Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 2043 – 2060
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doi:10.55730/1300-0098.3480

On the existence of 6-cycles for some families of difference equations of third order

Antonio LINERO BAS, Daniel NIEVES ROLDÁN*
Department of Mathematics, Faculty of Mathematics, University of Murcia, Murcia, Spain

Received: 29.12.2022 • **Accepted/Published Online:** 17.10.2023 • **Final Version:** 09.11.2023

Abstract: We prove that there are no 6-cycles of the form $x_{n+3} = x_i f(x_j, x_k)$, with $i, j, k \in \{n, n+1, n+2\}$ pairwise distinct, whenever $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ is a continuous symmetric function, that is, $f(x,y) = f(y,x)$, for all $x, y > 0$. Moreover, we obtain all the 6-cycles of potential form and present some open questions relative to the search of p-cycles whenever symmetry does not hold.

Key words: p-cycle, symmetric function, potential cycles, equilibrium point, iteration, homeomorphism

1. Introduction

Given the autonomous difference equation of order k

$$x_{n+k} = f(x_{n+k-1}, \dots, x_{n+1}, x_n), \quad n \geq 1, \tag{1.1}$$

where $f : \Omega \subseteq X^k \to X$ is defined on some subset Ω of a finite Cartesian product of a set X, the main objective, from the qualitative point of view, is to analyze the asymptotic behaviour of the sequences $(x_n)_n$ generated by the deterministic law of recurrence, Eq. (1.1), from certain initial conditions $x_1, \dots, x_k$ in X. In this sense, the most simple dynamics corresponds to periodic solutions. Recall that a solution $(x_n)_n$ is a periodic solution of the difference equation if $x_{n+m} = x_n$ for all $n \geq 1$ and some positive integer m. The smallest of such values m is called the period of the solution.

In this paper we are interested in the topic of global periodicity, namely, we require that every solution of the autonomous difference equation (1.1) be periodic. In this case, according to [19], the set of periods is bounded, so there exists a positive integer p that is the least common multiple of their periods. Then, we say that the difference equation (1.1) is a p-cycle or is globally periodic of period p. For instance, if $\Omega = X = (0,\infty)$, it is immediate to check that $x_{n+2} = \frac{1}{x_n}$ is a 4-cycle since all the solutions generated by recurrence from general initial conditions x_1, x_2 are periodic with period either 1 or 4, so $p = \mathrm{lcm}(1,4) = 4$. Furthermore, to reinforce the concept of global periodicity, we can highlight the following well-known p-cycles: Lyness' Equation $x_{n+2} = \frac{1+x_{n+1}}{x_n}$, a 5-cycle; Todd's Equation $x_{n+3} = \frac{1+x_{n+2}+x_{n+1}}{x_n}$, an 8-cycle; the piecewise linear difference equation $x_{n+2} = |x_{n+1}| - x_n + 1$, which is known as Gingerbreadman's Equation, a 126-cycle.

*Correspondence: daniel.nieves@um.es
2010 *AMS Mathematics Subject Classification:* Primary 39A23. Secondary 37E05, 39A06, 26A18

The reader is referred to [15] to consult another example as well as to have a first encounter with the subject of periodicity for difference equations.

Notice that in the non-autonomous case, $x_{n+k} = f_n(x_{n+k-1}, \ldots, x_{n+1}, x_n)$, contrary to the autonomous one, we can find globally periodic equations whose set of periods is unbounded (this is called an ∞-cycle), see [7].

The subject of global periodicity has attracted the attention of researchers in the last decades, especially from the nineties of the last century (see, for instance, [18], [14], [12], to mention a few of them). Furthermore, for a brief historical digression about global periodicity, see [17] or [11]. In the literature, we can find different approaches to attack the problem of global periodicity. Mainly, we have the following:

- The resolution of functional equations (see, for instance, [4], [9] and [18]).
- The application of techniques of discrete dynamical systems. In particular, when the maps have a certain degree of differentiability (consult, for example, [7], [8], [13] and [21]).
- The application of direct arguments of real analysis (for instance, see [1] and [23]).

Obviously, we could search for conditions breaking the global periodicity character. To this respect, the existence of solutions asymptotically stable avoids the existence of p-cycles (see [2] and [3] to consult results in this line).

An interesting question on the topic of global periodicity is the search for families of p-cycles exhibiting a determined typology. In this sense, in [4] the authors concentrated their attention on different equations of third order of type

$$x_{n+3} = x_i f(x_j, x_k), \tag{1.2}$$

where $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct and $f : (0, \infty)^2 \to (0, \infty)$ is continuous. In concrete, they proved the following:

- The unique 3-cycle of Eq. (1.2) is $x_{n+3} = x_n$.
- The unique 4-cycle is given by $x_{n+3} = x_n \frac{x_{n+2}}{x_{n+1}}$.
- There are two 5-cycles: $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^{\Phi}$ and $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^{\varphi}$, where $\Phi = \frac{1+\sqrt{5}}{2}$ and $\varphi = -\Phi^{-1}$.

It is worth pointing out that the difference equation (1.2) appears in different applied models. To this regard, let us mention an extension of the Baumol-Wolff productivity model appearing in [22], an economical model which establishes levels of production in sectors according to the $R\&D$ (research and development) information and the time period that each sector takes in order to integrate suitably this information; if y_n denotes the output level of the $R\&D$ sector in each period, the model leads to the difference equation $y_n = y_{n-1}\left(1 + g\left(h\left[\sum_{i=1}^{m} f_i(y_{n-i})\right]\right)\right)$, where g, h, f_i are appropriate continuous real functions (see [22] for more information; for the original model, consult [5]). Also, the classical Pielou's discrete equation with delay $N_{t+1} = \frac{aN_t}{1+bN_{t-k}}$, with k a non-negative integer, follows the pattern of (1.2) (see [20] and [6] for more details, here the reader will also find a large list containing rational difference equations of order three of type (1.2)).

In this paper, we extend the study to the case of 6-cycles having the form of Eq. (1.2). In Section 2 we prove that, whenever we impose the additional assumption on f of being symmetric, $f(x,y) = f(y,x)$, then Eq. (1.2) is not a 6-cycle. If one relaxes the above additional condition on the symmetry of f, it is possible to find the 6-cycle given by $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2$. In fact, this is the unique 6-cycle of potential form (see Section 3). Finally, in Section 4 we propose some open questions related to global periodicity for similar families of difference equations.

2. The symmetric case does not generate new 6-cycles

Unless otherwise stated, we assume that in each case the corresponding difference equation (1.2) is a 6-cycle and $f : (0,\infty)^2 \to (0,\infty)$ is continuous.

On the other hand, in the sequel, we will denote by x_1, x_2 and x_3 the initial conditions that will generate a solution. Moreover, every time we say *by global periodicity* we mean that, since we are assuming that Equation (1.2) is a 6-cycle, then, by the definition of 6-cycle, the following relations hold: $x_1 = x_7$, $x_2 = x_8$ and $x_3 = x_9$.

