag{1.8}$$

In [11], Koparal et al. showed that for any positive integer n,

$$\sum_{1\leq i\leq k\leq n} \frac{q^{i+k}}{[i]_q\,[k]_q} = \frac{1}{2}\left(\widetilde{H}_{n,2}(q) - (1-q)\widetilde{H}_n(q) + \widetilde{H}_n(q)^2\right),$$

and

$$\sum_{1\leq i\leq k\leq n} \frac{q^{i+2k}}{[i]_q\,[k]_q} = \frac{1}{2}\widetilde{H}_{n,2}(q) + \frac{1}{2}\widetilde{H}_n(q)^2 + \left(\frac{q-3}{2} + q^{n+1}\right)\widetilde{H}_n(q) + q(1-q^n). \tag{1.9}$$

In [15], Pan established that for an odd prime p,

$$\sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} \equiv -Q_p(2,q) + \frac{[p]_q}{2}\left(Q_p^2(2,q) + Q_p(2,q)\,(1-q) + \frac{p^2-1}{8}\,(1-q)^2\right) \pmod{[p]_q^2}, \tag{1.10}$$

$$\sum_{1\leq i\leq k\leq p-1} \frac{(-1)^k}{[i]_q\,[k]_q} \equiv Q_p^2(2,q) - (1-q)Q_p(2,q) - \frac{1}{24}\left(p^2-1\right)(1-q)^2 \pmod{[p]_q}, \tag{1.11}$$

$$\sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q^2} \equiv (1-q)\left(\frac{1-p^2}{24}(1-q) - Q_p(2,q)\right) \pmod{[p]_q}, \tag{1.12}$$

and for any prime p and any positive integer k,

$$q^{kp} \equiv 1 - k(1-q)[p]_q + \binom{k}{2}(1-q)^2 [p]_q^2 \pmod{[p]_q^3}. \tag{1.13}$$

In [9], He obtained that for any prime $p \geq 5$,

$$H_{p-1,2}(q) \equiv -\frac{(p-1)(p-5)}{12}(1-q)^2 \pmod{[p]_q}, \tag{1.14}$$

$$\widetilde{H}_{p-1,2}(q) \equiv -\frac{p^2-1}{12}(1-q)^2 \pmod{[p]_q}, \tag{1.15}$$

and

$$\begin{aligned} I_{p-1}(q) \quad \equiv \quad & [p]_q \left(Q_p^2(2,q) + Q_p(2,q)(1-q) + \frac{(p^2-1)}{12}(1-q)^2 \right) \\ & -2Q_p(2,q) - \frac{(p-1)(1-q)}{2} \pmod{[p]_q^2}. \end{aligned} \tag{1.16}$$

In [20], combining (1.4) and (1.14), Straub deduced that for any prime $p \geq 5$,

$$\sum_{1 \leq i \leq k \leq p-1} \frac{1}{[i]_q [k]_q} \equiv \frac{(p^2-1)(1-q)^2}{12} \pmod{[p]_q}. \tag{1.17}$$

Recently, some authors have investigated combinatorial and arithmetical properties of the binomial sums and $q-$analogues of these sums([4–6, 8, 10, 12, 13, 17, 18]).
In [3], Cai and Granville gave the following congruences: For any prime $p \geq 5$,

$$\sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k}^n \equiv \begin{cases} \binom{np-2}{p-1} \pmod{p^4} & \text{if } n \text{ is odd}, \\ 2^{n(p-1)} \pmod{p^3} & \text{if } n \text{ is even}. \end{cases} \tag{1.18}$$

In [17], Pan showed the generalization of Carlitz's congruence that for an odd prime p and any positive integer n,

$$\sum_{k=0}^{p-1} (-1)^{(n-1)k} \binom{p-1}{k}^n \equiv 2^{n(p-1)} + \frac{n(n-1)(3n-4)}{48} p^2 B_{p-3} \pmod{p^4},$$

where B_n is the nth Bernoulli number.
In [12], Liu et al. established the generalization of (1.18) proved by Cai and Granville as follows: For any positive odd integer n and positive integer a,

$$\sum_{k=0}^{n-1} (-1)^{(a-1)k} q^{a\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^a \equiv (-q;q)_{n-1}^{2a} + \frac{a(a-1)(n^2-1)}{24}(1-q)^2 [n]_q^2 \pmod{\Phi_n(q)^3},$$

where $\Phi_n(q)$ is the nth cyclotomic polynomial.
In [4], Cai and Shen obtained that for any prime $p \geq 7$ and nonnegative integer l,

$$\sum_{k=lp}^{(l+1)p-1} \binom{\alpha p-1}{k}^n \equiv \begin{cases} \binom{\alpha-1}{l}^n 2^{\alpha n(p-1)} \pmod{p^3} & \text{if } 2 \nmid n, \\ \binom{\alpha-1}{l}^n \binom{\alpha np-2}{p-1} \pmod{p^4} & \text{if } 2 \mid n, \end{cases}$$

where α and n are positive integers.
In [1], Abel's partial summation formula asserts that for every pair of families $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ of complex numbers, there is the relation

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) \sum_{j=1}^{k} b_j + a_n \sum_{j=1}^{n} b_j. \tag{1.19}$$

Motivated by the works mentioned, we mainly obtain the following results in this paper.

Theorem 1.1 *Let p be an odd prime and α be positive integer. For any integer n,*

$$\begin{aligned} &\sum_{k=1 \pmod 2}^{p-1} (-1)^{nk} \frac{q^{-\alpha npk+n\binom{k+1}{2}+2k}}{[k]_q} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \\ \equiv\ & Q_p(2,q) + \frac{1-q}{[2]_q} - \alpha n\, [p]_q \Bigg(\frac{(\alpha n)^{-1}-1}{2} Q_p^2(2,q) \\ & - \left(\frac{1}{[2]_q} + \frac{p-(\alpha n)^{-1}}{2} \right) Q_p(2,q)(1-q) - (1-q) \Bigg(\frac{5p^2(1-q)+17q+19}{24} - \frac{p}{2}q - \frac{113q+17}{48[2]_q^2} \\ & - \frac{7p^2(1-q^2)-24p(1+q^2)+31q^2}{48[2]_q} - \frac{(\alpha n)^{-1}(p^2(1-q^2)+q^2+47)}{48[2]_q} \Bigg) \Bigg) \pmod{[p]_q^2}. \end{aligned}$$

