Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2023) 47: 2028 – 2042
© TÜBİTAK
doi:10.55730/1300-0098.3479

Inequalities involving general fractional integrals of p-convex functions

İlknur YEŞİLCE IŞIK[1], Gültekin TINAZTEPE[2], Serap KEMALİ[2,*], Gabil ADİLOV[3]
[1]Department of Mathematics, Aksaray University, Aksaray, Turkiye
[2]Vocational School of Technical Sciences, Akdeniz University, Antalya, Turkiye
[3]Faculty of Education, Akdeniz University, Antalya, Turkiye

Received: 18.09.2022 • **Accepted/Published Online:** 09.08.2023 • **Final Version:** 09.11.2023

Abstract: The Hermite-Hadamard type inequalities involving fractional integral operations for p-convex functions with respect to another function are studied. Then, the inequalities via Riemann-Liouville and Hadamard fractional integrals are presented specially. Using the obtained results, some inequality relations among special functions including beta and incomplete beta functions, gamma and incomplete gamma functions, and hypergeometric functions are presented.

Key words: p-Convexity, Hermite-Hadamard inequality, generalized fractional integral

1. Introduction

Convex functions are mostly characterized by inequalities, some of which involve integral inequalities attributed to Hermite-Hadamard, Ostrowski, Fejer inequalities. In the last five decades, the characterization of convex functions has been improved by using more generalized integrals, namely, the fractional integrals, which have been used in the vast area of science from engineering to medicine [31]. The leading fractional integrals may be accepted as the Riemann-Liouville and Hadamard fractional operators among them. Also, there are many like Caputo, Katugampola, conformable and general fractional operations with respect to the other functions [19, 22, 25]. Recently, the Hermite-Hadamard inequalities and different inequalities for convex functions are generalized via fractional integrals including Riemann-Liouville, Hadamard, and other types of fractional integrals [6, 7, 12, 16, 23, 26, 27, 33].

During that time, on the other side, a lot of extensions and generalizations of convex functions are introduced. B-convex , B^{-1}-convex, s-convex in different senses, and p-convex functions can be counted among the most novel ones [2–5, 10, 21, 30]. The basic properties and characterizations via inequalities are studied by different authors [11, 20, 21, 32]. In this study, we cope with p-convex functions. Different convexity types called p-convexity are seen in the literature. The notion used in this paper is given in [15] with the definition of p-convex sets. Then, the p-convex functions are defined in [30]. It has applications to the fixed point theory, optimization, etc [13–15].

The characterization of different types of convex functions via Hermite-Hadamard inequality are studied by many researchers [1, 8, 9, 28, 29, 33]. Based on these results, more generalized versions of some of these inequalities via fractionals are given [16–18, 26, 33–36]. In this study, using some results based on

*Correspondence: skemali@akdeniz.edu.tr
2010 *AMS Mathematics Subject Classification:* 26B25, 52A40, 91B52

the Hermite-Hadamard inequality for p-convex functions, we set the Hermite-Hadamard type inequalities via general fractional operations p-convex functions with respect to another function. Since the used fractional integral operators are generalizations of the Riemann-Liouville and Hadamard fractional, the results including these main fractionals are also given with theorems. Furthermore, as applications, some inequality relations between special functions including beta, incomplete beta and gamma, incomplete gamma, and hypergeometric functions are presented by using the main results.

2. Preliminaries

2.1. Definitions and Notations of p-convexity

Before recalling definitions, the following notations which will be used throughout the paper should be given.

$\mathbb{R}^n$ denotes the n-dimensional vector space, i.e. $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i \in \{1, 2, .., n\}\}$.

Definition 2.1 *[30] Let $A \subset \mathbb{R}^n$ and $0 < p \leq 1$. The set A is called p-convex set if $\lambda x + \mu y \in A$, for all $x, y \in A$ and $\lambda, \mu \in [0,1]$ such that $\lambda^p + \mu^p = 1$.*

All real intervals including zero or accepting it as a boundary point are p-convex.

Definition 2.2 *[30] Let $A \subset \mathbb{R}^n$ be a p-convex set and $f : A \to \mathbb{R}$. If for all $x, y \in A$ and $\lambda, \mu \in [0,1]$ such that $\lambda^p + \mu^p = 1$;*

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \tag{2.1}$$

then f is called p-convex function.

