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Various types of continuity and their interpretations in ideal topological spaces

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Abstract: In this paper we work on preserving various types of continuity in ideal topological spaces. The accent will be on θ-continuity and weak continuity. We will give their translations in ideal topological spaces. As a consequence of those results, we will prove that every θ-continuous function is continuous if topologies are generated by θ-open sets and we will give an example of a weakly continuous function which is not τ_θ-continuous. This will complete the diagram of relations between continuous, τ_θ-continuous, θ-continuous, weakly continuous, and faintly continuous functions.

Key words: Ideal topological space, local function, local closure function, θ-open sets, continuity, θ-continuous function

1. Introduction

Continuity is almost as old as general topology. Both notions are first mentioned by Frèchet, topological structure in 1906 [8], and continuity in 1910 [9]. The importance of continuity in general topology does not need to be explained here. Later, several modifications of continuity were defined. Some of them are θ-continuity, weak continuity, faint continuity, almost continuity, and many others.

It is interesting that θ-continuity was defined before θ-open sets. It was done by Fomin [6] in 1942. Later, after Veličko [26] introduced θ-open and θ-closed sets, it turned out that those notions have some connections with θ-continuity. Topology defined by θ-open sets, the θ-topology, was later studied by Herrmann [11] and by Foroutan, Ganster and Steiner [7]. Weakly continuous functions were first mentioned by Levine in 1961 in [16]. There he proved that a weakly continuous function which is also weakly* continuous is continuous and vice versa.

Throughout history, some unintended overlapping occurred. For example, closure continuity was introduced by Andrew and Whittlesy [3] in 1966. and it turned out that it is equivalent to θ-continuity. Almost continuous mapping was presented the same year by Husain [12] and, by the same name, but with a slightly different definition, by Singal and Singal [24]. Different forms of faintly continuous functions can be found in [17] and [20]. Also, some weak forms of continuity were mentioned by Espelie and Joseph in [5].

Kuratowski was the first who considered ideals in general topology. In 1933 [14, 15] he defined the local function, generalization of closure by an ideal. About a decade later, Vaidyanathaswamy continued the research on this topic in [25]. Through the years ideal topological spaces became an interesting topic in topology, measure theory, etc. (see Freud [10], Scheinberg [23]). One of the most thorough papers on the local function and ideal

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topological spaces in general was written by Janković and Hamlett [13] in 1990. This survey paper was used later as a basis for further research, mostly for studying modifications of the local function. Thus in 2013, Al-Omari and Noiri [1] introduced the local closure function as a generalization of θ-closure in ideal topological spaces. In the same paper, they mentioned two new topologies obtained from the starting topology using the local closure function.

New variations of continuity were also defined in ideal topological spaces. Such examples can be found in the most recent works of Al-Omeri and Noiri [2], and of Powar, Mishra, and Bhadauria [22]. However, our work will consider some basic aspects of types of continuities and their natural interpretation in ideal topological spaces.

In Section 2 we will give basic definitions and notations. Also, we will give definitions of several topologies obtained in ideal topological spaces in which we will work. In Section 3 we will give definitions of continuity and its various types and present the current state of results considering relations between those types of continuity presented as a diagram. In the following two sections we will give results obtained as the continuation of the research started in [19] on preserving continuity in ideal topological spaces. Section 4 is reserved for results concerning θ-continuity and its consequences in ideal topological spaces. We will give a sufficient condition for ideals in order to θ-continuous function in topologies without ideals becomes continuous in σ, topology obtained by the local closure function. At the end of this section, we will prove that θ-continuity implies continuity in topologies consisting of θ-open sets, τ_θ-topology, which will add a new arrow on the diagram. In Section 5 we will deal with weakly continuous functions and consequences in ideal topological spaces. A condition on ideals when weakly continuous functions become, in ideal topological spaces, a continuous between τ^* and σ topologies, will be given. As a direct consequence of those results is an already known result that weak continuity implies faint continuity. We will prove that in case when at least one of sets (set of originals or set of images) is finite, weak continuity implies continuity in the topology of θ-open sets. Finally, we will give an infinite example of a weakly continuous function which is not continuous in the topology of θ-open sets, proving that those two types are incomparable in general. This example will complete the diagram in the sense that no new arrows can be added.