Theorem 1.2 *Let p be an odd prime and α be positive integer. For any integer n,*

$$\begin{aligned} &\sum_{k=1 \pmod 2}^{p-1} (-1)^{nk} q^{-\alpha npk+n\binom{k+1}{2}+k} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \widetilde{H}_k(q) \\ \equiv\ & 1 - \left(1 - \frac{q}{[2]_q} + \frac{1-q}{[2]_q^2} \frac{(p(p(1-q)+8)(q+1)+q(q-24)-9)}{16} \right) [p]_q \\ & + \frac{1}{[2]_q} \left(Q_p(2,q) - 1 + \frac{1-q}{2[2]_q} (p[2]_q + 1 - q) \right. \\ & + n\alpha [p]_q \left(-2q + 2 \left(\frac{1-q}{2[2]_q} (p[2]_q - 2q) - \frac{(n\alpha)^{-1}(1-q)}{4} \right) Q_p(2,q) \right. \end{aligned}$$

$$+\left(1-\frac{(n\alpha)^{-1}}{2}\right)Q_p^2(2,q)+\frac{1}{[2]_q}\left(4q^2\left(q^{-1}-[p]_q-\frac{q^2+2q-1}{[2]_q}\right.\right.$$
$$\left.+\frac{(q-1)\left(2p^2\left(q^2-1\right)-q^2\left(3p-25\right)+9q\left(p+3\right)+2\right)}{6}\right)$$
$$\left.\left.+[2]_q\left(\frac{p}{2}\left(1-q\right)+\frac{3q+1}{2}\right)\right)\right)\pmod{[p]_q^2}.$$

2. Proofs of Theorems 1.1 and 1.2

In this section, we will start with some lemmas and then derive our results about congruences.

Lemma 2.1 *Let α and k be any positive integers. For any prime p and any integer n,*

$$\begin{bmatrix}\alpha p-1\\ k\end{bmatrix}_q^n\equiv(-1)^{nk}\,q^{\alpha npk-n\binom{k+1}{2}}\left(1-\alpha n\,[p]_q\,\widetilde{H}_k(q)\right)\pmod{[p]_q^2}.$$

Proof By (1.2) and Binomial Theorem, the proof is clear. □

Lemma 2.2 *For any positive integer n, we have*

$$\sum_{k=1}^{n}\frac{q^{4k}}{[2k]_q}=(1-q)\left(1-n-[n+1]_{q^2}\right)+\frac{I_{2n}(q)+H_{2n}(q)}{2}.$$

Proof By equality $\frac{q^{4k}}{[2k]_q}=\frac{1}{[2k]_q}-(1-q)\left(q^{2k}+1\right)$, we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{q^{4k}}{[2k]_q}&=\sum_{k=1}^{n}\frac{1}{[2k]_q}-(1-q)\left(\sum_{k=1}^{n}q^{2k}+n\right)\\&=\frac{1}{2}\sum_{k=1}^{2n}\frac{1+(-1)^k}{[k]_q}-(1-q)\left(\frac{1-q^{2n+2}}{1-q^2}+n-1\right)\\&=\frac{1}{2}\left(\sum_{k=1}^{2n}\frac{1}{[k]_q}+\sum_{k=1}^{2n}\frac{(-1)^k}{[k]_q}\right)-(1-q)\left(\frac{1-q^{2n+2}}{1-q^2}+n-1\right)\\&=\frac{I_{2n}(q)+H_{2n}(q)}{2}-(1-q)\left([n+1]_{q^2}+n-1\right).\end{aligned}$$

Thus, we have the proof. □

Corollary 2.3 *For an odd prime p, we have*

$$\begin{aligned}\sum_{\substack{k=1\\ \pmod 2}}^{p-1}\frac{q^{2k}}{[k]_q}&\equiv Q_p(2,q)+\frac{1-q}{[2]_q}-\frac{Q_p^2(2,q)}{2}[p]_q\\&\quad+[p]_q\,(1-q)\left(\frac{1}{48}\left(1-p^2\right)(1-q)-\frac{Q_p(2,q)}{2}-\frac{1}{[2]_q}\right)\pmod{[p]_q^2}.\end{aligned}$$

Proof Taking $p-1$ replace of n in (1.6) and $(p-1)/2$ replace of n in Lemma 2.2, by (1.4), (1.5), (1.13) and (1.16), we have

$$\sum_{k=1}^{p-1} \frac{q^{2k}}{[k]_q} \equiv (1-q)\left(\frac{3-p}{2} + \left(\frac{(1-q)\left(p^2-1\right)}{24} - 1\right)[p]_q\right) \pmod{[p]_q^2}, \tag{2.1}$$

and

$$\begin{aligned}\sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} &\equiv -Q_p(2,q) - (1-q)\left(\frac{1}{1+q} + \frac{p-3}{2}\right) + \frac{Q_p^2(2,q)}{2}[p]_q \\ &+ \frac{[p]_q}{2}(1-q)\left(Q_p(2,q) + \frac{1}{8}\left(p^2-1\right)(1-q) - \frac{2q}{q+1}\right) \pmod{[p]_q^2},\end{aligned} \tag{2.2}$$

respectively. From (1.4) and (1.16), we get the result. □

Lemma 2.4 *For an odd prime p, we have*

$$\sum_{1\le i\le k\le p-1} \frac{q^{2k+i}}{[i]_q[k]_q} \equiv \frac{\left(p^2-1\right)(1-q)^2}{12} + q - 1 \pmod{[p]_q}.$$

Proof By using (1.5), (1.9), (1.13), and (1.15), the desired result is obtained. □

Lemma 2.5 *For an odd prime p, we have*

$$\begin{aligned}\sum_{k=0}^{(p-1)/2} \frac{1}{[2k+1]_q} &\equiv \frac{p-1}{2}(1-q) + Q_p(2,q) + \frac{1}{[p]_q} \\ &- \frac{1}{2}\left(Q_p^2(2,q) + Q_p(2,q)(1-q) + \frac{1}{24}\left(p^2-1\right)(1-q)^2\right)[p]_q \pmod{[p]_q^2},\end{aligned} \tag{2.3}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{1}{[2k+1]_q^2} \equiv Q_p(2,q)(1-q) + \frac{1}{[p]_q^2} - \frac{(p-1)(p-11)}{24}(1-q)^2 \pmod{[p]_q}. \tag{2.4}$$