Example 2.3 *[28] Let n is a positive integer. Then $f(x) = x^{2^n}$ is a p-convex function in $(0, +\infty)$.*

With some different notations, (2.1) can be expressed in the following forms. From the equation $\lambda^p + \mu^p = 1$, one has $\mu = (1 - \lambda^p)^{\frac{1}{p}}$; so (2.1) turns to

$$f(\lambda x + (1 - \lambda^p)^{\frac{1}{p}} y) \leq \lambda f(x) + (1 - \lambda^p)^{\frac{1}{p}} f(y) \ .$$

Also, instead of λ, one can be writen that $t^{\frac{1}{p}},\ t \in [0,1]$, thus (2.1) converts into the below inequality

$$f(t^{\frac{1}{p}} x + (1-t)^{\frac{1}{p}} y) \leq t^{\frac{1}{p}} f(x) + (1-t)^{\frac{1}{p}} f(y) \ .$$

In [9] where the Hermite-Hadamard inequality for p-convex is obtained, the following relations are also included.

Lemma 2.4 *Let $f : [0, \infty) \to \mathbb{R}$ be a non-decreasing p-convex function. If $a, b \in [0, \infty)$ with $a < b$, then*

$$f(\frac{a+b}{2^{\frac{2}{p}-1}}) \leq f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right) \tag{2.2}$$

and

$$f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right) \leq \frac{1}{2^{\frac{1}{p}}}\left[\left(f(a) + f(b)\right)\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right]. \tag{2.3}$$

2.2. Fractional integral types

In this subsection, the definition of the fractional integrals which are used for the results of the article are given. Before the definitions, it is useful to remind the gamma and beta functions which are used further of the article, respectively:

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx \quad \text{for} \quad \alpha > 0,$$
$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad \text{for} \quad \alpha, \beta > 0 .$$

Definition 2.5 *[25] Let $f : [a,b] \to \mathbb{R}$ where $a, b \in \mathbb{R}$ such that $a < b$ and $f \in L_1[a,b]$. The left-sided Riemann-Liouville integral $J_{a+}^{\alpha} f$ and the right-sided Riemann-Liouville integral $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \qquad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \qquad x < b$$

respectively.

Definition 2.6 *[25] Let $f : [a,b] \to \mathbb{R}$ where $a, b \in \mathbb{R}$ such that $a < b$ and $f \in L_1[a,b]$. The left-sided Hadamard fractional integral $\boldsymbol{J}_{a+}^{\alpha}$ of order $\alpha > 0$ of f is defined by*

$$\boldsymbol{J}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \qquad x > a$$

provided that the integral exists. The right-sided Hadamard fractional integral $\boldsymbol{J}_{b-}^{\alpha}$ of order $\alpha > 0$ of f is defined by

$$\boldsymbol{J}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \qquad x < b$$

provided that the integral exists.

Definition 2.7 *[25] Let $f : [a,b] \to \mathbb{R}$ where $a, b \in \mathbb{R}$ such that $a < b$ and $f \in L_1[a,b]$. Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive valued function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). The left-sided fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha > 0$ is defined by*

$$I_{a+;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{[g(x) - g(t)]^{1-\alpha}} dt, \qquad x > a$$

provided that the integral exists. The right-sided fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha > 0$ is defined by

$$I_{b-;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)}{[g(t) - g(x)]^{1-\alpha}} dt, \qquad x < b$$

provided that the integral exists.

3. Main results

The new types of Hermite-Hadamard inequalities via generalized fractional integrals will be proved in this section. Additionally, because these general fractional integrals can be turned to the Riemann-Liouville and the Hadamard fractional integrals that are the most famous fractional integrals, some new inequalities will be obtained by this relation of the integrals.

Theorem 3.1 *Let $f:[0,+\infty)\to\mathbb{R}$ be non-decreasing p-convex function where $a,b\geq 0$ such that $a<b$, $f\in L_1[0,+\infty)$ and $\alpha>0$. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive valued function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). The following left-sided fractional inequality holds:*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}\leq I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^{+};g}f\left(\frac{a+b}{2^{\frac{1}{p}}}\right)\leq\frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{2}{p}-1}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$

$$+\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int\limits_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{\alpha}du\ . \tag{3.1}$$

Proof Let $t\in\left[\frac{1}{2},1\right]$ to get positive values of the following coefficient. To prove (3.1), both side of (2.2) is multiplied by the coefficient

$$\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)\text{ then integrate over } t\text{, one has}$$

$$\int\limits_{\frac{1}{2}}^{1}f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)dt$$

$$\leq\int\limits_{\frac{1}{2}}^{1}f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)dt\ .$$

After the substitution by $\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)=u$ and division by $\Gamma(\alpha)$ to the both side of the inequality, one has

$$f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{1-\alpha}}du\leq\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{1-\alpha}}du$$

$$f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\frac{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\alpha}\leq\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{1-\alpha}}du$$

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}\leq I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^{+};g}f\left(\frac{a+b}{2^{\frac{1}{p}}}\right).$$