2. Basic definitions

By $\langle X, \tau \rangle$ we will denote a topological space, $\tau(x)$ will be the family of open neighbourhoods at the point x. Closure of the set A will be written as $\mathrm{Cl}_\tau(A)$ or, if it is clear, just by $\mathrm{Cl}(A)$. Similarly, the interior of A will be denoted by $\mathrm{Int}(A)$ or $\mathrm{Int}_\tau(A)$. An important part of this paper will be dedicated to **θ-topology**. This topology τ_θ consists of all θ-open sets: we say that a set U is **θ-open** if

$$\forall x \in U \ \exists V \in \tau(x) \ \ \mathrm{Cl}(V) \subseteq U.$$

$\mathrm{Int}_{\tau_\theta}(A)$ will denote the **interior in the topology of θ-open sets**. It is obvious that $\tau_\theta \subseteq \tau$. U is θ-open if and only if $\mathrm{Int}_\theta(U) = U$. Naturally, a set A is θ-closed if its complement $X \setminus A$ is θ-open.

θ-closure $\mathrm{Cl}_\theta(A)$ is an operator in the starting topology. It is defined by

$$\mathrm{Cl}_\theta(A) = \{x \in X : \mathrm{Cl}(U) \cap A \neq \emptyset \text{ for each } U \in \tau(x)\}.$$

A set A is **θ-closed** if and only if it is equal to its θ-closure. It is important to notice that θ-closure of a set does not have to be θ-closed, but it is always a closed set. We have $\mathrm{Cl}(A) \subseteq \mathrm{Cl}_\theta(A)$, for each set A. In order to distinguish closure in τ_θ from the operator Cl_θ, the prior will be denoted by $\mathrm{Cl}_{\tau_\theta}$.

We will use small Greek letters $\alpha, \beta, \gamma, \ldots, \omega, \ldots$ to denote ordinals. The family of all ordinals is denoted by ON. Letters λ and κ will be used for cardinals, while $\aleph_0$ is the first infinite cardinal.

An **ideal** on a nonempty set X is a family $\mathcal{I} \subset P(X)$ such that

(1) $\emptyset \in \mathcal{I}$,

(2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,

(3) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

If $\langle X, \tau \rangle$ is a topological space, then an **ideal topological space** is a triple $\langle X, \tau, \mathcal{I} \rangle$.

In an ideal topological space $\langle X, \tau, \mathcal{I} \rangle$, the **local function** (see [15]) can be defined as follows

$$A^*_{(\tau,\mathcal{I})} = \{x \in X : A \cap U \notin \mathcal{I} \text{ for each } U \in \tau(x)\}.$$

If it is clear which topology and ideal are considered, we write briefly A^*. It is monotone operator and $(A^*)^* \subseteq A^*$. Clearly, if $\mathcal{I} = \{\emptyset\}$, then $A^* = \mathrm{Cl}(A)$.

Basic properties of the local function can be found in the survey paper of Janković and Hamlett [13].

Using the local function, a new topology $\tau^*(\mathcal{I})$ can be defined using the closure operator $\mathrm{Cl}^*(A) = A \cup A^*$. Therefore, $\tau^*(\mathcal{I})$ can be described as

$$\tau^*(\mathcal{I}) = \{U \subseteq X : \mathrm{Cl}^*(X \setminus U) = X \setminus U\}.$$

Note that $\tau \subseteq \tau^* \subseteq P(X)$.

Several modifications of the local function have been studied throughout history. We will deal with the one given by Al-Omari and Noiri [1]. They defined the **local closure function** as a generalization of θ-closure in ideal topological spaces. The local closure function in an ideal topological space $\langle X, \tau, \mathcal{I} \rangle$ is defined as

$$\Gamma_{(\tau,\mathcal{I})}(A) = \{x \in X : \mathrm{Cl}(U) \cap A \notin \mathcal{I} \text{ for each } U \in \tau(x)\}.$$

If the topology and the ideal are given, we write briefly $\Gamma(A)$. It is a monotone operator, but there is no general relation between A and $\Gamma(A)$, and it is not idempotent. Notice that if $\mathcal{I} = \{\emptyset\}$ then, for each set A, we have $\Gamma(A) = \mathrm{Cl}_\theta(A)$.

Some basic properties of the local closure function can be found in [1], and further analysis of its properties and relations with the local function in [21] and [18].