Before starting the study of each difference equation $x_{n+3} = x_i f(x_j, x_k)$, let us recall that $\overline{x}$ is an equilibrium point if and only if $\overline{x}$ is a constant solution, which occurs when $\overline{x} = \overline{x} f(\overline{x}, \overline{x})$, i.e. $f(\overline{x}, \overline{x}) = 1$. Thus, the set of equilibrium points of Equation (1.2) can be described by the closed set

$$\mathcal{F} := \{x > 0 : f(x,x) = 1\}.$$

Notice that belonging to $\mathcal{F}$ is equivalent to be an equilibrium point of the difference equation, or a fixed point of the discrete dynamical system F associated with the difference equation, namely

$$F : (0,\infty)^3 \to (0,\infty)^3,\ F(x_1,x_2,x_3) = \big(x_2, x_3, x_i f(x_j, x_k)\big),$$

where $i,j,k \in \{1,2,3\}$ are pairwise distinct.

Since the difference equation is a 6-cycle, then $F^6 \equiv \mathrm{Id}|_{(0,\infty)}$, where $\mathrm{Id}|_{(0,\infty)}$ is meant the identity map on $(0,\infty)$, so F is a periodic homeomorphism. In that case, being F defined in a space homeomorphic to $\mathbb{R}^3$, it is well-known that F possesses a fixed point (see [16], where the reader will find a brief explanation on the fact that any periodic homeomorphism on $\mathbb{R}^n$ has fixed points if $n \leq 4$), that is, the initial difference equation has an equilibrium point, and $\mathcal{F} \neq \emptyset$. Therefore, we obtain the non-emptiness character of $\mathcal{F}$:

Lemma 2.1 $\mathcal{F} \neq \emptyset$.

Given $x > 0$, by $f_x : (0,\infty) \to (0,\infty)$ we understand the *fiber map* defined by

$$f_x(y) = f(x,y), \text{ for all } y > 0. \tag{2.1}$$

For some cases of Equation (1.2), we can show that the fiber map f_x is bijective.

Lemma 2.2 *Suppose that $x_{n+3} = x_{n+2} f(x_{n+1}, x_n)$ or $x_{n+3} = x_{n+1} f(x_{n+2}, x_n)$ are 6-cycles. In both cases, f_x is bijective for all $x > 0$.*

Proof Firstly, we focus on $x_{n+3} = x_{n+2}f(x_{n+1}, x_n)$. Let us fix $x > 0$ and consider the initial conditions $x_1 = x, x_2 = 1, x_3 = z$. If the difference equation is a 6-cycle, $z = x_3 = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6) = f(x, x_6)$, which proves the surjectivity of f_x since the value z is arbitrarily taken. On the other hand, to prove the injectivity of the fiber map, suppose that $f(x, y) = f(x, z)$ for some $y, z > 0$. Then $y = z$ since the initial conditions $y_1 = y, y_2 = x, y_3 = 1$, and $z_1 = z, z_2 = x, z_3 = 1$ generate the same solution, namely, $y_4 = f(x, y) = f(x, z) = z_4$, $y_5 = y_4 f(y_3, y_2) = z_4 f(z_3, z_2) = z_5$, ... and so on. Therefore, $z = z_7 = y_7 = y$.

Now, the proof for $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$ is analogous and will be omitted. In this case, it suffices to take the initial conditions $x_1 = 1, x_2 = x, x_3 = y$ for the surjectivity, and the groups of initial conditions $u_1 = y, u_2 = 1, u_3 = x$ and $v_1 = z, v_2 = 1, v_3 = x$ for the injectivity. □

Corollary 2.3 *Suppose that $x_{n+3} = x_{n+2}f(x_{n+1}, x_n)$ or $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$ are 6-cycles. In both cases, f is surjective.*

Remark 2.4 *Notice that Lemma 2.2 and Corollary 2.3 also hold for any p-cycle, $p \geq 4$. Moreover, no additional condition on the symmetry of f is assumed.*

We will use the notation $f_x^{(n)}(\cdot)$, $n \geq 1$, to denote the iteration

$$(f_x \circ f_x \circ \underbrace{\ldots}_{n\,\text{times}} \circ f_x)(\cdot).$$

In the sequel, we divide our study into three subsections, one for each of the three possible difference equations displaying the form of Equation (1.2).

2.1. The case $x_{n+3} = x_{n+2}f(x_{n+1}, x_n)$

We start studying the third-order difference equation

$$x_{n+3} = x_{n+2}f(x_{n+1}, x_n). \tag{2.2}$$

Before the analysis of symmetric maps f, we can establish some general results without the assumption of symmetry for f.

Lemma 2.5 *Consider Equation (2.2). It holds:*

(a) The inverse map of $f_x(\cdot)$ is given by

$$\varphi_x(z) = zf(1, x) \cdot f(z, 1) \cdot f(zf(1, x), z).$$

As a particular case, the inverse of $f_1(\cdot)$ is

$$\varphi_1(z) = zf(1, 1) \cdot f(z, 1) \cdot f(zf(1, 1), z).$$

(b) If $f(\alpha, \beta) = f(\beta, \alpha) = 1$ for some $\alpha, \beta \in (0, \infty)$, then $\alpha = \beta$.

Proof (a) Take initial conditions $x_1 = x$, $x_2 = 1$ and $x_3 = z$. Bearing in mind that Equation (2.2) is a 6-cycle, by global periodicity, $x_3 = x_9$, so $x_3 = z = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6) = f(x, x_6)$, and we obtain

$$z = f\big(x,\ zf(1,x) \cdot f(z,1) \cdot f(zf(1,x), z)\big) = f_x\big(zf(1,x) \cdot f(z,1) \cdot f(zf(1,x), z)\big).$$

(b) If $x_1 = \alpha, x_2 = \beta, x_3 = \alpha$, then $x_4 = \alpha, x_5 = \alpha, x_6 = \alpha f(\alpha, \alpha)$. From the global periodicity, $\alpha = x_1 = x_7 = x_6 f(x_5, x_4) = \alpha f(\alpha, \alpha) f(\alpha, \alpha) = \alpha(f(\alpha, \alpha))^2$. Therefore, $f(\alpha, \alpha) = 1$. To obtain that $f(\beta, \beta) = 1$, proceed analogously by taking $y_1 = \beta, y_2 = \alpha, y_3 = \beta$. Finally, if we set the initial conditions $z_1 = \alpha, z_2 = \alpha, z_3 = \beta$, we have $z_4 = \beta, z_5 = \beta, z_6 = \beta, \ldots,\ z_j = \beta$ for all $j \geq 3$. Since $z_1 = z_7$, we conclude that $\alpha = \beta$. □

Remember that $\mathcal{F} = \{x > 0 : f(x, x) = 1\}$ is the set of equilibrium points of Equation (1.2), with $\mathcal{F} \neq \emptyset$. Here, in particular, we focus on the set of Equilibrium points of Eq. (2.2).

Lemma 2.6 *If $x \in \mathcal{F}$, then $\frac{1}{x} = f\big(x, f(1,x) \cdot f(1,1)\big)$.*

Proof Use the initial conditions $x_1 = x_2 = x$, $x_3 = 1$, to generate the periodic sequence of period 6 $(x_1, x_2, \cdots, x_6)$. We iterate the considered initial conditions by (2.2) in order to obtain the remaining terms of the 6-cycle, that are, $x_4 = f(x,x) = 1, x_5 = f(1,x), x_6 = f(1,x) \cdot f(1,1)$. Then, by global periodicity,

$$1 = x_3 = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6) = xf\big(x, f(1,x) \cdot f(1,1)\big),$$

so it follows that $\frac{1}{x} = f\big(x, f(1,x) \cdot f(1,1)\big)$. □

Lemma 2.7 *Consider Equation (2.2). It holds:*

(a) $xf(x,1) \cdot f(x,x) \cdot f\big(xf(x,1), x\big) = [f(1,1)]^2 \cdot f\big(f(1,1), 1\big)$, *for all* $x > 0$.