To prove the other side of (3.1), using the same coefficient and the procedure as the first part of the proof to the inequality (2.3) the following is obtained

$$\int_{\frac{1}{2}}^{1}f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)dt$$

$$\leq\int_{\frac{1}{2}}^{1}\frac{1}{2^{\frac{1}{p}}}\left[(f(a)+f(b))\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right]\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)dt$$

with the same substitution $\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)=u$ and use the integration by parts,

$$\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{1-\alpha}}du\leq\frac{(f(a)+f(b))}{a+b}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{ug'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{1-\alpha}}du$$

$$\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{1-\alpha}}du\leq\frac{(f(a)+f(b))}{a+b}\left[\frac{a+b}{\alpha 2^{\frac{2}{p}-1}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}\right.$$

$$\left.+\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{\alpha}}{\alpha}du\right]$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)\,g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{1-\alpha}}du\leq\frac{(f(a)+f(b))}{\alpha\Gamma(\alpha)\,2^{\frac{2}{p}-1}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$

$$+\frac{(f(a)+f(b))}{\alpha\Gamma(\alpha)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{\alpha}du$$

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^{+};g}f\left(\frac{a+b}{2^{\frac{1}{p}}}\right)\leq\frac{f(a)+f(b)}{\Gamma(\alpha+1)\,2^{\frac{2}{p}-1}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$

$$+\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{\alpha}du\ .$$

Combining the both of the obtained sides, we have the deduced inequality. □

Using the left part of (3.1), we have the following:

Corollary 3.2 *Let f and g be given in Theorem 3.1. For $x>0$ and $0<p<1$,*

$$\frac{f\left(\frac{x}{2^{\frac{1}{p}-1}}\right)\left[g(x)-g\left(\frac{x}{2^{\frac{1}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}\leq I^{\alpha}_{\left(\frac{x}{2^{\frac{1}{p}-1}}\right)^{+};g}f(x)\ . \tag{3.2}$$

To prove (3.2), $x=\frac{a+b}{2^{\frac{1}{p}}}$ is taken in (3.1).

Theorem 3.3 *Let $a,b\in\mathbb{R}_+$, $a<b$ and $g:[a,b]\to\mathbb{R}$ be an increasing and positive valued function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). Let $f:[0,+\infty)\to\mathbb{R}$ be non-decreasing p-convex function, $f\in L_1[0,+\infty)$ and $\alpha>0$. The following right sided general fractional inequality holds:*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}\leq I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{-};g}f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\leq\frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$

$$-\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}du\ . \tag{3.3}$$

Proof Let $t\in\left[\frac{1}{2},1\right]$. The p-convexity of the function f yields (2.2), if this inequality is multiplied by

$\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)$ and integrated over $t\in\left[\frac{1}{2},1\right]$, we have the left side of (3.3) following the same way as in the first part of proof of the Theorem 3.1. To prove the right side of (3.3), the coefficient $\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)$ is multiplied by (2.3) and integration of both sides over $t\in\left[\frac{1}{2},1\right]$ gives

$$\int_{\frac{1}{2}}^{1} f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)dt$$

$$\leq\int_{\frac{1}{2}}^{1}\frac{1}{2^{\frac{1}{p}}}\left[(f(a)+f(b))\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right]\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)dt$$

with the changing variable with $\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)=u$, we deduce that

$$\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)\,g'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}du\leq\frac{(f(a)+f(b))}{a+b}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{ug'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}du$$

$$\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)\,g'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}du\leq\frac{(f(a)+f(b))}{a+b}\left[\frac{a+b}{\alpha 2^{\frac{1}{p}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}\right.$$
$$\left.-\frac{1}{\alpha}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\alpha}du\right]$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\frac{f(u)\,g'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{1-\alpha}}du\leq\frac{(f(a)+f(b))}{\alpha\Gamma(\alpha)\,2^{\frac{1}{p}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$
$$-\frac{(f(a)+f(b))}{\alpha\Gamma(\alpha)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}du$$

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{-};g} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$

$$-\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}du\,.$$

Finally, with the consideration of the inequalities together, (3.3) is proved. □

The fractional integrals with respect to the function g are given the Riemann-Liouville fractional. Inequalities involving the Riemann-Liouville fractional integrals of p-convex functions are given below.