Al-Omari and Noiri [1] also studied a variant of θ-interior in ideal topological spaces. They denoted this operator by $\psi_\Gamma(A)$ and defined it by

$$\psi_\Gamma(A) = X \setminus \Gamma(X \setminus A).$$

Using $\psi_\Gamma(A)$ they defined a new topology σ using the operator ψ_Γ:

$$A \in \sigma \Leftrightarrow A \subseteq \psi_\Gamma(A).$$

F is a closed set in the topology σ iff $\Gamma(F) \subseteq F$. It is important to point out that $\tau_\theta \subseteq \sigma$, and if $\mathcal{I} = \{\emptyset\}$, we have $\tau_\theta = \sigma$.

Since we are dealing with functions, we will always deal with two topologies. To distinguish them, sometimes we will put the index of the set next to the topology, like τ_X, or σ_Y. But, when it is clear what is the carrier set of the topology we are talking about, that index will be omitted, especially when the name of the topology has to be part of the closure or interior operator.

3. Several types of continuity

In this section, we will give definitions of various types of continuity and their known relations. All are defined in classical topological spaces without ideals.

The notation is standard. If $f : X \to Y$, for $A \subseteq X$ and $B \subseteq Y$, direct image of the set A is defined by $f[A] = \{f(x) : x \in A\}$ and preimage of B is defined by $f^{-1}[B] = \{x \in X : f(x) \in B\}$.

The following definition belongs to the folklore of general topology.

Definition 3.1 *A function $f : X \to Y$ is **continuous** at the point $x \in X$ if and only if for each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that*

$$f[U] \subseteq V.$$

$f : X \to Y$ is continuous if and only if f is continuous at each point $x \in X$.

Proposition 3.2 *[4, Proposition 1.4.1] For $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ the following conditions are equivalent*

a) f is continuous.

b) For each $O \in \tau_Y$ we have $f^{-1}[O] \in \tau_X$.

c) For each $A \subseteq X$ we have $f[\mathrm{Cl}(A)] \subseteq \mathrm{Cl}(f[A])$.

d) For each $B \subseteq Y$ we have $\mathrm{Cl}(f^{-1}[B]) \subseteq f^{-1}[\mathrm{Cl}(B)]$.

e) For each $B \subseteq Y$ we have $f^{-1}[\mathrm{Int}(B)] \subseteq \mathrm{Int}(f^{-1}[B])$.

Definition 3.3 (Levine, [16]) *A function $f : X \to Y$ is **weakly continuous** at the point $x \in X$ if and only if for each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f[U] \subseteq \mathrm{Cl}(V)$. A function $f : X \to Y$ is weakly continuous if and only if f is weakly continuous at each point $x \in X$.*

An equivalent condition for weak continuity can be given in terms of preimage.

Theorem 3.4 (Levine, [16]) *A function $f : X \to Y$ is **weakly continuous** if and only if $f^{-1}[V] \subseteq \mathrm{Int}(f^{-1}[\mathrm{Cl}[V]])$ for each open subset V of Y.*

Definition 3.5 (Fomin, [6]) *A function $f : X \to Y$ is θ**-continuous** in $x_0 \in X$ iff for each open neighbourhood V of $f(x_0)$ there exists open neighbourhood U of x_0 such that $f[\mathrm{Cl}(U)] \subseteq \mathrm{Cl}(V)$. The same definition is given in [3], but there it is called **closure continuity**.*

It is important to mention that θ-continuity is not the same as continuity in topologies of θ-open sets. Therefore, to make a difference, the second type of continuity we will call τ_θ**-continuity**. The following result gives a sufficient condition for preserving θ-continuity when topology τ on the domain is replaced with the finer topology τ^*.

Theorem 3.6 (Janković, Hamlett, [13]) *If $X = X^*$ then $f : \langle X, \tau\rangle \to Y$ is θ-continuous iff $f : \langle X, \tau^*\rangle \to Y$ is θ-continuous.*

Definition 3.7 (Long and Herrington, [17]) *A function $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is **faintly continuous** at the point $x \in X$ if and only if for each θ-open neighbourhood V of $f(x)$ there is an open neighbourhood U of x such that*

$$f[U] \subseteq V.$$

$f : X \to Y$ is faintly continuous if and only if f is faintly continuous at each point $x \in X$.