(b) $y = f\big(x,\ y \cdot f(1,x) \cdot f(y,1) \cdot f(yf(1,x), y)\big)$ *for all* $x, y > 0$.

Proof (a) Regard the initial conditions $x_1 = 1, x_2 = 1, x_3 = 1$ to obtain $x_4 = f(1,1),\ x_5 = [f(1,1)]^2$, and $x_6 = [f(1,1)]^2 \cdot f(f(1,1),1)$. Then $1 = x_3 = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6) = f\big(1, [f(1,1)]^2 \cdot f\,(f(1,1),1)\,\big)$.

On the other hand, if we iterate $y_1 = 1, y_2 = x, y_3 = x$, we find $y_4 = xf(x,1), y_5 = xf(x,1) \cdot f(x,x)$, $y_6 = xf(x,1) \cdot f(x,x) \cdot f\,(xf(x,1), x)$. Hence,

$$x = y_3 = y_9 = y_8 f(y_7, y_6) = y_2 f(y_1, y_6) = xf(1, xf(x,1) \cdot f(x,x) \cdot f\,(xf(x,1), x)),$$

and $1 = f(1, xf(x,1) \cdot f(x,x) \cdot f\,(xf(x,1), x))$. Taking into account that the fiber map $f_1(\cdot) = f(1, \cdot)$ is bijective by Lemma 2.5, we deduce that $xf(x,1) \cdot f(x,x) \cdot f\,(xf(x,1), x) = [f(1,1)]^2 \cdot f\,(f(1,1),1)$.

(b) It suffices to set $x_1 = x, x_2 = 1, x_3 = y$ and use that $y = x_3 = x_2 f(x_1, x_6) = f(x, x_6)$. The details are left in charge of the reader. □

Once we know some general properties for the 6-cycles of the form $x_{n+3} = x_{n+2} f(x_{n+1}, x_n)$, independently on the symmetry of f, from now on we assume that f is symmetric.

Lemma 2.8 *In the symmetric case, the following properties hold:*

(a) *If* $f(x,y)=1$*, then* $x=y$.

(b) $f(1,1)=1$.

(c) $f(x,x)=1$ *for all* $x>0$*, that is,* $\mathcal{F}=(0,\infty)$.

(d) $\frac{1}{x}=f(x,f(1,x))$ *for all* $x>0$.

Proof (a) Use the symmetry of f and Lemma 2.5-(b).

(b) Put $x_1=x_2=x_3=1$, and denote $\lambda:=f(1,1)$. Then, iterating (2.2), we get $x_4=\lambda$, $x_5=\lambda^2$, $x_6=\lambda^2 f(\lambda,1)$. Since (2.2) is a 6-cycle, recall that $x_8=x_2=1$, which implies that $1=x_7 f(x_6,x_5)=x_1 f\big(\lambda^2 f(\lambda,1),\lambda^2\big)=f\big(\lambda^2 f(\lambda,1),\lambda^2\big)$. Thus, $1=f\big(\lambda^2 f(\lambda,1),\lambda^2\big)$. By Part (a), $\lambda^2 f(\lambda,1)=\lambda^2$, that is, $1=f(\lambda,1)$. Again by Part (a), $\lambda=1$.

(c) Take initial conditions $x_1=1, x_2=1, x_3=x$. By Part (b), $x_4=x$, $x_5=xf(x,1)$, $x_6=xf(x,1)f(x,x)$. Therefore, by global periodicity, $1=x_2=x_8=x_7 f(x_6,x_5)=f\big(xf(x,1)\cdot f(x,x),xf(x,1)\big)$. Applying Part (a), $xf(x,1)\cdot f(x,x)=xf(x,1)$, that yields to $f(x,x)=1$, as desired.

(d) Once we know that $\mathcal{F}=(0,\infty)$ according to Part (c), we use Lemma 2.6 to deduce that $\frac{1}{x}=f(x,f(1,x)\cdot f(1,1))=f(x,f(1,x))$. □

We are now in a position to prove the main result of this subsection.

Proposition 2.9 *There are no* 6*-cycles of third order having the form* $x_{n+3}=x_{n+2}f(x_{n+1},x_n)$*, whenever* $f:(0,\infty)^2\to(0,\infty)$ *is a symmetric continuous map.*

Proof Let $x>0$ be an arbitrary positive real number. Taking into account that f is symmetric, by Lemma 2.8-(c) we know that $f(z,z)=1$ for all $z>0$. Then Lemma 2.7-(a) yields

$$xf(1,x)\cdot f(xf(1,x),x)=1. \tag{2.3}$$

On the other hand, Lemma 2.7-(b), with $x=y$, implies

$$x=f\big(x,\,xf(1,x)\cdot f(1,x)\cdot f(xf(1,x),x)\big). \tag{2.4}$$

By replacing (2.3) into (2.4), we deduce that $x=f(x,f(1,x))$. Finally, from Lemma 2.8-(d), we have $x=\frac{1}{x}$, so $x=1$, which contradicts that x was arbitrarily chosen. □

It is an open question to determine whether the result remains true or not when we suppress the assumption on the symmetry of $f(x,y)$.

2.2. The case $x_{n+3}=x_{n+1}f(x_{n+2},x_n)$

In this subsection we consider the particular case of Equation (1.2) given by

$$x_{n+3}=x_{n+1}f(x_{n+2},x_n). \tag{2.5}$$

As we have proceeded in the previous study, our first results in this subsection will be established without the symmetric assumption on f. Recall that $\mathcal{F}=\{x>0: f(x,x)=1\}\neq\emptyset$ (see Lemma 2.1). We assume that Eq. (2.5) is a 6-cycle and that f is continuous.

Lemma 2.10 *Let $x \in \mathcal{F}$. Then*

(a) $1 = f\big(xf(1,1), x\big) \cdot f\big(x, xf(1,1)\big)$.

(b) $1 = f(1,1) \cdot f\big(f\big(xf(1,1), x\big), 1\big)$.

(c) $1 = f\big(1, f(xf(1,1), x)\big)$.

Proof Let $x \in \mathcal{F}$ and consider the initial conditions $x_1 = x, x_2 = 1, x_3 = x$ to generate $x_4 = 1, x_5 = xf(1,1)$, and $x_6 = f(xf(1,1), x)$. Then

$$\begin{aligned} x &= x_1 = x_7 = x_5 f(x_6, x_4) = xf(1,1) \cdot f\left(f\big(xf(1,1), x\big), 1\right), \\ 1 &= x_2 = x_8 = x_6 f(x_7, x_5) = x_6 f(x_1, x_5) = f\big(xf(1,1), x\big) \cdot f\big(x, xf(1,1)\big), \\ x &= x_3 = x_9 = x_7 f(x_8, x_6) = x_1 f(x_2, x_6) = xf\big(1, f(xf(1,1), x)\big), \end{aligned}$$

so we deduce that

$$1 = f(1,1) \cdot f\big(f(xf(1,1), x), 1\big), \;\; 1 = f\big(xf(1,1), x\big) \cdot f\big(x, xf(1,1)\big), \;\; 1 = f\left(1, f\big(xf(1,1), x\big)\right),$$

respectively. □

Next, we need to impose to f the condition of symmetry $f(x,y) = f(y,x)$, for every $x, y > 0$.