Theorem 3.4 *Let $a,b \geq 0$ such that $a < b$, $f : [0,+\infty) \to \mathbb{R}$ be non-decreasing p-convex function, $f \in L_1[0,+\infty)$ and $\alpha > 0$. The following inequality holds:*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}}{\Gamma(\alpha+1)2^{\frac{2\alpha}{p}}} \leq J^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{(f(a)+f(b))(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}\left(2\alpha+2^{\frac{1}{p}}\right)}{\Gamma(\alpha+2)2^{\frac{2\alpha+2}{p}}}\,. \tag{3.4}$$

Proof To prove (3.4), we use Theorem 3.1. In (3.1), if the function $g(x) = x$ is taken, then (3.4) is obtained. □

Theorem 3.5 *Let $a,b \geq 0$ such that $a < b$, Let $f : [0,+\infty) \to \mathbb{R}$ be non-decreasing p-convex function, $f \in L_1[0,+\infty)$ and $\alpha > 0$. The following inequality holds*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}}{\Gamma(\alpha+1)2^{\frac{2\alpha}{p}}} \leq J^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{(f(a)+f(b))(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}\left(2+\alpha 2^{\frac{1}{p}}\right)}{\Gamma(\alpha+2)2^{\frac{2\alpha+2}{p}}}\,. \tag{3.5}$$

Proof Taking $g(x) = x$ in Theorem 3.3 yields the desired result. □

In the case of $\alpha = 1$, (3.4) and (3.5) give the following inequality

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left(2^{\frac{1}{p}}-2\right)}{2^{\frac{2}{p}}} \leq \frac{1}{a+b}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} f(t)\,dt \leq \frac{(f(a)+f(b))\left(2^{\frac{2}{p}}-4\right)}{2^{\frac{4}{p}+1}}\,. \tag{3.6}$$

The used general fractional integral operator in the article gives special essential fractional integrals, one of these is Hadamard fractional integrals. Thus, inequalities for p-convex functions including Hadamard fractional integral operators are proven in the following theorems.

Theorem 3.6 *Let $a,b \geq 0$ such that $a < b$, $f : [0,+\infty) \to \mathbb{R}$ be non-decreasing p-convex function, $f \in$*

$L_1[0,+\infty)$ *and* $\alpha>0$. *The following inequality via Hadamard fractional integrals holds*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)(1-p)^{\alpha}(\ln 2)^{\alpha}}{\Gamma(\alpha+1)p^{\alpha}} \leq \boldsymbol{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{(f(a)+f(b))}{\Gamma(\alpha+1)}\left[\frac{(\ln 2)^{\alpha}(1-p)^{\alpha}}{2^{\frac{2}{p}-1}p^{\alpha}}+\frac{1}{a+b}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left(\ln\left(\frac{a+b}{2^{\frac{1}{p}}u}\right)^{\alpha}\right)du\right].$$

Proof Taking $g(x)=\ln x$ in (3.1), which is increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)=\frac{1}{x}$ on (a,b), then one has

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-\ln\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} \leq \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{2}{p}-1}}\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-\ln\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$

$$+\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-\ln(u)\right]^{\alpha}du .$$

Afterwards,

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} \leq \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{2}{p}-1}}\left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^{\alpha}$$

$$+\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}u}\right)\right]^{\alpha}du .$$

Thus, the desired inequality is obtained. □

Theorem 3.7 *Let* $a,b\geq 0$ *such that* $a<b$, $f:[0,+\infty)\to\mathbb{R}$ *be non-decreasing p-convex function,* $f\in L_1[0,+\infty)$ *and* $\alpha>0$. *The following inequality for p-convex functions including Hadamard fractional integral operator holds*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)(1-p)^{\alpha}(\ln 2)^{\alpha}}{\Gamma(\alpha+1)p^{\alpha}} \leq \boldsymbol{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{(f(a)+f(b))}{\Gamma(\alpha+1)}\left[\frac{(1-p)^{\alpha}(\ln 2)^{\alpha}}{2^{\frac{1}{p}}p^{\alpha}}-\frac{1}{a+b}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left(\ln\left(\frac{2^{\frac{2}{p}-1}u}{a+b}\right)\right)^{\alpha}du\right]. \tag{3.7}$$

Proof The general fractional integral operator turns to the Hadamard fractional when $g(x)=\ln x$. By using this in (3.3), the following is obtained.

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-\ln\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} \leq \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}}\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-\ln\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}$$

$$-\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[\ln(u)-\ln\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha} du\ .$$

Then,

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} \leq \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}}\left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^{\alpha}$$

$$-\frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[\ln\left(\frac{2^{\frac{2}{p}-1}u}{a+b}\right)\right]^{\alpha} du\ .$$

Hence, (3.7) is obtained after some editing. □

4. Applications

Some relations involving special functions can be deduced using the results in Section 3. In the following propositions, we exemplified some of them pertaining to the functions which are expressed via integrals. The integral calculations of the propositions have been made by Mathematica 13.1.