Directly from the definition follows that $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is faintly continuous iff $f : \langle X, \tau_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous. In the same paper, it is proved that continuity implies τ_θ-continuity.

Theorem 3.8 (Long and Herrington, [17]) *If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is continuous then $f : \langle X, (\tau_\theta)_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous.*

The following result is obvious, but it is given since it will represent one arrow at the diagram which will be given at the end of the section.

Theorem 3.9 (Long and Herrington, [17]) *If $f : \langle X, (\tau_\theta)_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous then $f : \langle X, \tau_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous, i.e. $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is faintly continuous.*

Theorem 3.10 (Long and Herrington, [17]) *If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a weakly continuous function then $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is faintly continuous.*

Trivially, θ-continuous function is weakly continuous. So, so far, the following diagram, presented in Figure 1, illustrates currently known relations between various types of continuity. It is also known that opposite implications do not hold in general.

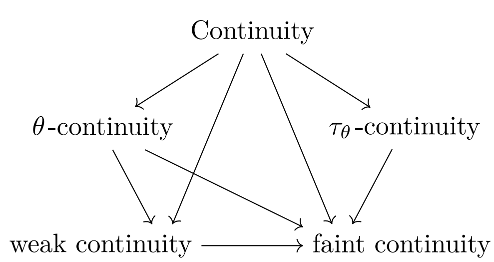

Figure 1. Current state of knowledge concerning various types of continuity

4. θ-continuity and local closure function

Theorem 4.1 *Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a θ-continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold the following equivalent conditions:*

a) $\forall A \subseteq X \ f[\Gamma(A)] \subseteq \Gamma(f[A])$;

b) $\forall B \subseteq Y \ \Gamma(f^{-1}[B]) \subseteq f^{-1}[\Gamma(B)]$.

Proof Let us prove that a) holds. Suppose that there exists $A \subseteq X$ such that there exists $y \in f[\Gamma(A)] \setminus \Gamma(f[A])$. So, there exists $x \in \Gamma(A)$ such that $f(x) = y$ and

$$\forall U \in \tau_X(x)\ \mathrm{Cl}(U) \cap A \notin \mathcal{I}_X. \tag{4.1}$$

Since $y \notin \Gamma(f[A])$, there exists $W \in \tau_Y(y)$ such that $\mathrm{Cl}(W) \cap f[A] \in \mathcal{I}_Y$. By θ-continuity, there exists $V \in \tau_X(x)$ such that $f[\mathrm{Cl}(V)] \subseteq \mathrm{Cl}[W]$. So $f[\mathrm{Cl}(V)] \cap f[A] \in \mathcal{I}_Y$, implying $f^{-1}[f[\mathrm{Cl}(V)] \cap f[A]] \in \mathcal{I}_X$, and since we have

$$\mathrm{Cl}(V) \cap A \subseteq f^{-1}[f[\mathrm{Cl}(V)]] \cap f^{-1}[f[A]] \subseteq f^{-1}[f[\mathrm{Cl}(V)] \cap f[A]],$$

we conclude $\mathrm{Cl}(V) \cap A \in \mathcal{I}_X$, which contradicts (4.1). This proves a).

Let us show that b) is equivalent to a). Suppose a) holds and let $B \subseteq Y$. Then $f[\Gamma(f^{-1}[B])] \subseteq \Gamma(f[f^{-1}[B]]) \subseteq \Gamma(B)$. Now we have $\Gamma(f^{-1}[B]) \subseteq f^{-1}[f[\Gamma(f^{-1}[B])]] \subseteq f^{-1}[\Gamma(B)]$.

Now suppose b) holds. Then $f^{-1}[\Gamma(f[A])] \supseteq \Gamma(f^{-1}[f[A]]) \supseteq \Gamma(A)$. By taking the image by f of both sets we obtain $\Gamma(f[A]) \supseteq f[f^{-1}[\Gamma(f[A])]] \supseteq f[\Gamma(A)]$. □

In the following theorem, we will show how closure in σ topology can be obtained by transfinite recursion.