Proof Consider that

$$H_{p-1}(q) = \sum_{k=0}^{(p-1)/2} \frac{1}{[2k+1]_q} + \sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} - \frac{1}{[p]_q},$$

and

$$H_{p-1,2}(q) = \sum_{k=0}^{(p-1)/2} \frac{1}{[2k+1]_q^2} + \sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q^2} - \frac{1}{[p]_q^2}.$$

From here, by (1.4), (1.10) and by (1.12), (1.14), respectively, we have the proofs of congruences. □

Lemma 2.6 *For any prime $p \geq 5$, we have*

$$\sum_{1\leq i\leq k\leq (p-1)/2} \frac{1}{[2k]_q\,[2i]_q} \equiv \frac{1}{2}\left(Q_p^2(2,q) - Q_p(2,q)(1-q) - \frac{p^2-1}{24}(1-q)^2\right) \pmod{[p]_q}, \tag{2.5}$$

and

$$\sum_{1\leq i\leq k\leq (p-1)/2} \frac{1}{[2k+1]_q[2i]_q} \equiv -\frac{1}{2}Q_p^2(2,q) + \left(\frac{1}{2}(2-p)(1-q) - \frac{1}{[p]_q}\right)Q_p(2,q) + \frac{1}{48}\left(p^2-1\right)(1-q)^2 \pmod{[p]_q}. \tag{2.6}$$

Proof By exchanging sums, we have

$$\begin{aligned}\sum_{1\leq i\leq k\leq (p-1)/2} \frac{1}{[2k]_q\,[2i]_q} &= \sum_{i=1}^{(p-1)/2}\frac{1}{[2i]_q}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q} - \sum_{i=1}^{(p-1)/2}\frac{1}{[2i]_q}\sum_{k=1}^{i-1}\frac{1}{[2k]_q}\\ &= \sum_{i=1}^{(p-1)/2}\frac{1}{[2i]_q}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q} - \sum_{1\leq i\leq k\leq (p-1)/2}\frac{1}{[2k]_q\,[2i]_q} + \sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q^2}.\end{aligned}$$

From here, we get

$$\sum_{1\leq i\leq k\leq (p-1)/2} \frac{1}{[2k]_q\,[2i]_q} = \frac{1}{2}\left(\left(\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}\right)^2 + \sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q^2}\right).$$

By (1.10) and (1.12), the proof of the first congruence is finished.
Observe that

$$\begin{aligned}&\sum_{1\leq i\leq k\leq (p-1)/2} \frac{1}{[2k]_q[2i+1]_q}\\ &= \sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}\left(H_{2k+1}(q) - \sum_{i=1}^{k}\frac{1}{[2i]_q} - 1\right)\\ &= \sum_{k=1}^{(p-1)/2}\frac{H_{2k}(q)}{[2k]_q} + \sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q[2k+1]_q} - \sum_{1\leq i\leq k\leq (p-1)/2}\frac{1}{[2k]_q[2i]_q} - \sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q},\end{aligned}$$

and

$$\begin{aligned}&\sum_{1\leq i\leq k\leq (p-1)/2} \frac{1}{[2k+1]_q[2i]_q}\\ &= \sum_{i=1}^{(p-1)/2}\frac{1}{[2i]_q}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q} - \sum_{1\leq i\leq k\leq (p-1)/2}\frac{1}{[2k]_q[2i+1]_q} + \sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q[2k+1]_q}.\end{aligned}$$

By combining these equations, we write

$$\sum_{1\le i\le k\le (p-1)/2} \frac{1}{[2k+1]_q[2i]_q} \tag{2.7}$$

$$= \sum_{i=1}^{(p-1)/2} \frac{1}{[2i]_q} \sum_{k=1}^{(p-1)/2} \frac{1}{[2k+1]_q} + \sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} - \sum_{k=1}^{(p-1)/2} \frac{H_{2k}(q)}{[2k]_q} + \sum_{1\le i\le k\le (p-1)/2} \frac{1}{[2k]_q\,[2i]_q}.$$

Note that

$$2\sum_{k=1}^{(p-1)/2} \frac{H_{2k}(q)}{[2k]_q} = \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} H_k(q) + \sum_{1\le i\le k\le p-1} \frac{1}{[i]_q[k]_q}.$$

(1.11) and (1.17) yield that

$$\sum_{k=1}^{(p-1)/2} \frac{H_{2k}(q)}{[2k]_q} \equiv \frac{1}{2}\left(Q_p^2(2,q) - (1-q)Q_p(2,q) + \frac{1}{24}\left(p^2-1\right)(1-q)^2\right) \pmod{[p]_q}. \tag{2.8}$$

Similarly, using (1.10), (2.5) and (2.8) in (2.7), the proof of (2.6) is obtained. Thus, we complete the proof of Lemma (2.6). □

Lemma 2.7 *For an odd prime p, we have*

$$\sum_{k=1}^{(p-1)/2} \frac{k}{[2k]_q} \equiv -\frac{1}{4}\left(\frac{1}{4}(p-1)^2(1-q) + pQ_p(2,q) + \frac{p-(q^{1-p};q)_{p-1}}{[p]_q}\right) \pmod{[p]_q}, \tag{2.9}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{k}{[2k+1]_q} \equiv \frac{1}{4}(p-2)Q_p(2,q) - \frac{2-p-(q^{1-p};q)_{p-1}}{4[p]_q} + \frac{1}{16}(1-q)\left(p^2-6p+5\right) \pmod{[p]_q}.$$

Proof Observe that

$$\sum_{k=1}^{p-1} \frac{k}{[k]_q}$$

$$= 2\sum_{k=1}^{(p-1)/2} \frac{k}{[2k]_q} + 2\sum_{k=1}^{(p-1)/2} \frac{k}{[2k+1]_q} + \sum_{k=0}^{(p-1)/2} \frac{1}{[2k+1]_q} - \frac{p}{[p]_q}$$

$$= 2\sum_{k=1}^{(p-1)/2} \frac{k}{[2k]_q} + 2\left(\frac{(p-1)}{2}\sum_{k=0}^{(p-3)/2} \frac{1}{[p-2k]_q} - \sum_{k=0}^{(p-3)/2} \frac{k}{[p-2k]_q}\right) + \sum_{k=0}^{(p-1)/2} \frac{1}{[2k+1]_q} - \frac{p}{[p]_q}$$

$$= 2\left(\sum_{k=1}^{(p-1)/2} \frac{k}{[2k]_q} - \sum_{k=0}^{(p-1)/2} \frac{k}{[p-2k]_q}\right) + (p-1)\sum_{k=0}^{(p-3)/2} \frac{1}{[p-2k]_q} + \sum_{k=1}^{(p-1)/2} \frac{1}{[2k+1]_q} - \frac{p}{[p]_q} + p.$$