Lemma 2.11 *Suppose that $f : (0,\infty)^2 \to (0,\infty)$ is continuous and symmetric. Then:*

(a) $f(x,x) = 1$ *for all* $x > 0$, *or equivalently* $\mathcal{F} = (0,\infty)$.

(b) If $f(x,y) = 1$, *then* $x = y$.

Proof (a) By Lemma 2.1, $\mathcal{F} \neq \emptyset$, so we can take a $z \in \mathcal{F}$. Next, by Lemma 2.10-(b)-(c), if additionally f is symmetric, we can deduce that $f(1,1) = 1$. Now, take an arbitrary point $x > 0$, and set $x_1 = 1, x_2 = x, x_3 = 1$. Then $x_4 = xf(1,1) = x$, $x_5 = f(x,x)$, $x_6 = xf\big(f(x,x), 1\big)$. Since Eq. (2.5) is a 6-cycle, $x = x_2 = x_8 = x_6 f(x_7, x_5) = x_6 f(x_1, x_5) = xf\big(f(x,x), 1\big) \cdot f\big(1, f(x,x)\big)$, thus $1 = [f\big(1, f(x,x)\big)]^2$, which implies $1 = f(1, f(x,x))$. Finally, recall that $f_1(\cdot)$ is a bijective map (see Lemma 2.2), and also $1 = f(1,1)$, to deduce that $f(x,x) = 1$.

(b) Let $f(x,y) = 1$. At the same time, by Part (a), $f(x,x) = 1$, and due to the bijectivity of the fiber map f_x obtained in Lemma 2.2, it follows that $x = y$. □

The following result is immediate and we leave its proof in charge of the reader.

Lemma 2.12 *Let $\varphi : (0,\infty) \to (0,\infty)$ be an increasing (decreasing) homeomorphism. Then:*

(a) $\varphi^{(3)}$ *is an increasing (decreasing) homeomorphism.*

(b) The map $g(x) := \frac{1}{\varphi(x)}$ *is a decreasing (increasing) homeomorphism.*

Lastly, we are able to prove the main result of this subsection.

Proposition 2.13 *There are no* 6*-cycles of third order having the form* $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$*, whenever* $f : (0,\infty)^2 \to (0,\infty)$ *is a continuous symmetric map.*

Proof Suppose that $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$ is a 6-cycle. By taking initial conditions $x_1 = x, x_2 = x_3 = 1$:

$$x_4 = f(1,x) = f_1(x), \quad x_5 = f(1, f_1(x)) = f_1^{(2)}(x), \quad x_6 = f_1(x)f(1, f_1^{(2)}(x)) = f_1(x)\cdot f_1^{(3)}(x).$$

From here, and the fact that Eq. (2.5) is assumed to be globally periodic, we infer that

$$1 = x_3 = x_9 = x_7 f(x_8, x_6) = x_1 f(x_2, x_6) = x f_1\big(f_1(x)\cdot f_1^{(3)}(x)\big).$$

So, $\frac{1}{x} = f_1\big(f_1(x)\cdot f_1^{(3)}(x)\big)$. Now, notice that $g(x) = \frac{1}{x}$ is a decreasing map. On the one hand, if $f_1(x)$ is increasing, by Lemma 2.12, $f_1^{(3)}$ is also increasing. Thus, $f_1(x)\cdot f_1^{(3)}(x)$ and $f_1\big(f_1(x)\cdot f_1^{(3)}(x)\big)$ are increasing too. On the other hand, if $f_1(x)$ is decreasing, by Lemma 2.12, $f_1^{(3)}$ is also decreasing. So, $f_1(x)\cdot f_1^{(3)}(x)$ is decreasing, which implies that $f_1\big(f_1(x)\cdot f_1^{(3)}(x)\big)$ is increasing. In both cases, we get that $g(x) = \frac{1}{x}$ and $f_1\big(f_1(x)\cdot f_1^{(3)}(x)\big)$ do not present the same type of monotony. This contradiction gives finally the non-existence of 6-cycles of the form $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$. □

It is worth mentioning that the problem of determining the 6-cycles of the form $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$ when f is not a symmetric map is still open.

2.3. The case $x_{n+3} = x_n f(x_{n+2}, x_{n+1})$

At last, we study the remaining case of Eq. (1.2), that is,

$$x_{n+3} = x_n f(x_{n+2}, x_{n+1}). \tag{2.6}$$

As usual, unless otherwise stated, we assume that the difference equation is a 6-cycle. Before imposing the additional condition on the symmetry of the continuous map $f : (0,\infty)^2 \to (0,\infty)$, we can establish some general results which are also true without the symmetric restriction. Again, $\mathcal{F}$ denotes the set of equilibrium points of Eq. (2.6). In the first result, we employ the notation $\mathrm{Im}(\cdot)$ to denote the image of a map.

Lemma 2.14 *It holds* $1 \in \mathrm{Im}(f)$ *and* $\mathrm{Im}(f)$ *adopts one of the following forms*

$$\left[m, \frac{1}{m}\right], \quad \left(m, \frac{1}{m}\right), \quad (0,\infty),$$

where $0 < m < 1$.

Proof We assume that f is not constant, since on the contrary, the difference equation reduces to the 3-cycle $x_{n+3} = x_n$, because $\mathcal{F} \neq \emptyset$, so $f(x,x) = 1$ for some $x > 0$ and, consequently, $f \equiv 1$. Now, by Lemma 2.1, we get $1 \in \mathrm{Im}(f)$. We take arbitrary initial conditions $x_1 = x, x_2 = y, x_3 = z$. By Eq. (2.6), we get $x_4 = xf(z,y)$, $x_5 = yf(x_4, z)$, $x_6 = zf(x_5, x_4)$. As a consequence of the global periodicity, we obtain

$$z = x_3 = x_9 = x_6 f(x_8, x_7) = x_6 f(x_2, x_1) = zf(x_5, x_4)\cdot f(y, x).$$

Therefore $1 = f(x_5, x_4) \cdot f(y, x)$. By using a direct argument of continuity and connectivity, if $f(y, x) \leq 1$ (the other case $f(y, x) > 1$ is analogous) it is immediate to conclude that the closed interval $\left[f(y, x), \frac{1}{f(y,x)}\right]$ is included in $\mathrm{Im}(f)$. Let $\mu = \sup\{f(x, y) : x, y > 0\}$. It is clear that $\mu > 1$. With similar reasoning as in the above paragraph, it is immediate to prove that $\mathrm{Im}(f) = \left\langle \frac{1}{\mu}, \mu \right\rangle$, where $\langle , \rangle$ denotes either an open interval $(,)$ or a compact one $[,]$, and μ can be finite or infinite. $\square$

Lemma 2.15 *It holds* $1 \in \mathrm{Im}(f_1) \cap \mathrm{Im}(f^1)$, *where* $f_1(\cdot) = f(1, \cdot)$ *and* $f^1(\cdot) = f(\cdot, 1)$.