In the following proposition, a relation between beta and incomplete beta function is obtained via Theorem 3.4

Proposition 4.1 *Let* $n \in \mathbb{N},\ m = 2^n + 1,\ \alpha > 0,\ 0 < x < 1$. *Then*

$$\frac{x^{m-1}}{2^{\alpha}\alpha}\left(x-x^2\right)^{\alpha} \leq B(m,\alpha) - B_x(m,\alpha) \leq \frac{\left(x-x^2\right)^{\alpha}(\alpha x+1)}{(\alpha+1)\,x^{m-2}2^{2+\alpha-m}} \tag{4.1}$$

where $B\_(\cdot,\cdot)$ *is incomplete beta function, namely,*

$$B_{\theta}(q,r) = \int_0^{\theta}(1-t)^{r-1}t^{q-1}dt \quad (0<\theta<1,\ 0<q,r).$$

Proof Let us write $f(x) = x^{m-1}$ in (3.4), which is nondecreasing p-convex function on $[0,\infty)$ by Example 2.3 . Using the fact

$$\int (x-t)^{\alpha-1}t^{m-1}dt = x^{\alpha+m-1}B_{\frac{t}{x}}(m,\alpha), \qquad (t<x)$$

we have

$$J^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}}-1}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) = \frac{(a+b)^{\alpha+m-1}}{2^{\frac{\alpha+m-1}{p}}\Gamma(\alpha)}\left(B(m,\alpha) - B_{2^{1-\frac{1}{p}}}(m,\alpha)\right).$$

To obtain left piece of (4.1), let us use the left part of (3.4). Making some easy simplifications, the following inequality obtained

$$\frac{2^{(m-1)(1-\frac{1}{p})}}{\alpha}\left(\frac{2^{\frac{1}{p}}-2}{2^{\frac{2}{p}}}\right)^{\alpha} \leq B(m,\alpha) - B_{2^{1-\frac{1}{p}}}(m,\alpha). \tag{4.2}$$

In a similar way, to show right piece of (4.1), we use right part of (3.4). Making some algebraic manipulations, we have

$$B(m,\alpha) - B_{2^{1-\frac{1}{p}}}(m,\alpha) \leq \frac{\left(a^{m-1}+b^{m-1}\right)\left(\frac{2^{\frac{1}{p}}-2}{2^{\frac{2}{p}}}\right)^{\alpha}\left(2\alpha+2^{\frac{1}{p}}\right)}{(a+b)^{m-1}\,2^{\frac{3-m}{p}}(\alpha+1)}.$$

Using $\frac{a^{m-1}+b^{m-1}}{(a+b)^{m-1}} < 1$, one can write

$$B(m,\alpha) - B_{2^{1-\frac{1}{p}}}(m,\alpha) \leq \frac{\left(\frac{2^{\frac{1}{p}}-2}{2^{\frac{2}{p}}}\right)^{\alpha}\left(2\alpha+2^{\frac{1}{p}}\right)}{2^{\frac{3-m}{p}}(\alpha+1)}. \tag{4.3}$$

Letting $x = 2^{1-\frac{1}{p}}$ in (4.2) and (4.3) where $x \in (0,1)$ and combining them, we have the desired result. □

The following propositions that set some relations between gamma and incomplete gamma (or upper gamma) functions are obtained via Theorem 3.6.

Proposition 4.2 *Let* $\alpha > 0$. *If* $x > 0$, *then*

$$2^{-2x}x^{\alpha} + \alpha\Gamma(\alpha,x) \leq \Gamma(\alpha+1)$$

where $\Gamma(\cdot,\cdot)$ *denotes incomplete gamma functions:*

$$\Gamma(\alpha,x) = \int_{x}^{\infty} e^{-t}t^{\alpha-1}dt.$$

Proof Putting $f(x) = x^2$ at left-hand side piece of (3.6), then using

$$\int\left(\ln\frac{x}{t}\right)^{\alpha-1}tdt = \frac{x^2}{2^{\alpha}}\left(\Gamma\left(\alpha, 2\ln\frac{x}{t}\right)\right)$$

and making some simplifications, we have

$$\mathbf{J}^{\alpha}_{a+}\left(x^2\right) = \frac{1}{\Gamma(\alpha)}\frac{x^2}{2^{\alpha}}\left(\Gamma(\alpha) - \Gamma\left(\alpha, 2\ln\frac{x}{a}\right)\right). \tag{4.4}$$