By using the congruence $[p-2k]_q \equiv -q^{-2k}[2k]_q \pmod{[p]_q}$ for $1 \le k \le (p-1)/2$, we get

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{k}{[k]_q} &\equiv 2\left(\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q}+\sum_{k=0}^{(p-1)/2}\frac{q^{2k}k}{[2k]_q}\right)-(p-1)\sum_{k=1}^{(p-3)/2}\frac{q^{2k}}{[2k]_q}\\ &+\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q}-\frac{1}{[p]_q}+p \pmod{[p]_q}.\end{aligned}$$

With the help of $\frac{q^{2k}}{[2k]_q}=\frac{1}{[2k]_q}-(1-q)$, we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{k}{[k]_q} &\equiv 4\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q}-p\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}+\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}\\ &+\sum_{k=1}^{(p-3)/2}\frac{1}{[2k+1]_q}+\frac{p-1}{[p-1]_q}+p+\frac{(p-1)(p-7)}{4}(1-q)\\ &=4\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q}-p\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}+\sum_{k=1}^{p-1}\frac{1}{[k]_q}\\ &+\frac{p-1}{[p-1]_q}+p-1+\frac{(p-1)(p-7)}{4}(1-q) \pmod{[p]_q}.\end{aligned}$$

Thus, by (1.4), we get

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q} &\equiv \frac{1}{4}\left(p\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}+\sum_{k=1}^{p-1}\frac{k}{[k]_q}-\frac{p-1}{[p-1]_q}\right.\\ &\left.-\frac{1}{4}(p-1)(p+5q-pq-1)\right) \pmod{[p]_q}.\end{aligned} \tag{2.10}$$

Note that, using $a_k=\frac{1}{[k]_q}$ and $b_k=k$ in (1.19), by (1.3) and (1.4), we have

$$\sum_{k=1}^{p-1}\frac{k}{[k]_q}\equiv -\frac{p-\left(q^{1-p};q\right)_{p-1}}{[p]_q} \pmod{[p]_q}. \tag{2.11}$$

From here, (2.10) yields that

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q} &\equiv \frac{1}{4}\left(p\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}-\frac{p-\left(q^{1-p};q\right)_{p-1}}{[p]_q}-\frac{p-1}{[p-1]_q}\right.\\ &\left.-\frac{1}{4}(p-1)(p+5q-pq-1)\right) \pmod{[p]_q}.\end{aligned}$$

Finally, by (1.1) and (1.10), the first congruence is obtained.

Similarly, for the last congruence, consider that

$$\sum_{k=1}^{p-1}\frac{k}{[k]_q}=2\left(\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q}+\sum_{k=1}^{(p-1)/2}\frac{k}{[2k+1]_q}\right)-\frac{p}{[p]_q}+1.$$

Then, by (2.9) and (2.11), the congruence is obtained. Thus, the proof of Lemma (2.7) is complete. □

Corollary 2.8 *For an odd prime p, we have*

$$\sum_{1\le i\le k\le (p-1)/2}\frac{1}{[2i]_q}\equiv -\frac{1}{2}\left(\frac{p}{2}+1\right)Q_p(2,q)+\frac{1}{4}\left(\frac{1}{4}(p-1)^2(1-q)+\frac{p-(q^{1-p};q)_{p-1}}{[p]_q}\right)\pmod{[p]_q}.$$

Proof By observing that

$$\sum_{1\le i\le k\le (p-1)/2}\frac{1}{[2i]_q}=\frac{p+1}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}-\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q},$$

and by (1.10) and (2.9), the congruence is obtained. □

Lemma 2.9 *For an odd prime p, we have*

$$\sum_{k=1}^{(p-1)/2}\widetilde{H}_{2k}(q)\equiv\frac{1}{2}\left(\frac{p-\left(q^{1-p};q\right)_{p-1}}{[p]_q}+\frac{1}{2}\left((1-p)(1-q)-2Q_p(2,q)\right)\right)\pmod{[p]_q},\tag{2.12}$$

and

$$\sum_{k=1}^{p-1}q^k\widetilde{H}_k^2(q)\equiv-\frac{p}{2}(1-q)-\frac{1+3q}{2}\pmod{[p]_q}.\tag{2.13}$$

Proof Observe that

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\widetilde{H}_{2k}(q)\\ =&\sum_{1\le i\le k\le p-1}q^i\frac{1+(-1)^k}{[i]_q}=\frac{1}{2}\left(\sum_{1\le i\le k\le p-1}\frac{q^i}{[i]_q}+\sum_{1\le i\le k\le p-1}(-1)^k\frac{q^i}{[i]_q}\right)\\ =&\frac{1}{2}\left(\sum_{1\le i\le k\le p-1}\frac{q^i}{[i]_q}+\sum_{i=1}^{p-1}\frac{q^i}{[i]_q}\left(\sum_{k=1}^{p-1}(-1)^k-\sum_{k=1}^{i-1}(-1)^k\right)\right)\end{aligned}$$

$$= \frac{1}{2}\left(\sum_{1\le i\le k\le p-1}\frac{q^i}{[i]_q}+\widetilde{H}_{p-1}(q)\left(\sum_{k=1}^{p-1}(-1)^k+1\right)-\frac{1}{2}\sum_{k=1}^{p-1}q^k\frac{1-(-1)^k}{[k]_q}\right)$$

$$= \frac{1}{2}\left(\sum_{1\le i\le k\le p-1}\frac{q^i}{[i]_q}+\frac{1}{2}\widetilde{H}_{p-1}(q)+\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-q)^k}{[k]_q}\right).$$

Then, by (1.7), we get

$$\sum_{k=1}^{(p-1)/2}\widetilde{H}_{2k}(q)=\frac{1}{2}\left(\sum_{1\le i\le k\le p-1}\frac{q^i}{[i]_q}+\frac{1}{2}\widetilde{H}_{p-1}(q)+\frac{1}{2}I_{p-1}(q)\right).$$

Finally, using (1.3), (1.5) and (1.16), we have the proof of (2.12).