Theorem 4.2 *Let* $\mathrm{CL}\Gamma^0(A) = A$, $\mathrm{CL}\Gamma^{\alpha+1}(A) = \mathrm{CL}\Gamma^{\alpha}(A) \cup \Gamma(\mathrm{CL}\Gamma^{\alpha}(A))$, *and* $\mathrm{CL}\Gamma^{\gamma}(A) = \bigcup_{\alpha<\gamma} \mathrm{CL}\Gamma^{\alpha}(A)$, *for any* $A \subset X$, *any ordinal* α *and limit ordinal* γ. *Then*

a) For each $\alpha < \beta$, $\mathrm{CL}\Gamma^{\alpha}(A) \subseteq \mathrm{CL}\Gamma^{\beta}(A)$.

b) For each $\alpha \in ON$, $\mathrm{CL}\Gamma^{\alpha}(A) \subseteq \mathrm{Cl}_\sigma(A)$.

c) If there exists $\alpha_0 \in ON$ *such that* $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha_0+1}(A)$, *then* $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha}(A)$ *for each* $\alpha \geq \alpha_0$.

d) There exists $\alpha_0 \in ON$ *such that* $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha}(A)$ *for each* $\alpha \geq \alpha_0$.

e) For such α_0 *(and all ordinals larger than it)* $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{Cl}_\sigma(A)$.

Proof a) Obviously, $\mathrm{CL}\Gamma^{\alpha}(A) \subseteq \mathrm{CL}\Gamma^{\alpha}(A) \cup \Gamma(\mathrm{CL}\Gamma^{\alpha}(A)) = \mathrm{CL}\Gamma^{\alpha+1}(A)$ and $\mathrm{CL}\Gamma^{\gamma}(A) \supseteq \mathrm{CL}\Gamma^{\alpha}(A)$ for limit ordinal γ and each $\alpha < \gamma$. So, $\langle \mathrm{CL}\Gamma^{\alpha}(A) : \alpha \in ON \rangle$ is nondecreasing sequence indexed by the class of all ordinals.

b) Obviously $\Gamma(\mathrm{Cl}_\sigma(A)) \subseteq \mathrm{Cl}_\sigma(A)$ and $A = \mathrm{CL}\Gamma^0(A) \subseteq \mathrm{Cl}_\sigma(A)$. Applying Γ on the last inclusion we get $\Gamma(\mathrm{CL}\Gamma^0(A)) \subseteq \Gamma(\mathrm{Cl}_\sigma(A)) \subseteq \mathrm{Cl}_\sigma(A)$. So, $\mathrm{CL}\Gamma^1(A) \subseteq \mathrm{Cl}_\sigma(A)$. Suppose that for each $\alpha < \beta$ holds $\mathrm{CL}\Gamma^{\alpha}(A) \subseteq \mathrm{Cl}_\sigma(A)$. Let us prove it for β. If β is a limit ordinal, then it holds directly from the property of union, and if $\beta = \delta + 1$ for some $\delta \in ON$, then the proof is similar to the case of $\mathrm{CL}\Gamma^1(A)$.

c) Suppose that, for each $\alpha \in [\alpha_0, \beta)$ we have $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha}(A)$, where $\beta > \alpha_0$. Let us prove that it holds for β.

If $\beta = \delta + 1$, then $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\delta}(A)$, so $\Gamma(\mathrm{CL}\Gamma^{\alpha_0}(A)) = \Gamma(\mathrm{CL}\Gamma^{\delta}(A))$, implying $\mathrm{CL}\Gamma^{\alpha_0+1}(A) = \mathrm{CL}\Gamma^{\delta+1}(A)$, so $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\beta}(A)$.

If β is a limit ordinal, then, for each $\alpha \in [\alpha_0, \beta)$ we have $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha}(A)$, and, due to the increasing property, $\mathrm{CL}\Gamma^{\beta}(A) = \bigcup^{\alpha<\beta} \mathrm{CL}\Gamma^{\alpha}(A) = \bigcup \mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha_0}(A)$.

d) Since $\langle \mathrm{CL}\Gamma^{\alpha}(A) : \alpha \in ON \rangle$ is a nondecreasing sequence, it can not strictly increase forever, since there are no more than $|P(X)|$ different sets. So, there exists α_0 such that $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha_0+1}(A)$, and d) follows from c).