Making use of (4.4) and $\Gamma(\alpha+1) = (\alpha)\Gamma(\alpha)$ thus simplifying the expressions in left-hand side piece of (3.6), one has

$$\frac{2^{\alpha-2(\frac{1}{p}-1)}(\ln 2)^{\alpha}}{\alpha}\left(\frac{1}{p}-1\right)^{\alpha} \leq \Gamma(\alpha) - \Gamma\left(\alpha, 2(\frac{1}{p}-1)\ln 2\right).$$

Taking $x = 2(\ln 2)(\frac{1}{p}-1)$ where $x > 0$ for $0 < p < 1$, we have the desired result. □

Proposition 4.3 *Let $\alpha > 0$. If $x > 0$, then*

$$2^{-\frac{x}{\ln 2}-\alpha-1}\left(\Gamma(\alpha+1) - \alpha\Gamma(\alpha, 2x)\right) \leq 2^{-\frac{x}{\ln 2}}x^{\alpha} + \Gamma(\alpha+1) - \Gamma(\alpha+1, x)$$

where gamma and incomplete gamma functions are given as in Proposition 4.2.

Proof Putting $f(x) = x^2$ at right-hand side piece of (3.6) and using

$$\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left(\ln\left(\frac{a+b}{2^{\frac{1}{p}}u}\right)\right)^{\alpha} du = \frac{a+b}{2^{\frac{1}{p}}}\left(\alpha\Gamma(\alpha) - \Gamma\left(\alpha+1, (\frac{1}{p}-1)\ln 2\right)\right)$$

and making some simplifications, we have

$$\frac{1}{2^{\alpha}2^{\frac{1}{p}}}\left(\Gamma(\alpha) - \Gamma\left(\alpha, 2(\frac{1}{p}-1)\ln 2\right)\right)(a+b)^2 \leq \frac{a^2+b^2}{\alpha}\left[\frac{(\ln 2)^{\alpha}(1-p)^{\alpha}}{2^{\frac{1}{p}-1}p^{\alpha}} + \left(\alpha\Gamma(\alpha) - \Gamma\left(\alpha+1, (\frac{1}{p}-1)\ln 2\right)\right)\right].$$

Dividing each side by $(a+b)^2$ and using $\frac{a^2+b^2}{(a+b)^2} < 1$, we have

$$\frac{1}{2^{\alpha}2^{\frac{1}{p}}}\left(\Gamma(\alpha) - \Gamma\left(\alpha, 2(\frac{1}{p}-1)\ln 2\right)\right) \leq \frac{1}{\alpha}\left[\frac{(\ln 2)^{\alpha}(1-p)^{\alpha}}{2^{\frac{1}{p}-1}p^{\alpha}} + \left(\alpha\Gamma(\alpha) - \Gamma\left(\alpha+1, (\frac{1}{p}-1)\ln 2\right)\right)\right].$$

After some algebraic manipulations and simplifications and using $\Gamma(\alpha+1) = (\alpha)\Gamma(\alpha)$, one has

$$2^{-(\frac{1}{p}-1)-\alpha-1}\left(\Gamma(\alpha+1) - \alpha\Gamma\left(\alpha, 2(\frac{1}{p}-1)\ln 2\right)\right) \leq 2^{-(\frac{1}{p}-1)}\left((\frac{1}{p}-1)\ln 2\right)^{\alpha} + \Gamma(\alpha+1) - \Gamma\left(\alpha+1, (\frac{1}{p}-1)\ln 2\right).$$

Taking $x = (\ln 2)(\frac{1}{p}-1)$ where $x > 0$ for $0 < p < 1$, we have the desired result. □

Using Theorem 3.1, the following result is obtained, which shows a relation between two hypergeometric series with same coefficients.

Proposition 4.4 *Let n, m, r positive integers such that $m = 2^r$ and a, b, p, α positive real numbers with $a < b$ and $0 < p < 1$. Suppose $2(1 + \log_2(\frac{a+b}{b}))^{-1} < p$. Then*

$$(a+b)^{m+n}\, 2^{-\frac{2m+n}{p}+m}\left(1 - 2^{n(1-\frac{1}{p})}\right)^{\alpha}\frac{m+n}{n\alpha} \leq {}_2F_1(1-\alpha, \frac{m}{n}+1; \frac{m}{n}+2; 1)$$
$$- {}_2F_1(1-a, \frac{m}{n}+1; \frac{m}{n}+2; \left(\frac{b}{a+b}2^{\frac{2}{p}-1}\right)^{n})$$

where F denotes the hypergeometric function which is defined by

$$ {}_2F_1(\beta,\gamma;\eta;z) = 1 + \frac{\beta\gamma}{\eta}z + \frac{\beta(\beta+1)\gamma(\gamma+1)}{\eta(\eta+1)}\frac{z^2}{2!} + \cdots + \frac{(\beta)_n(\gamma)_n}{(\eta)_n}\frac{z^n}{n!} + \cdots $$

on $|z| \le 1$ where $(\cdot)_n$ is the Pochhammer symbol, i.e.