Proof By Lemma 2.14, $1 \in \mathrm{Im}(f)$, so there exists $(\alpha, \beta) \in (0, \infty)^2$ such that $f(\alpha, \beta) = 1$. Let us consider $x_1 = 1, x_2 = \beta, x_3 = \alpha$. Hence $x_4 = 1$, $x_5 = \beta f(1, \alpha)$, $x_6 = \alpha f\left(\beta f(1, \alpha), 1\right)$, and

$$\beta = x_2 = x_8 = x_5 f(x_7, x_6) = \beta f(1, \alpha) \cdot f\big(x_1, \alpha f\left(\beta f(1, \alpha), 1\right)\big) = \beta f(1, \alpha) \cdot f\big(1, \alpha f\left(\beta f(1, \alpha), 1\right)\big).$$

From here, $1 = f(1, \alpha) \cdot f\big(1, \alpha f\left(\beta f(1, \alpha), 1\right)\big)$, which means, $1 = f_1(\alpha) \cdot f_1\big(\alpha f\left(\beta f(1, \alpha), 1\right)\big)$. Then, the continuity of f_1 yields to the existence of a point $z > 0$ for which $f_1(z) = 1$, as desired. On the other hand, we find

$$\alpha = x_3 = x_9 = x_6 f(x_8, x_7) = \alpha f\big(\beta f(1, \alpha), 1\big) \cdot f(x_2, x_1) = \alpha f\big(\beta f(1, \alpha), 1\big) \cdot f(\beta, 1),$$

that is, $1 = f^1\big(\beta f(1, \alpha)\big) \cdot f^1(\beta)$. From here, we deduce that $f^1(w) = 1$ for some $w > 0$. $\square$

Lemma 2.16 *Let* $(\alpha, \beta) \in (0, \infty)^2$ *satisfy* $f(\alpha, \beta) = f(\beta, \alpha) = 1$. *Then* $f(\alpha, \alpha) = f(\beta, \beta) = 1$, $\alpha, \beta \in \mathcal{F}$.

Proof Set $x_1 = \alpha$, $x_2 = \alpha$, $x_3 = \beta$ and apply the recursion defined by the 6-cycle. Then,

$$x_4 = \alpha,\ x_5 = \alpha,\ x_6 = \beta f(\alpha, \alpha).$$

In this situation, $\beta = x_3 = x_9 = x_6 f(x_8, x_7) = x_6 f(x_2, x_1) = \beta [f(\alpha, \alpha)]^2$. Consequently, $1 = [f(\alpha, \alpha)]^2$, so $f(\alpha, \alpha) = 1$, $\alpha \in \mathcal{F}$. Finally, if we take $y_1 = \beta, y_2 = \beta, y_3 = \alpha$, in an analogous way we obtain $f(\beta, \beta) = 1$, $\beta \in \mathcal{F}$. $\square$

From now on, we will move into the symmetric case.

Lemma 2.17 *Let* $f : (0, \infty)^2 \to (0, \infty)$ *be a symmetric continuous map. Then,* $f(1, 1) = 1$, *or* $1 \in \mathcal{F}$.

Proof By Lemma 2.15, there exists $\alpha > 0$ such that $f(1, \alpha) = 1$. Since f is symmetric, also $f(\alpha, 1) = 1$, and in that case Lemma 2.16 implies that $f(1, 1) = 1$. $\square$

Lemma 2.18 *Let* $f : (0, \infty)^2 \to (0, \infty)$ *be a symmetric continuous map. If* $f(1, z) = 1$ *for all* $z > 0$, *then* $f(x, y) = 1$ *for all* $x, y > 0$.

Proof Let $x_1 = y, x_2 = x, x_3 = 1$. Then, $x_4 = y f(1, x) = y$, $x_5 = x_2 f(x_4, x_3) = x f(y, 1) = x$, and $x_6 = x_3 f(x_5, x_4) = f(x, y)$. Since Eq. (2.6) is a 6-cycle, $1 = x_3 = x_9 = x_6 f(x_8, x_7) = f(x, y) f(x_2, x_1) = f(x, y) f(x, y) = [f(x, y)]^2$. Therefore, $f(x, y) = 1$. $\square$

As a consequence of the last lemma, if we look for 6-cycles, notice that it must be $f_1(\cdot) = f(1, \cdot) \not\equiv 1$, otherwise we would arrive to the difference equation $x_{n+3} = x_n$, which is a 3-cycle.

Lemma 2.19 *If* $f : (0,\infty)^2 \to (0,\infty)$ *is a symmetric continuous map, for each* $x > 0$ *there exists a* $z = z(x) > 0$ *such that* $f(x, z(x)) = 1$.

Proof By taking initial conditions $x_1 = x, x_2 = x_3 = 1$ and using that $f(1,1) = 1$ (see Lemma 2.17), it follows $x_4 = x,\ x_5 = f(x,1),\ x_6 = f(f(x,1), x)$, and the following requisite imposed by global periodicity:

$$1 = x_3 = x_9 = x_6 f(x_8, x_7) = f\big(f(x,1), x\big) \cdot f(1,x).$$

Thus, $1 = f_x\big(f(1,x)\big) \cdot f_x(1)$. By the continuity of the fiber map f_x, it is immediate to deduce the existence of a point $z = z(x)$ such that $1 = f_x(z) = f(x, z(x))$. □

Lemma 2.20 *Let* $f : (0,\infty)^2 \to (0,\infty)$ *be a symmetric continuous map. Then,* $\mathcal{F} = (0,\infty)$.

Proof Given an arbitrary $x > 0$, by Lemma 2.19 we find $z = z(x) > 0$ such that $f(x, z(x)) = 1$. Then, the symmetry of f and Lemma 2.16 imply that $f(z(x), z(x)) = f(x,x) = 1$, which ends the proof. □

Lemma 2.21 *If* $f : (0,\infty)^2 \to (0,\infty)$ *is a symmetric continuous map, it holds*

$$f(x,y) \cdot f\big(x, yf(x,y)\big) = 1 \text{ for all } x, y > 0.$$

In particular,

$$f(1,y) \cdot f\big(1, yf(1,y)\big) = 1 \text{ for all } y > 0.$$

Proof Put $x_1 = x, x_2 = y, x_3 = y$; by Lemma 2.20 and global periodicity of Eq. (2.6):

$$x_4 = x,\ \ x_5 = yf(x,y),\ \ x_6 = yf\big(yf(x,y), x\big),$$

jointly with $y = x_3 = x_9 = x_6 f(x_8, x_7) = yf\big(yf(x,y), x\big) \cdot f(y,x)$. By simplifying, $1 = f\big(yf(x,y), x\big) f(y,x)$. □

Lemma 2.22 *Assume that* $f : (0,\infty)^2 \to (0,\infty)$ *is a symmetric continuous map. Then, for all* $y > 0$ *it holds*

$$\frac{1}{f(1,y)} = f\left(y, \frac{1}{f(1,y)}\right).$$

Proof Put $z = f(1,y)$. If we set $x_1 = 1,\ x_2 = 1,\ x_3 = y$, we get $x_4 = z,\ x_5 = f(z,y),\ x_6 = yf\big(z, f(z,y)\big)$. Consequently, since Eq. (2.6) is a 6-cycle,

$$\begin{aligned}
1 &= x_1 = x_7 = x_4 f(x_6, x_5) = zf\big(yf\big(z, f(z,y)\big), f(z,y)\big),\\
1 &= x_2 = x_8 = x_5 f(x_7, x_6) = x_5 f(x_1, x_6) = f(z,y) \cdot f\big(1, yf\big(z, f(z,y)\big)\big),\\
y &= x_3 = x_9 = x_6 f(x_8, x_7) = x_6 f(x_2, x_1) = yf\big(z, f(z,y)\big) \cdot f(1,1) = yf\big(z, f(z,y)\big).
\end{aligned}$$

From the last equality, $1 = f(z, f(z,y))$. Hence, the first equations can be rewritten as

$$1 = zf(y, f(z,y)), \tag{2.7}$$

$$1 = f(z,y) \cdot f(1,y), \tag{2.8}$$

respectively. Thus, Equation (2.8) gives $\frac{1}{f(1,y)} = f(z,y) = f\big(y, f(1,y)\big)$. Then, if in Equation (2.7) we replace $f(y,z) = f\big(y, f(1,y)\big)$ by $\frac{1}{f(1,y)}$, we obtain $\frac{1}{z} = f\big(y, \frac{1}{f(1,y)}\big)$, that is, $\frac{1}{f(1,y)} = f\big(y, \frac{1}{f(1,y)}\big)$. □

Lemma 2.23 *Assume that $f : (0,\infty)^2 \to (0,\infty)$ is a symmetric continuous map. Given $a > 0$, let b be a fixed point of the fiber map $f_a(\cdot)$. Then*

a) $f\big(a, b^2\big) = \frac{1}{b}$.

b) $f(1, b^2) = 1$.