e) Obviously $A \subseteq \mathrm{CL}\Gamma^{\alpha_0}(A) \subseteq \mathrm{Cl}_\sigma(A)$. If we prove that $\mathrm{CL}\Gamma^{\alpha_0}(A)$ is a closed set in topology σ, the proof is over. Since $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{CL}\Gamma^{\alpha_0+1}(A) = \mathrm{CL}\Gamma^{\alpha_0}(A) \cup \Gamma(\mathrm{CL}\Gamma^{\alpha_0}(A))$, we have $\Gamma(\mathrm{CL}\Gamma^{\alpha_0}(A)) \subseteq \mathrm{CL}\Gamma^{\alpha_0}(A)$, witnessing that $\Gamma(\mathrm{CL}\Gamma^{\alpha_0}(A))$ is closed. □

Theorem 4.3 *Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a θ-continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold:*

a) $\forall A \subseteq X$ $f[\mathrm{CL}\Gamma^\alpha(A)] \subseteq \mathrm{CL}\Gamma^\alpha(f[A])$, for each ordinal α.

b) $\forall A \subseteq X$ $f[\mathrm{Cl}_\sigma(A)] \subseteq \mathrm{Cl}_\sigma(f[A])$;

c) $f : \langle X, \sigma_X\rangle \to \langle Y, \sigma_Y\rangle$ is a continuous function.

Proof a) By definition of $\mathrm{CL}\Gamma^0$, it holds for $\alpha = 0$. Suppose it holds for every $\beta < \alpha$. Let us prove that it holds for α. Ih α is a consecutive ordinal, then $\alpha = \delta + 1$. So, using Theorem 4.2, we have

$$\begin{aligned} f[\mathrm{CL}\Gamma^\alpha(A)] = f[\mathrm{CL}\Gamma^{\delta+1}(A)] &= f[\mathrm{CL}\Gamma^\delta(A) \cup \Gamma(\mathrm{CL}\Gamma^\delta(A))] \\ &= f[\mathrm{CL}\Gamma^\delta(A)] \cup f[\Gamma(\mathrm{CL}\Gamma^\delta(A))] \\ &\subseteq \mathrm{CL}\Gamma^\delta(f[A]) \cup \Gamma(f[\mathrm{CL}\Gamma^\delta(A)]) \\ &\subseteq \mathrm{CL}\Gamma^\delta(f[A]) \cup \Gamma(\mathrm{CL}\Gamma^\delta(f[A])) \\ &= \mathrm{CL}\Gamma^{\delta+1}(f[A]) \\ &= \mathrm{CL}\Gamma^\alpha(f[A]). \end{aligned}$$

If α is a limit ordinal, then

$$\begin{aligned} f[\mathrm{CL}\Gamma^\alpha(A)] &= f[\bigcup_{\gamma<\alpha} \mathrm{CL}\Gamma^\gamma(A)] = \bigcup_{\gamma<\alpha} f[\mathrm{CL}\Gamma^\gamma(A)] \\ &\subseteq \bigcup_{\gamma<\alpha} \mathrm{CL}\Gamma^\gamma f[A] = \mathrm{CL}\Gamma^\alpha f[A]. \end{aligned}$$

b) Since, by Theorem 4.2 e), there exists an ordinal α_0 such that $\mathrm{CL}\Gamma^{\alpha_0}(A) = \mathrm{Cl}_\sigma(A)$ and ordinal α_1 such that $\mathrm{CL}\Gamma^{\alpha_1}(f[A]) = \mathrm{Cl}_\sigma(f[A])$, so, for $\beta = \max\{\alpha_0, \alpha_1\}$ holds

$$f[\mathrm{Cl}_\sigma(A)] = f[\mathrm{CL}\Gamma^\beta(A)] \subseteq \mathrm{CL}\Gamma^\beta(f[A]) = \mathrm{Cl}_\sigma(f[A]).$$

c) is equivalent to b). □

If we in the previous theorem take $\mathcal{I} = \{\emptyset\}$, we obtain a relation between θ-continuous functions and τ_θ-continuity.

Corollary 4.4 *If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a θ-continuous function then $f : \langle X, (\tau_\theta)_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous.*

This is an improvement of the result obtained by Long and Herrington [17, Th. 8], stated in Theorem 3.8, which says that continuity implies τ_θ-continuity.

It is well known that the opposite of the previous corollary does not have to be true.