$$ (\mu)_n = \mu(\mu+1)(\mu+2)\cdots(\mu+n-1). $$

Proof Let $f(x) = x^m$ and $g(x) = x^n$ in Theorem 3.1, which satisfy the conditions of the theorem. Using

$$ \int \frac{nt^{n-1}t^m}{(x^n - t^n)^{1-\alpha}}dt = \frac{n}{m+n}t^{n+m}x^{n(\alpha-1)} \times_2 F_1(1-\alpha, \frac{m}{n}+1; \frac{m}{n}+2; \left(\frac{t}{x}\right)^n) $$

we have

$$ \begin{aligned} I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^+;g} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) &= \frac{1}{\Gamma(\alpha)}\frac{n}{d+n}\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{n(\alpha-1)}\left(F(1-\alpha, \frac{d}{n}+1; \frac{d}{n}+2; 1)\right. \\ &\left. -F(1-a, \frac{d}{n}+1; \frac{d}{n}+2; \left(\frac{b}{a+b}2^{\frac{2}{p}-1}\right)^n\right). \end{aligned} \tag{4.5} $$

Note that ${}_2F_1(\beta,\gamma;\eta;z)$ is defined (i.e. series converges) on $-1 \le z \le 1$ provided that $\beta+\gamma-\eta < 0$ and η is not a negative integer or zero [24]. Since $2(1+\log_2(\frac{a+b}{b}))^{-1} < p$, it is clear that $\left(\frac{b}{a+b}2^{\frac{2}{p}-1}\right)^n < 1$. Thus, the hypergeometric functions in (4.5) satisfy the convergency conditions. Making use of the left part of (3.1) and making some algebraic manipulations we have the desired result. □

References

[1] Adilov G, Kemali S. Abstract convexity and Hermite-Hadamard type inequalities. Journal of Inequalities and Applications 2009; 2009: Article ID 943534, 1-13.https://doi.org/10.1155/2009/943534

[2] Adilov G, Yeşilce I. $\mathbb{B}^{-1}-$convex functions. Journal of Convex Analysis 2017; 24 (2): 505-517. https://doi.org/10.22075/IJNAA.2017.1621.1427

[3] Adilov G, Yeşilce I. On generalization of the concept of convexity. Hacettepe Journal of Mathematics and Statistics 2012; 41 (5): 723-730.

[4] Briec, W, Horvath, CD, Rubinov AM. Seperation in $\mathbb{B}-$convexity. Pacific Journal of Optimization; 1: 13-30.

[5] Briec W, Horvath C. B-Convexity. Optimization, 2004; 53 (2): 103-127. https://doi.org/10.1080/02331930410001695283

[6] Bohner M, Kashuri A, Mohammed P, Valdes JEN. Hermite-Hadamard-type inequalities for conformable integrals. Hacettepe Journal of Mathematics and Statistics 2022; 51 (3): 775-786. https://doi.org/10.15672/hujms.946069

[7] Dahmani Z. On Minkowski and Hermite-Hadamard integral inequalities via fractional integration. Annals of Functional Analysis 2010; 1: 51-58.

[8] Dragomir SS, Pearce, CEM. Selected topics on Hermite-Hadamard inequalities and applications. 2000; RGMIA Monographs, Victoria University.

[9] Eken Z, Kemali S, Tınaztepe G, Adilov G. The Hermite-Hadamard inequalities for p-convex functions. Hacettepe Journal of Mathematics and Statistics 2021; 50 (5): 1268-1279. https://doi.org/10.15672/hujms.775508

[10] Eken Z, Sezer S, Tınaztepe G, Adilov G. s-Convex functions in the fourth sense and some of their properties. Konuralp Journal of Mathematics 2021; 9 (2): 260-267.

[11] Eken, Z. Fejer inequality for s-convex functions in the fourth sense. Journal of Mathematical Inequalities 2022; 16 (1): 19-34. https://doi.org/10.7153/jmi-2022-16-02

[12] Fernandez A, Mohammed P. Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. Mathematical Methods in the Applied Sciences 2021; 44 (10): 8414-8431. https://doi.org/10.1002/mma.6188

[13] Gholizadeh L, Karapınar E, Roohi M. Some fixed point theorems in locally p-convex spaces. Fixed Point Theory and Applications 2013; 2013 (1): 312.

[14] Guedon O, Litvak AE. Euclidean projections of a p-convex body. In Geometric aspects of functional analysis 2000; 95-108. Springer, Berlin, Heidelberg.