Similarly, by exchanging sums, using (1.8) and (2.1), the proof of (2.13) is obtained. Thus, we have the proof.
□

Lemma 2.10 *For an odd prime p, we have*

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}q^{2k}\widetilde{H}_{2k}(q)\\
\equiv\ &\frac{1}{[2]_q}\Bigg(-Q_p(2,q)-\frac{2}{[2]_q}+2+\frac{1}{2}(1-p)(1-q)\\
&+\frac{[p]_q}{2}Q_p^2(2,q)+\frac{[p]_q}{2}Q_p(2,q)(1-q)-\frac{q[p]_q}{2}(2+p(1-q))\\
&+\frac{[p]_q}{16[2]_q}\left(p^2(q+1)-9q-1\right)(1-q)^2\Bigg)\pmod{[p]_q^2}.
\end{aligned}$$

Proof By taking with $a^k=q^{2k}$ and $b_k=\widetilde{H}_{2k}(q)$ in (1.19), we have

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}q^{2k}\widetilde{H}_{2k}(q)\\
=\ &\widetilde{H}_{p+1}(q)\sum_{k=1}^{(p-1)/2}q^{2k}+\sum_{k=1}^{(p-1)/2}\left(\widetilde{H}_{2k}(q)-\widetilde{H}_{2k+2}(q)\right)\sum_{i=1}^{k}q^{2i}\\
=\ &\widetilde{H}_{p+1}(q)\sum_{k=1}^{(p-1)/2}q^{2k}+\sum_{k=1}^{(p-1)/2}\left(\widetilde{H}_{2k}(q)-\widetilde{H}_{2k+2}(q)\right)\left(\frac{1-q^{2k+2}}{1-q^2}-1\right)\\
=\ &\widetilde{H}_{p+1}(q)\sum_{k=1}^{(p-1)/2}q^{2k}-\sum_{k=1}^{(p-1)/2}\left(\frac{q^{2k+1}}{[2k+1]_q}+\frac{q^{2k+2}}{[2k+2]_q}\right)\left(\frac{[2k+2]_q}{[2]_q}-1\right).
\end{aligned}$$

Then, by some elementary operations, we get

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}(q) \\
=\ & \left(\widetilde{H}_{p-1}(q)+\frac{q^p}{[p]_q}+\frac{q^{p+1}}{[p+1]_q}\right)\sum_{k=1}^{(p-1)/2} q^{2k}+\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q} \\
&+\frac{q}{[2]_q}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q}-\frac{2q^2}{[2]_q}\sum_{k=1}^{(p-1)/2} q^{2k}-\frac{1}{[2]_q}-\frac{2q+1}{2[2]_q}(1-q)(p-1)+\frac{1}{[p+1]_q}.
\end{aligned}$$

By using (1.5), (1.13) and the congruence $\frac{1}{[p+1]_q}\equiv 1-q[p]_q \pmod{[p]_q^2}$, we have

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}(q) \\
\equiv\ & \frac{q}{[2]_q}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q}+\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}+\frac{1}{2}(1-p)(1-q)+\frac{q^2+2q-1}{[2]_q^2} \\
&-\frac{q}{[2]_q[p]_q}+\frac{1}{[2]_q}+\frac{q}{[2]_q}\left((1-p)\frac{1-q}{2}-\frac{p^2-1}{24}(1-q)^2-\frac{1-q}{[2]_q}\right)[p]_q \pmod{[p]_q^2}.
\end{aligned}$$

Finally, by (1.10) and (2.3), the desired congruence is obtained. □

Lemma 2.11 *For an odd prime p, we have*

$$\begin{aligned}
\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+1]_q} \equiv\ & \frac{1}{2}\Big(-Q_p^2(2,q)+(1-p)(1-q)Q_p(2,q) \\
&+\frac{(p+13)}{24}(p-1)(1-q)^2+(1-q)\frac{1-(q^{1-p};q)_{p-1}}{[p]_q}\Big) \pmod{[p]_q},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+2]_q} \equiv\ & \frac{1}{2}\left(Q_p^2(2,q)-(4+(1-p)(1-q))\,Q_p(2,q)\right) \\
&-\frac{(1-p)(1-q)}{16}(3p(1-q)+q-17)+(1-q)\frac{p-(q^{1-p};q)_{p-1}}{2[p]_q} \pmod{[p]_q}.
\end{aligned}$$

Proof Observe that

$$\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+1]_q}=\sum_{k=0}^{(p-1)/2}\frac{\widetilde{H}_{p-2k-1}(q)}{[p-2k]_q}.$$

Then, from the congruence $\widetilde{H}_{p-k}(q) \equiv \widetilde{H}_{p-1}(q) + H_{k-1}(q) \pmod{[p]_q}$ for $1 \leq k \leq p$ and some elementary operations, (1.1) yields that

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} \frac{\widetilde{H}_{2k}(q)}{[2k+1]_q} \\
\equiv \quad & -\sum_{k=1}^{(p-1)/2} \frac{H_{2k}(q)}{[2k]_q} + (1-q)\left(\sum_{k=1}^{(p-1)/2} \widetilde{H}_{2k}(q) + (1-q)\frac{p^2-1}{4}\right) + \frac{q^{p-1}}{[p-1]_q} H_{p-1}(q) \\
& + \widetilde{H}_{p-1}(q)\left((1-q)^2\frac{p^2-1}{4} + \frac{1}{[p]_q} - \sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} + (1-q)\frac{p-1}{2} + \frac{q^{p-1}}{[p-1]_q}\right) \pmod{[p]_q}.
\end{aligned}$$

By (1.1), (1.4) and (1.5), we obtain

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} \frac{\widetilde{H}_{2k}(q)}{[2k+1]_q} \\
\equiv \quad & -\sum_{k=1}^{(p-1)/2} \frac{H_{2k}(q)}{[2k]_q} + (1-q)\sum_{k=1}^{(p-1)/2} \widetilde{H}_{2k}(q) - \frac{p-1}{2[p]_q}(1-q) \\
& + (p-1)\frac{1-q}{2}\sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} + \frac{(p+13)(p-1)}{24}(1-q)^2 \pmod{[p]_q}.
\end{aligned}$$

Finally, using (1.10), (2.12) and (2.8), the first congruence is obtained.