Example 4.5 *[τ_θ-continuity does not imply θ-continuity] [17, Ex. 2] Let $X = \{0,1\}$ with topology $\tau_X = \{\emptyset, \{1\}, \{0,1\}\}$ and let $Y = \{a,b,c\}$ with topology $\tau_Y = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ and $f : X \to Y$ is defined by $f(0) = a$ and $f(1) = b$. Let $x_0 = 0$. Then $V = \{a\}$ be a neighbourhood of $f(x_0) = a$, and $\mathrm{Cl}(V) = \{a,c\}$. On the other hand, the only neighbourhood of the point $0 \in X$ is $U = \{0,1\}$, which is, at the same time, its closure. But $f[\mathrm{Cl}(U)] = f[\{0,1\}] = \{a,b\} \not\subseteq \{a,c\}$, so, f is not θ-continuous. But, the only nonempty θ-open set in Y is Y, and its preimage is X, which is also θ-open, implying that $f : \langle X, (\tau_\theta)_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous.*

5. Weakly continuous functions and local closure function

Theorem 5.1 *Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a weakly continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold the following equivalent conditions:*

a) $\forall A \subseteq X\ f[A^*] \subseteq \Gamma(f[A])$;

b) $\forall B \subseteq Y\ (f^{-1}[B])^* \subseteq f^{-1}[\Gamma(B)]$.

Proof Let us prove that a) holds. Suppose that there exists $A \subseteq X$ such that there exists $y \in f[A^*] \setminus \Gamma(f[A])$. So, there exists $x \in A^*$ such that $f(x) = y$. So,

$$\forall U \in \tau_X(x),\ U \cap A \notin \mathcal{I}_X. \tag{5.1}$$

Since $y \notin \Gamma(f[A])$, there exists $W \in \tau_Y(y)$ such that $\mathrm{Cl}(W) \cap f[A] \in \mathcal{I}_Y$, and by weak continuity, there exists $V \in \tau_X(x)$ such that $f[V] \subseteq \mathrm{Cl}[W]$. So $f[V] \cap f[A] \in \mathcal{I}_Y$, implying $f^{-1}[f[V] \cap f[A]] \in \mathcal{I}_X$, and since we have

$$V \cap A \subseteq f^{-1}[f[V]] \cap f^{-1}[f[A]] \subseteq f^{-1}[f[V] \cap f[A]],$$

we conclude $V \cap A \in \mathcal{I}_X$, which contradicts (5.1). This proves a).

Let us show that b) is equivalent to a). Suppose a) holds and let $B \subseteq Y$. Then $f[(f^{-1}[B])^*] \subseteq \Gamma(f[f^{-1}[B]]) \subseteq \Gamma(B)$. Now we have $(f^{-1}[B])^* \subseteq f^{-1}[f[(f^{-1}[B])^*]] \subseteq f^{-1}[\Gamma(B)]$.

Now suppose b) holds. Then $f^{-1}[\Gamma(f[A])] \supseteq (f^{-1}[f[A]])^* \supseteq A^*$. By taking image by f of both sets we obtain $\Gamma(f[A]) \supseteq f[f^{-1}[\Gamma(f[A])]] \supseteq f[A^*]$. □

Theorem 5.2 *Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a weakly continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$. Then $f : \langle X, \tau_X^*\rangle \to \langle Y, \sigma_Y\rangle$ is a continuous function.*

Proof Let $A \subset X$. Then its closure in τ_X^* equals $A \cup A^*$, and by Theorem 4.2 b) we have that closure of $f[A]$ contains $A \cup \Gamma(A)$. By the previous theorem we have that for each A holds $f[A^*] \subseteq \Gamma(f[A])$. Therefore

$$\begin{aligned} f[\mathrm{Cl}_{\tau_X^*}(A)] &= f[A \cup A^*] = f[A] \cup f[A^*] \\ &\subseteq f[A] \cup \Gamma(f[A]) \\ &\subseteq \mathrm{Cl}_\sigma(f[A]). \end{aligned}$$

□

For $\mathcal{I} = \{\emptyset\}$, as a consequence, we obtain an already known result.

Corollary 5.3 *[17, Th. 10] If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a weakly continuous function then $f : \langle X, \tau_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous, which is equivalent to faint continuity of $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$.*

Example 5.4 (τ_θ-continuity does not imply weak continuity) *Example 4.5 also witnesses that continuity of $f : \langle X, (\tau_\theta)_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ does not imply that f is weakly continuous.*

Now, the only open question which needed to be answered to completely fill the diagram given at the end of Section 3 states: Does weak continuity imply τ_θ-continuity?