Proof a) Regard the initial conditions $x_1 = a, x_2 = b, x_3 = b$. Then, by Lemma 2.20, we have $x_4 = a$, $x_5 = bf(a,b) = b^2$, and $x_6 = bf(b^2, a)$. By global periodicity,

$$b = x_3 = x_9 = x_6 f(x_8, x_7) = x_6 f(x_2, x_1) = bf(b^2, a) f(b, a) = b^2 f(b^2, a).$$

Thus, $\frac{1}{b} = f\left(a, b^2\right)$.

b) On the other hand,

$$a = x_1 = x_7 = x_4 f(x_5, x_6) = af\big(b^2, bf(b^2, a)\big).$$

So, $1 = f(b^2, bf(b^2, a))$. Now, by the symmetry of f, $1 = f(b^2, bf(a, b^2))$, and applying Part a), $1 = f\left(b^2, b \cdot \frac{1}{b}\right) = f(b^2, 1)$. Finally, the symmetry of f yields $f(1, b^2) = 1$. □

Define

$$F := \left\{ \frac{1}{f(1,x)} : x > 0 \right\}. \tag{2.9}$$

Obviously, $F \neq \emptyset$. Moreover, we say that $z \in F^2$ if and only if $z = u^2$ for some $u \in F$.

Lemma 2.24 *F is connected, with non-empty interior, and $f_1|_{F^2} \equiv 1$.*

Proof F is connected inasmuch f is a continuous map. Moreover, F cannot be the singleton $\{1\}$, otherwise $f_1 \equiv 1$ and Lemma 2.18 would imply that $f(u,v) = 1$ for all u, v, reducing the initial difference equation to $x_{n+3} = x_n$, a 3-cycle. We now prove that $f(1,z) = 1$ for all $z \in F^2$. Let $z \in F^2$, $z = \frac{1}{(f(1,x))^2}$. By Lemma 2.22, $f\left(x, \frac{1}{f(1,x)}\right) = \frac{1}{f(1,x)}$, and then Lemma 2.23 (with $a = x$ and $b = \frac{1}{f(1,x)}$) implies that $f\left(1, \frac{1}{(f(1,x))^2}\right) = 1$, that is, $f(1,z) = 1$. □

Lemma 2.25 *Assume that Eq. (2.6) is a 6-cycle, with $f : (0,\infty)^2 \to (0,\infty)$ a symmetric continuous map. Let $u, v \in (0,\infty)$, $u < v$, satisfy $f(1,u) = f(1,v) = 1$. Then, there exists $w \in (u,v)$ such that $f(1,w) = 1$.*

Proof Suppose, on the contrary, that either $f(1,x) > 1$ for all $x \in (u,v)$ or $f(1,x) < 1$ for all $x \in (u,v)$. Define $g(x) := xf(1,x), x > 0$ and observe that $g(u) = u, g(v) = v$.

On the one hand, if $f(1,x) > 1$ for all $x \in (u,v)$, then $g(x) > u$ in (u,v), and the continuity of $g(x)$ allows us to find a value z sufficiently close to u in such a way that $u < g(z) < v$, that is, $u < zf_1(z) < v$ (see Figure 1).

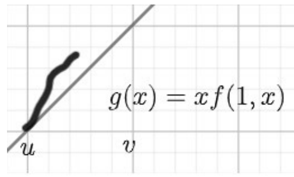

Figure 1. u is repeller from the right for the map $g(x) = xf_1(x)$.

Therefore, $f(1, zf(1,z)) > 1$, and $f(1,z) \cdot f(1, zf_1(z)) > 1$, in contradiction with Lemma 2.21.

On the other hand, if $f(1,x) < 1$ for all $x \in (u,v)$, now it suffices to take a value y in (u,v) sufficiently close to v so that $g(y) > u$. Then $yf(1,y) > u$ and $yf(1,y) < v \cdot 1 = v$, so $u < yf_1(y) < v$ (see Figure 2).

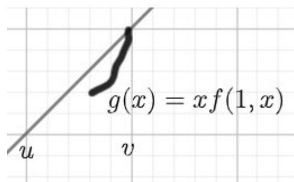

Figure 2. v is attractor from the left for the map $g(x) = xf_1(x)$.

In this situation, we have $f(1,y) \cdot f(1, yf_1(y)) < 1$, which contradicts again Lemma 2.21. □

Now, we can show the non-existence of 6-cycles of the form of Eq. (2.6), whenever we add the extra assumption of f being a symmetric continuous map.

Proposition 2.26 *There are no* 6*-cycles of third order having the form* $x_{n+3} = x_n f(x_{n+2}, x_{n+1})$, *whenever* $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ *is a symmetric continuous map.*

Proof Consider the sets F and F^2 given by Equation (2.9). Notice that by Lemma 2.14, if $w \in \mathrm{Im}(f_1)$, then also $\frac{1}{w} \in \mathrm{Im}(f_1)$. Furthermore, by Lemma 2.21 we infer that $w \in \mathrm{Im}(f_1)$ if and only if $w \in F$. Consequently, $w \in F$ if and only if $\frac{1}{w} \in F$. Since $1 \in \mathrm{Int}(F) \neq \emptyset$ and F is connected by Lemma 2.24, we get that

$$F = \langle \alpha, \omega \rangle = \mathrm{Im}(f_1), \tag{2.10}$$

where $\alpha \geq 0, \alpha < 1$, $\omega = \frac{1}{\alpha}$ (if $\alpha = 0$, we understand that $\omega = \infty$), and $\langle \cdot \rangle$ is meant either an open interval $(\cdot)$ or a compact interval $[\cdot]$. At the same time, $F^2 = \left\langle \alpha^2, \frac{1}{\alpha^2} \right\rangle$.

If $\alpha = 0$ and $F = F^2 = (0,\infty)$, then Lemma 2.24 implies that the fiber map f_1 is constant, $f_1 \equiv 1$, and Lemma 2.18 gives $f(x,y) = 1$ for all $x, y > 0$, but then the initial difference equation reads as $x_{n+3} = x_n$, which is not a 6-cycle. Therefore, if the difference equation is a 6-cycle, necessarily $0 < \alpha < 1$.