For the second congruence, observe that

$$\sum_{k=1}^{(p-1)/2} \frac{\widetilde{H}_{2k}(q)}{[2k+2]_q} = \sum_{k=2}^{(p+1)/2} \frac{\widetilde{H}_{2k-2}(q)}{[2k]_q}.$$

From here, by some elementary operations, we find that

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} \frac{\widetilde{H}_{2k}(q)}{[2k+2]_q} \\
= \quad & \sum_{k=1}^{(p-1)/2} \frac{H_{2k}(q)}{[2k]_q} - 2(1-q)\sum_{k=1}^{(p-1)/2} \frac{k}{[2k]_q} - \sum_{k=1}^{(p-1)/2} \frac{1}{[2k+1]_q} - \sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q^2} \\
& + (2-q)\sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} - \frac{1}{[p+1]_q^2} + \frac{\widetilde{H}_{p+1}(q)}{[p+1]_q} + \frac{2-q}{[p+1]_q} - 1,
\end{aligned}$$

and using (1.10), (1.12), (2.8) and (2.9),

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} \frac{\widetilde{H}_{2k}(q)}{[2k+2]_q} \\
\equiv\ & \frac{1}{2}Q_p(2,q)\left(Q_p(2,q)+p\left(1-q\right)+q-3\right)+\left(1-q\right)\frac{p-\left(q^{1-p};q\right)_{p-1}}{2[p]_q}-\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q} \\
&-\frac{1}{[p+1]_q^2}+\frac{\widetilde{H}_{p+1}(q)}{[p+1]_q}+\frac{(2-q)}{[p+1]_q}-1+\frac{1}{16}\left(1-q\right)^2\left(3p^2-4p+1\right) \pmod{[p]_q}.
\end{aligned}$$

Hence, by (1.5), (1.13), (2.3) and the congruence $\frac{1}{[p+1]_q} \equiv 1-q[p]_q \pmod{[p]_q^2}$, the proof of the second congruence is obtained. Thus, we have the proof. □

Lemma 2.12 *For an odd prime p, we have*

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}^2(q) \\
\equiv\ & \frac{1}{1-q^2}\Bigg(Q_p(2,q)\left(\frac{-3q^3+5q^2+3q+3}{[2]_q}+\left(p+1\right)\left(1-q\right)^2-4\right) \\
&+(1-q)Q_p^2(2,q)-2q\left(1-q\right)-\frac{4q^2\left(1-q\right)}{[2]_q}\left(\frac{q^2+2q-1}{[2]_q}-q^{-1}+[p]_q\right) \\
&-\left(1-q\right)^2\frac{\left(2p^2\left(q^2-1\right)-q^2\left(3p-25\right)+9q\left(p+3\right)+2\right)}{6[2]_q}\Bigg) \pmod{[p]_q}.
\end{aligned}$$

Proof Observe that

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}^2(q)+q^{p+1}\widetilde{H}_{p+1}^2(q) \\
=\ & \sum_{k=1}^{(p+1)/2} q^{2k}\widetilde{H}_{2k}^2(q)=\sum_{k=0}^{(p-1)/2} q^{2k+2}\widetilde{H}_{2k+2}^2(q) \\
=\ & \sum_{k=0}^{(p-1)/2} q^{2k+2}\left(\widetilde{H}_{2k}(q)+\frac{q^{2k+1}}{[2k+1]_q}+\frac{q^{2k+2}}{[2k+2]_q}\right)^2 \\
=\ & 2\left(\sum_{k=0}^{(p-1)/2} q^{2k+2}\frac{q^{2k+1}}{[2k+1]_q}+\sum_{k=0}^{(p-1)/2} q^{2k+2}\frac{q^{2k+2}}{[2k+2]_q}\right)\widetilde{H}_{2k}(q) \\
&+\sum_{k=1}^{(p-1)/2} q^{2k+2}\widetilde{H}_{2k}^2(q)+q^2\sum_{k=0}^{(p-1)/2} q^{2k+2}\frac{q^{4k}}{[2k+1]_q^2}
\end{aligned}$$

$$+2\sum_{k=0}^{(p-1)/2} q^{2k+2}\frac{q^{4k+3}}{[2k+1]_q[2k+2]_q}+\sum_{k=0}^{(p-1)/2} q^{2k+2}\frac{q^{4k+4}}{[2k+2]_q^2}.$$

Then, by some elementary operations, we get

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}^2(q)\\
=\;&\frac{1}{1-q^2}\Bigg(2q\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+1]_q}+2\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+2]_q}+\sum_{k=1}^{(p+1)/2}\frac{1}{[2k]_q^2}\\
&-4q^2(1-q)\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}(q)+q^2(1-q)^2\left(3+q^{-2}\right)\sum_{k=0}^{(p-1)/2} q^{2k}\\
&+q\sum_{k=0}^{(p-1)/2}\frac{1}{[2k+1]_q^2}+\left(2q^2-3q(1-q)\right)\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q}+q(5q-3)\\
&-(5-3q)\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}-2(1-q^2)\sum_{k=1}^{(p-1)/2}\widetilde{H}_{2k}(q)-\frac{5-3q}{[p+1]_q}\\
&+\left(q^{p+1}-1\right)(1-q)^2+2(p+1)(1-q)^2(q+1)-q^{p+1}\widetilde{H}_{p+1}(q)\Bigg).
\end{aligned}$$

(1.10), (1.12), (2.3) and the congruence $\frac{1}{[p+1]_q}\equiv 1-q[p]_q \pmod{[p]_q^2}$ yield that

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}^2(q)\\
\equiv\;&\frac{1}{1-q^2}\Bigg(2q\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+1]_q}-4q^2(1-q)\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}(q)\\
&+2\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+2]_q}-2(1-q^2)\sum_{k=1}^{(p-1)/2}\widetilde{H}_{2k}(q)+q\sum_{k=0}^{(p-1)/2}\frac{1}{[2k+1]_q^2}\\
&+\left(5q^2-5q+4\right)Q_p(2,q)+\left(2q^2-3q(1-q)\right)\frac{1}{[p]_q}\\
&+\frac{(1-q)^2}{[2]_q}\left(2p\left(1+q^2\right)+4q(p+1)+5q^2+3\right)\\
&+\left(q^{p+1}-1\right)(1-q)^2+3q-4-q^{p+1}\widetilde{H}_{p+1}(q)\\
&+\frac{(p(1-q)+(35-60q)q+1)}{24}(1-p)(1-q)\Bigg)\pmod{[p]_q}.
\end{aligned}$$