[15] Jesus Bastero JB, Pena A. The Theorems of Caratheodory and Gluskin for $0 < p < 1$. Proceedings of the American Mathematical Society 1995; 123 (1): 141-144.

[16] Jleli M, Oregan D, Samet B. On Hermite-Hadamard type inequalities via generalized fractional integrals. Turkish Journal of Mathematics 2016; 40 (6): 1221-1230. https://doi.org/10.3906/mat-1507-79

[17] Jleli M, Samet B. (2016). On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function. Journal of Nonlinear Science and Applications, 2016; 9 (3): 1252-1260.

[18] Kara, H, Ali, MA, Budak, H. Hermite-Hadamard-type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals.Mathematical Methods in the Applied. 2021; 44: 104-123. https://doi.org/10.1002/mma.6712

[19] Katugampola UN. A new approach to generalized fractional derivatives. Bulletin of Mathematical Analysis and Applications. 2014; 6: 1-15.

[20] Kemali S, Tinaztepe G, Adilov G. New type inequalities for $\mathbb{B}^{-1}$-convex functions involving Hadamard fractional integral. Facta Universitatis. Series: Mathematics and Informatics. 2018; 33: 697-704. https://doi.org/10.22190/FUMI1805697K

[21] Kemali S. Hermite-Hadamard type inequality for s-convex functions in the fourth sense. Turkish Journal of Mathematics and Computer Science 2021; 13(2): 287-293. https://doi.org/10.47000/tjmcs.925182

[22] Kilbas AA, Srivastava HM, Trujillo JJ, Theory and applications of fractional differential equations, Vol. 204. Elsevier 2006.

[23] Mohammed, P. Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. Mathematical Methods in the Applied Sciences 2021; 44: 2314-2324. https://doi.org/10.1002/mma.5784

[24] Ryzhik IS, Gradshteyn IM, Jeffrey A, Zwillinger D. Table of integrals, series, and products: Seventh edition. In Table of Integrals, Series, and Products: Seventh Edition. Academic Press, 2007.

[25] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives. Yverdon-les-Bains, Vo.l 1. Gordon and Breach, Switzerland 1993.

[26] Sarikaya MZ, Budak H. New inequalities of Opial type for conformable fractional integrals. Turkish Journal of Mathematics 2017; 41(5): 1164-1173. https://doi.org/10.3906/mat-1606-91

[27] Set E, Sarikaya MZ, Ozdemir ME, Yıldırım H. The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results. Journal of Applied Mathematics, Statistics and Informatics 2014; 10 (2): 69-83.

[28] Sezer S. Hermite-Hadamard type inequalities for the functions whose absolute values of first derivatives are p-convex. Fundamental Journal of Mathematics and Applications 2021; 4 (2): 88-99. https://doi.org/10.33401/fujma.881979

[29] Sezer S. The Hermite-Hadamard inequality for s-Convex functions in the third sense. AIMS Mathematics 2021; 6 (7): 7719-7732. https://doi.org/10.3934/math.2021448

[30] Sezer S, Eken Z, Tınaztepe G, Adilov G. p-convex functions and their some properties. Numerical Functional Analysis and Optimization 2021; 42 (4): 443-459.https://doi.org/10.1080/01630563.2021.1884876

[31] Sun H, Zhang Y, Baleanu D, Chen W, Chen Y. A new collection of real world applications of fractional calculus in science and engineering. Communications in Nonlinear Science and Numerical Simulation 2018; 64: 213-231. https://doi.org/10.1016/j.cnsns.2018.04.019

[32] Tınaztepe G, Sezer S, Eken Z, Sezer Evcan S. The Ostrowski inequality for s-convex functions in the third sense. AIMS Mathematics 2022; 7 (4): 5605-5615. https://doi.org/10.3934/math.2022310

[33] Tınaztepe G, Yeşilce Işık I, Kemali S, Adilov, G. Certain inequalities for s-convex functions via Caputo fractional derivative and Caputo-Fabrizio integral operator. Journal of Mathematical Extension, 2021; 15 (23): 1-18.

[34] Wang J, Feckan M. Fractional Hermite-Hadamard Inequalities, Berlin, Boston: De Gruyter, 2018.

[35] Yeşilce I. Some inequalities for $\mathbb{B}^{-1}$-convex functions via fractional integral operator, TWMS Journal of Applied and Engineering Mathematics. TWMS Journal of Applied and Engineering Mathematics 2019; 9 (3): 620-625.

[36] Yeşilce I, Adilov G. Fractional integral inequalities for $\mathbb{B}$-convex functions. Creative Mathematics and Informatics 2017; 26 (3): 345-351. https://doi.org/10.37193/CMI.2017.03.12