We already know that $f_1(z) = 1$ for all $z \in F^2 = \left\langle \alpha^2, \frac{1}{\alpha^2} \right\rangle$ (see Lemma 2.24). We are going to prove that, in fact, $f_1(x) = 1$ for all $x > 0$, which will lead us to derive a contradiction, and the proof of the present proposition will end. Realize that if $0 < \alpha < 1$, then, from the continuity of the fiber map, we have that at least $f_1(x) = 1$ for all $x \in \left[\alpha^2, \frac{1}{\alpha^2}\right]$. Our first step is to prove that there exists a point $z_1 \in (0, \alpha^2)$ such that $f(1, z_1) = 1$. Suppose, on the contrary, that either $f_1(x) > 1$ for all $x \in (0, \alpha^2)$ or $f_1(x) < 1$ for all $x \in (0, \alpha^2)$.

– In the first situation, if $f_1(x) \in (1,\omega)$ or $f_1(x) \in (1,\omega]$ for all $x \in (0,\alpha^2)$ (we use (2.10)), we can choose a point y such that $0 < yf(1,y) \leq y\omega < \alpha^2$ (it suffices to take $y < \alpha^3$). Then, we use Lemma 2.21 to obtain $1 = f(1,y) \cdot f(1, yf(1,y))$, and consequently $f(1, yf(1,y)) = \frac{1}{f(1,y)} < 1$, with $0 < yf(1,y) < \alpha^2$, which contradicts our hypothesis on $f_1(x) > 1$ in $(0,\alpha^2)$.

– In the second situation, if $\alpha \leq f_1(x) < 1$ for all $x \in (0,\alpha^2)$, now $xf(1,x) < x < \alpha^2$. Again by Lemma 2.21, an analogous reasoning as in the previous case gives $f(1,yf(1,y)) = \frac{1}{f(1,y)} > 1$, with $0 < yf(1,y) < \alpha^2$, a contradiction.

Thus, we conclude that there exists at least a point $z_1 \in (0,\alpha^2)$ such that $f(1,z_1) = 1$.

Next, we are going to prove that $f(1,x) = 1$ for all $x \in [z_1,\alpha^2]$. To do it, firstly by Lemma 2.25 there exists z_2 in (z_1,α^2) holding $f(1,z_2) = 1$. By applying repeatedly Lemma 2.25, we find that at the same time there exist points $z_3 \in (z_1,z_2)$ and $z_4 \in (z_2,\alpha^2)$ for which $f(1,z_j) = 1, j = 3,4$. An argument of density leads us to finally conclude that $f(1,x) = 1$ for all $x \in [z_1,\alpha^2]$ (notice that if some subsequence of values $(z_n)_n$ accumulates in an interior point, we can continue the extension to new points x for which $f_1(x) = 1$ due to the continuity of the fiber map f_1 and Lemma 2.25).

Let $m := \inf\{x > 0 : f(1,x) = 1\}$. From the above discussion, $m \leq z_1$. Moreover, if $m > 0$ (so $f(1,m) = 1$ by continuity), the same reasoning we used to prove the existence of the point $z_1 \in (0,\alpha^2)$ satisfying $f(1,z_1) = 1$ can be translated to the interval $(0,m)$ and we could find a positive value $q < m$ such that $f(1,q) = 1$, in contradiction with the definition of m as an infimum. Therefore, $m = 0$ and the argument of density explained to prove that $f(1,x) = 1$ for all $x \in [z_1,\alpha^2]$ can be extended to the whole interval $(0,\alpha^2)$. We then infer that $f(1,x) = 1$ for all $x \in (0,\alpha^2)$. In fact, we already know that $f(1,x) = 1$ for all $x \in (0,\omega^2)$ (see Figure 3).

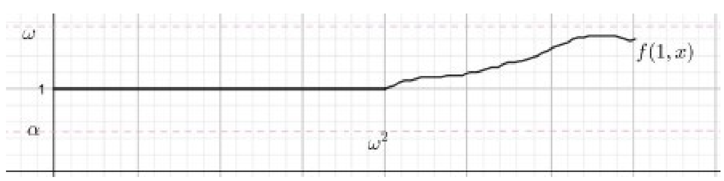

Figure 3. At least, $f(1,x) = 1$ in $(0,\omega^2]$.

Next, we show that there exists a point $w_1 > \omega^2$ such that $f(1,w_1) = 1$. At the beginning of the proof, we have seen that $\mathrm{Im}(f_1) = \langle\alpha,\omega\rangle$. Being $f_1 \equiv 1$ in the interval $(0,\omega^2]$, with $\alpha < 1 < \omega$, there exist values $u,v \in (\omega^2,\infty)$ such that $f(1,u) < 1$ and $f(1,v) > 1$, and then the continuity of the fiber map f_1 implies the existence of a point $w_1 > \omega^2$ satisfying $f(1,w_1) = 1$. In fact, $f_1(x) = 1$ for all (ω^2,w_1), since it suffices to consider Lemma 2.25 to obtain it by an argument of density.

Put $M := \sup\{x > 0 : f(1,x) = 1\}$. By similar considerations to the above paragraph, we have $M = \infty$, because in the contrary, if $M < \infty$, we would deduce by Lemma 2.25 and by density that $f_1 \equiv 1$ in $(0,M]$, and the fact that $\mathrm{Im}(f_1) = \langle\alpha,\omega\rangle$ would imply the existence of a point $x > M$ for which $f(1,x) = 1$, which would contradict the definition of M.

Finally, given an arbitrary point $z > 0$, from our study we can find two values x,y such that $x < z < y$, $f(1,x) = f(1,y) = 1$. A repeated application of Lemma 2.25 gives us a sequence $(q_n)_n$ of positive values tending to z with $f(1,q_n)=1$, and by continuity we conclude that $f(1,z)=1$. Therefore, $f_1\equiv 1$ on $(0,\infty)$. This means that really the difference equation can be written as $x_{n+3}=x_n$, a 3-cycle. This contradiction ends the proof. □

It is worth mentioning that if we suppress the assumption of symmetry, we can find a 6-cycle of potential form (see Section 3), namely,

$$x_{n+3}=x_n\left(\frac{x_{n+2}}{x_{n+1}}\right)^2. \tag{2.11}$$

Now, a natural question that arises is if we can find more 6-cycles of the form $x_{n+3}=x_nf(x_{n+2},x_{n+1})$, and in this case, if they are topological conjugate to (2.11).

As colophon to this section, we gather Propositions 2.9-2.13-2.26 in order to establish our main result:

Theorem 2.27 *There are no 6-cycles of third order having the form $x_{n+3}=x_if(x_j,x_k)$, whenever $f:(0,\infty)^2\to(0,\infty)$ is a symmetric continuous map, and $i,j,k\in\{n,n+1,n+2\}$ are pairwise disjoint.*

Notice that if the order k of the difference equation is not equal to three necessarily, and if by a symmetric map $f:(0,\infty)^{k-1}\to(0,\infty)$, we understand that $f(x_1,\ldots,x_{k-1})=f\big(\sigma(x_1,\ldots,x_{k-1})\big)$, where σ represents any permutation of the variables, then it is possible to find 6-cycles. For instance, consider the 5-order difference equation

$$x_{n+5}=x_{n+4}\frac{x_nx_{n+2}}{x_{n+1}x_{n+3}}. \tag{2.12}$$

Here $f(x,y,z,t)=\frac{yt}{xz}$, $\sigma(x,y,z,t)=(z,t,x,y)$ and it is easy to check that it is a 6-cycle (see [8]).