By (2.4) and (2.12), we obtain

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}^2(q) \\
\equiv\ & \frac{1}{1-q^2}\Bigg(-4q^2(1-q)\sum_{k=1}^{(p-1)/2} q^{2k}\widetilde{H}_{2k}(q) - 2(1-q^2)\frac{p-\left(q^{1-p};q\right)_{p-1}}{2[p]_q} \\
&+2\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+2]_q} + 2q\sum_{k=1}^{(p-1)/2}\frac{\widetilde{H}_{2k}(q)}{[2k+1]_q} + \left(3q^2-4q+5\right)Q(p) \\
&+\left(q^{p+1}-1\right)(1-q)^2 + \left(2q^2-3q(1-q)+\frac{q}{[p]_q}\right)\frac{1}{[p]_q} \\
&+(1-q)^2\,\frac{2p\left(1+q^2\right)+4q(p+1)+5q^2+3}{[2]_q} - q^{p+1}\widetilde{H}_{p+1}(q) \\
&+\frac{(1-q)^2}{[2]_q}\left(2p\left(1+q^2\right)+4q(p+1)+5q^2+3\right)+3q-4 \\
&-\frac{\left(p\left(q^2-1\right)-(24-37q)\,q+11\right)}{24}(1-p)(1-q)\Bigg) \pmod{[p]_q}.
\end{aligned}$$

By Lemmas 2.10 and 2.11, this completes the proof. □

Lemma 2.13 *For any prime $p\geq 5$, we have*

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q}\widetilde{H}_{2k}(q) \\
\equiv\ & \frac{Q_p^2(2,q)}{2} + (1-q)\Bigg(\left(\frac{1}{[2]_q}+\frac{p}{2}\right)Q_p(2,q) + \frac{7\left(p^2-1\right)(1-q)}{24} \\
&-\frac{q(p-1)}{2} - \frac{113q+17}{48\,[2]_q^2} - \frac{7p^2\left(1-q^2\right)-24p\left(1+q^2\right)+31q^2}{48\,[2]_q}\Bigg) \pmod{[p]_q}.
\end{aligned}$$

Proof By taking with $a_k=\frac{q^{4k}}{[2k]_q}$ and $b_k=\widetilde{H}_{2k}(q)$ in (1.19), we get

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q}\widetilde{H}_{2k}(q) \\
=\ & \widetilde{H}_{p+1}(q)\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q} + \sum_{k=1}^{(p-1)/2}\left(\widetilde{H}_{2k}(q)-\widetilde{H}_{2k+2}(q)\right)\sum_{i=1}^{k}\frac{q^{4i}}{[2i]_q} \\
=\ & \widetilde{H}_{p+1}(q)\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q} - \sum_{1\leq i\leq k\leq (p-1)/2}\frac{q^{2k+1}}{[2k+1]_q}\frac{q^{4i}}{[2i]_q} - \sum_{1\leq i\leq k\leq (p-1)/2}\frac{q^{2k+2}}{[2k+2]_q}\frac{q^{4i}}{[2i]_q}.
\end{aligned}$$

Then, with the help of Lemma 2.2, we write

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} \widetilde{H}_{2k}(q) \\
&= \widetilde{H}_{p+1}(q) \sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} + (1-q) \left(\frac{q^2}{q+1} \sum_{k=1}^{(p-1)/2} \frac{q^{2k+1}}{[2k+1]_q} [2k]_q + \sum_{k=1}^{(p-1)/2} \frac{q^{2k+1}}{[2k+1]_q} k \right) \\
&- \sum_{1 \le i \le k \le (p-1)/2} \frac{q^{2k+1}}{[2k+1]_q [2i]_q} - \sum_{1 \le i \le k \le (p-1)/2} \frac{q^{2k+2}}{[2k+2]_q [2i]_q} \\
&+ (1-q) \left(\frac{q^2}{q+1} \sum_{k=1}^{(p-1)/2} \frac{q^{2k+2}}{[2k+2]_q} [2k]_q + \sum_{k=1}^{(p-1)/2} \frac{q^{2k+2}}{[2k+2]_q} k \right),
\end{aligned}$$

and from the definition of $[n]_q$, equals

$$\begin{aligned}
&\widetilde{H}_{p+1}(q) \sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} - \sum_{1 \le i \le k \le (p-1)/2} \frac{1}{[2k+1]_q [2i]_q} \\
&- \sum_{1 \le i \le k \le (p-1)/2} \frac{1}{[2k+2]_q [2i]_q} + 2(1-q) \sum_{1 \le i \le k \le (p-1)/2} \frac{1}{[2i]_q} \\
&+ (1-q) \left(\frac{q^2}{[2]_q} \left(2 \sum_{k=1}^{(p-1)/2} q^{2k} - \frac{(1-p)(1-q)}{2q^2} (2q+1) \right.\right. \\
&\left. - q^{-1} \sum_{k=1}^{(p-1)/2} \frac{1}{[2k+1]_q} \right) - \sum_{k=1}^{(p-1)/2} \frac{1}{[2k+2]_q} \\
&\left. + \sum_{k=1}^{(p-1)/2} \frac{k}{[2k+1]_q} + \sum_{k=1}^{(p-1)/2} \frac{k}{[2k+2]_q} - 2(1-q) \sum_{k=1}^{(p-1)/2} k \right).
\end{aligned}$$

By using some elementary operations, we have

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} \widetilde{H}_{2k}(q) \\
= \;& \widetilde{H}_{p+1}(q) \sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} - \sum_{1 \le i \le k \le (p-1)/2} \frac{1}{[2k+1]_q [2i]_q} + \sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q^2} \\
&- \left(\frac{1}{[p+1]_q} + (1-q)(1-p) \right) \sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} - \sum_{1 \le i \le k \le (p-1)/2} \frac{1}{[2k]_q [2i]_q}
\end{aligned}$$

$$-(q-1)\left(\sum_{k=1}^{(p-1)/2}\frac{k}{[2k+1]_q}-\sum_{k=1}^{(p-1)/2}\frac{k}{[2k]_q}+\frac{p-3}{2[p+1]_q}\right.$$
$$+\frac{q^2}{[2]_q}\left(2\sum_{k=1}^{(p-1)/2}q^{2k}-q^{-1}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q}\right)$$
$$\left.+\frac{1}{2\,[2]_q}\left(p+1+(p-1)\,q\,(1-2q)\right)+(1-q)\frac{1-p^2}{4}\right).$$