3. Potential cycles

Following the ideas of [10], if we are looking for p-cycles of potential form $x_{n+3}=x_{n+2}^{\alpha_2}x_{n+1}^{\alpha_1}x_n^{\alpha_0}$, by linearizing the difference equation (put $y_j=\log(x_j)$) we arrive to the p-cycle $y_{n+3}=\alpha_2y_{n+2}+\alpha_1y_{n+1}+\alpha_0y_n$, a linear finite difference equation whose characteristic equation is given by

$$\lambda^3-\alpha_2\lambda^2-\alpha_1\lambda-\alpha_0=0. \tag{3.1}$$

Since all the solutions of a p-cycle are periodic, all the roots μ_1,μ_2,μ_3 of the characteristic polynomial $P(\lambda)=\lambda^3-\alpha_2\lambda^2-\alpha_1\lambda-\alpha_0$ must lie in the boundary of the unit disk, $|\mu_j|=1$ for $j=1,2,3$, and they have to be simple.

Taking into account that $P(\lambda)$ is a third-degree polynomial, necessarily one of its roots, say μ_1, is real. For the other roots, by force, they have to be conjugated complex roots of the unity. Even more, if the difference equation is a 6-cycle, the complex roots are the sixth roots of the unity. Denoting by $r_1=1$ and $r_4=-1$ the real sixth roots of the unity, the other four roots are

$$r_2=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}+i\frac{\sqrt{3}}{2},\qquad r_6=\overline{r_2}=\frac{1}{2}-i\frac{\sqrt{3}}{2},$$

$$r_3=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+i\frac{\sqrt{3}}{2},\qquad r_5=\overline{r_3}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

Therefore, in order to obtain 6-cycles, we have the following possible combinations for the roots of the characteristic equation (3.1):

(i) $\{r_1, r_2, r_6\} = \left\{1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$. Here, it is easily seen that

$$P(\lambda) = (\lambda - 1) \cdot \left(\lambda - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right) \cdot \left(\lambda - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right) = \lambda^3 - 2\lambda^2 + 2\lambda - 1,$$

so $\alpha_2 = 2, \alpha_1 = -2, \alpha_0 = 1$, and the difference equation is

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

(ii) $\{r_1, r_3, r_5\} = \left\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$. Now, $P(\lambda) = \lambda^3 - 1$, so, in fact, we obtain a 3-cycle, $x_{n+3} = x_n$, instead of a 6-cycle. Notice that r_1, r_3, r_5 are third roots of the unity.

(iii) $\{r_4, r_3, r_5\} = \left\{-1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$. The characteristic polynomial is $P(\lambda) = \lambda^3 + 2\lambda^2 + 2\lambda + 1$, and the associate potential cycle is

$$x_{n+3} = \frac{1}{x_n \left(x_{n+1} x_{n+2}\right)^2}.$$

(iv) $\{r_4, r_2, r_6\} = \left\{-1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$. In this case, $P(\lambda) = \lambda^3 + 1$, thus $\alpha_2 = \alpha_1 = 0, \alpha_0 = -1$ and we obtain the potential 6-cycle

$$x_{n+3} = \frac{1}{x_n}.$$

Summarizing, we have:

Proposition 3.1 *The unique 6-cycles of third order and potential form are given by*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2, \quad x_{n+3} = \frac{1}{x_n \left(x_{n+2} x_{n+1}\right)^2}, \quad x_{n+3} = \frac{1}{x_n}.$$

As an immediate consequence, we obtain:

Corollary 3.2 *The unique potential 6-cycle of the form $x_{n+3} = x_i f(x_j, x_k)$, with $i, j, k \in \{n, n+1, n+2\}$ pairwise distinct and $f : (0,\infty)^2 \to (0,\infty)$ continuous, is given by*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

4. Forthcoming lines of research

As we have seen, one of the possible directions in the study of the global periodic phenomenon for difference equations consists in searching for new families of p-cycles. In this sense, we have proved that there are no 6-cycles exhibiting the form $x_{n+3} = x_i f(x_j, x_k)$ whenever $i, j, k \in \{n, n+1, n+2\}$ are pairwise disjoint and $f : (0, \infty)^2 \to (0, \infty)$ is a continuous symmetric map, that is, $f(x, y) = f(y, x)$. Throughout the work, we have been emphasizing some open questions related to the developed results. In order to clarify, we gathered them in the following statement:

Open Problem 1 *Consider the family of third order difference equations $x_{n+3} = x_i f(x_j, x_k)$, with $i, j, k \in \{n, n+1, n+2\}$ pairwise distinct, where $f : (0, \infty)^2 \to (0, \infty)$ is continuous. If we do not assume the extra condition of being f symmetric, does the result of Theorem 2.27 remain true?*

In particular, if the difference equation is of type $x_{n+3} = x_i h(x_j) g(x_k)$, with $i, j, k \in \{n, n+1, n+2\}$ pairwise distinct and $h, g : (0, \infty) \to (0, \infty)$ are continuous, we announce here that the only 6-cycle is just the potential one described above. In this way, we extend a result of [9], where the case $i = n, j = n+1, k = n+2$ was already treated. We hope we can present in brief the proof in a forthcoming paper.

Furthermore, assuming extra conditions to f, such as the differentiability of the map, could help to solve Open Problem 1.

As we have seen in Section 3, we have the potential 6-cycle $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2$, but it is an open question whether there exist or not another 6-cycle displaying the form $x_{n+3} = x_i f(x_j, x_k)$. In the affirmative case, it would be interesting to determine if it is topological conjugate to the above potential one. Even, it is an open question:

Open Problem 2 *To determine if the unique p-cycles of the form given by Equation (1.2), with $p \geq 6$, are precisely the potential ones.*

Also, in view of the example given by Equation (2.12), we propose to extend the above open questions to difference equations of order $k \geq 4$:

Open Problem 3 *Consider the difference equation*

$$x_{n+k} = x_{j_1} f(x_{j_2}, \ldots, x_{j_k}), \tag{4.1}$$

where $j_1, \ldots, j_k \in \{n, \ldots, n+k-1\}$ are pairwise distinct and $f : (0, \infty)^{k-1} \to (0, \infty)$ is a continuous map satisfying that $f(u_1, \ldots, u_{k-1}) = f\big(\sigma(u_1, \ldots, u_{k-1})\big)$ for all $u_1, \ldots, u_{k-1} \in (0, \infty)$, where σ is any non-trivial permutation of the variables. Determine the possible p-cycles displaying the form (4.1). In particular, solve the problem when $k = 4$ and $p \leq 6$.

Apart from $x_{n+3} = x_i f(x_j, x_k)$, we can consider difference equations of the form $x_{n+3} = \frac{1}{x_i} f(x_j, x_k)$ or, more general, $x_{n+3} = x_i^{\alpha} f(x_j, x_k)$, where $\alpha \in \mathbb{R}$, and propose the same questions for the searching of 6-cycles (see [6] for an extensive amount of literature concerning dynamical aspects of third order rational equations presenting this form; notice that in [9], for $\alpha = -1$, the case $p \leq 5$ was completely described).

Evidently, there are some other open questions relative to global periodicity. To know some of them, the reader is referred to [17] or [11], where on the other hand he/she can explore the different approaches to the global periodicity problem.

Acknowledgements

This work has been supported by the grant MTM2017-84079-P funded by MCIN/AEI/10.13039/501100011033 and by ERDF "A way of making Europe", by the European Union.

The authors wish to sincerely express their thanks to the referee for his/her valuable comments and suggestions.

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