With the help of Lemmas 2.6 and 2.7, we get

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q}\widetilde{H}_{2k}(q)\\
\equiv\ &\widetilde{H}_{p+1}(q)\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q}-Q_p(2,q)\left((1-p)\,(1-q)-\frac{1}{[p]_q}\right)\\
&-\left(\frac{1}{[p+1]_q}+(1-q)\,(1-p)\right)\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q}+\sum_{k=1}^{(p-1)/2}\frac{1}{[2k]_q^2}\\
&+(1-q)\left(\frac{q^2}{[2]_q}\left(2\sum_{k=1}^{(p-1)/2}q^{2k}-\frac{1}{q}\sum_{k=1}^{(p-1)/2}\frac{1}{[2k+1]_q}\right)\right.\\
&\qquad+\frac{p-1}{2\,[p]_q}+\frac{1}{2\,[2]_q}\left(p+1+q\,(p-1)\,(1-2q)\right)\\
&\qquad\left.-\frac{1}{8}\,(1-q)\left(p^2+4p-5\right)+\frac{p-3}{2}\right)\pmod{[p]_q}.
\end{aligned}$$

(1.10), (1.12) and (2.3) yield that

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q}\widetilde{H}_{2k}(q)\\
\equiv\ &\widetilde{H}_{p+1}(q)\sum_{k=1}^{(p-1)/2}\frac{q^{4k}}{[2k]_q}+Q_p(2,q)\left(\frac{1}{[p]_q}+2\frac{q^2}{[2]_q}\right)\\
&-(q-1)\left(\frac{q^2}{[2]_q}\left(2\sum_{k=1}^{(p-1)/2}q^{2k}-q^{-1}H_{p-1}(q)\right)+\left(\frac{p-1}{2}-\frac{q}{[2]_q}\right)\frac{1}{[p]_q}\right.\\
&\left.-\frac{q^2\,(p-1)}{[2]_q}+\frac{2\,(2q+1)-p\,(1-q)}{6}\,(p-1)\right)\pmod{[p]_q}.
\end{aligned}$$

Finally, by using (1.4), (1.5), (1.13) and (2.2), we complete the required congruence. □

Proof of Theorem 1.1. Observe that by Lemma 2.1

$$\begin{aligned}
&\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} (-1)^{nk} \frac{q^{-\alpha npk+n\binom{k+1}{2}+2k}}{[k]_q} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \\
\equiv &\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} \frac{q^{2k}}{[k]_q} - \alpha n\, [p]_q \sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} \frac{q^{2k}}{[k]_q} \widetilde{H}_k(q) \\
= &\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} \frac{q^{2k}}{[k]_q} - \alpha n\, [p]_q \left(\sum_{k=1}^{p-1} \frac{q^{2k}}{[k]_q} \widetilde{H}_k(q) - \sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} \widetilde{H}_{2k}(q) \right) \pmod{[p]_q^2}.
\end{aligned}$$

Then, Corollary 2.3 yields that

$$\begin{aligned}
&\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} (-1)^{nk} \frac{q^{-\alpha npk+n\binom{k+1}{2}+2k}}{[k]_q} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \\
\equiv\ & Q_p(2,q) + \frac{1-q}{[2]_q} + [p]_q \left(-\frac{1}{2} \left(Q_p^2(2,q) + Q_p(2,q)(1-q) \right) \right. \\
&\left. - \frac{1-q}{[2]_q} - \frac{1}{48} \left(p^2-1\right)(1-q)^2 \right) - \alpha n\, [p]_q \left(\sum_{k=1}^{p-1} \frac{q^{2k}}{[k]_q} \widetilde{H}_k(q) - \sum_{k=1}^{(p-1)/2} \frac{q^{4k}}{[2k]_q} \widetilde{H}_{2k}(q) \right) \pmod{[p]_q^2}.
\end{aligned}$$

Finally, by using Lemmas 2.4 and 2.13, the desired congruence is obtained.

Proof of Theorem 1.2. Observe that by Lemma 2.1

$$\begin{aligned}
&\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} (-1)^{nk} q^{-\alpha npk+n\binom{k+1}{2}+k} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \widetilde{H}_k(q) \\
\equiv &\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} q^k \widetilde{H}_k(q) - n\alpha [p]_q \sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} q^k \widetilde{H}_k^2(q) \\
= &\sum_{k=1}^{p-1} q^k \widetilde{H}_k(q) - \sum_{k=1}^{(p-1)/2} q^{2k} \widetilde{H}_{2k}(q) - n\alpha [p]_q \left(\sum_{k=1}^{p-1} q^k \widetilde{H}_k^2(q) - \sum_{k=1}^{(p-1)/2} q^{2k} \widetilde{H}_{2k}^2(q) \right) \pmod{[p]_q^2}.
\end{aligned}$$

Then, by (1.5), (1.8) and Lemma 2.10, we obtain that

$$\begin{aligned}
&\sum_{\substack{k=1 \\ (\text{mod } 2)}}^{p-1} (-1)^{nk} q^{-\alpha npk+n\binom{k+1}{2}+k} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q^n \widetilde{H}_k(q) \\
\equiv\ & 1 + \frac{Q_p(2,q)}{[2]_q} - [p]_q \left(\frac{1}{2[2]_q} \left(Q_p^2(2,q) + Q_p(2,q)(1-q) \right) \right. \\
& - \frac{1-q}{[2]_q^2} \frac{\left(1+9q-p^2(q+1)\right)(1-q) + 8pq(q+1)}{16} + (p-1)\frac{1-q}{2} + 1 \Bigg) \\
& - \frac{1}{[2]_q} \left(\frac{q^2+2q-1}{[2]_q} + \frac{1}{[p+1]_q} + \frac{1}{2}\left(1-3q-p(1-q)\right) \right)
\end{aligned}$$

$$-n\alpha[p]_q\left(\sum_{k=1}^{p-1}q^k\widetilde{H}_k^2(q)-\sum_{k=1}^{(p-1)/2}q^{2k}\widetilde{H}_{2k}^2(q)\right)\pmod{[p]_q^2},$$

and so, using (2.13), the congruence $\frac{1}{[p+1]_q}\equiv 1-q[p]_q\pmod{[p]_q^2}$ and Lemma 2.12, we have the proof.

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