We will show that when either X or Y is finite, we have a positive answer to the previous question.

Theorem 5.5 *If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is weakly continuous and not τ_θ-continuous, then both X and Y have to be infinite.*

Proof Let $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ be weakly continuous and not continuous as a function of their θ- topologies. Therefore there exists a set $A \subseteq X$ such that $f[\mathrm{Cl}_{\tau_\theta}(A)] \not\subset \mathrm{Cl}_{\tau_\theta}(f[A])$. Since σ from Theorem 4.2 is equal to τ_θ for the trivial ideal $\{\emptyset\}$, and since $f[A] = f[\mathrm{CL}\Gamma^0(A)] \subset \mathrm{Cl}_{\tau_\theta}(f[A])$, there exists $\alpha \in ON$ such that $f[\mathrm{CL}\Gamma^\alpha(A)] \subset \mathrm{Cl}_{\tau_\theta}(f[A])$ and $f[\mathrm{CL}\Gamma^{\alpha+1}(A)] \not\subset \mathrm{Cl}_{\tau_\theta}(f[A])$. So, there exists $x_1 \in \mathrm{CL}\Gamma^{\alpha+1}(A) = \mathrm{CL}\Gamma^\alpha(A) \cup \mathrm{Cl}_\theta(\mathrm{CL}\Gamma^\alpha(A)) = \mathrm{Cl}_\theta(\mathrm{CL}\Gamma^\alpha(A))$ such that $y_1 = f(x_1) \notin \mathrm{Cl}_{\tau_\theta}(f[A])$. For that $y_1 \in Y \setminus \mathrm{Cl}_{\tau_\theta}(f[A]) \in (\tau_\theta)_Y$, there exists $V_1 \in \tau_Y(y_1)$ such that $y_1 \in V_1 \subset \mathrm{Cl}(V_1) \subset Y \setminus \mathrm{Cl}_{\tau_\theta}(f[A])$. Due to weak continuity of f there exists $U_1 \in \tau_X(x_1)$ such that

$$f[U_1] \subseteq \mathrm{Cl}(V_1). \tag{5.2}$$

From $x_1 \in \mathrm{Cl}_\theta(\mathrm{CL}\Gamma^\alpha(A))$, we conclude that $\mathrm{Cl}(U_1) \cap \mathrm{CL}\Gamma^\alpha(A) \neq \emptyset$. Namely, let $x_2 \in \mathrm{Cl}(U_1) \cap \mathrm{CL}\Gamma^\alpha(A)$. Since $x_2 \in \mathrm{Cl}(U_1)$, we know that

$$\forall U \in \tau_X(x_2) \quad U \cap U_1 \neq \emptyset, \tag{5.3}$$

and if $\tilde{x} \in U \cap U_1$, then $f(\tilde{x}) \notin \mathrm{Cl}_{\tau_\theta}(f[A])$.

Let $y_2 = f(x_2)$. Suppose that there exists $V \in \tau_Y(y_2)$ such that $V \subseteq \mathrm{Cl}_{\tau_\theta}(f[A])$. Then, since f if weakly continuous, there exists $U_2 \in \tau_X(x_2)$ such that $f[U_2] \subseteq \mathrm{Cl}(V) \subseteq \mathrm{Cl}_{\tau_\theta}(f[A])$ (since the last one is closed). But this is in contradiction with (5.3) and the remark right after it. So, for each $V \in \tau_Y(y_2)$ there holds

$$V \setminus \mathrm{Cl}_{\tau_\theta}(f[A]) \in \tau_Y \setminus \{\emptyset\}$$

i.e. $V \setminus \mathrm{Cl}_{\tau_\theta}(f[A])$ is an nonempty open set disjoint with $\mathrm{Cl}_{\tau_\theta}(f[A])$.

Let us consider the intersection of all $V \in \tau_Y(y_2)$, denoted by O. Such set contains y_2. Let us prove that O is not open. If we assume that it is open, we have two possibilities. Firstly $O \subset \mathrm{Cl}_{\tau_\theta}(f[A])$, which we already discussed is impossible. So, there exists $y \in Y \setminus \mathrm{Cl}_{\tau_\theta}(f[A])$ such that each neighbourhood of y_2 intersects $\{y\}$, implying $y_2 \in \mathrm{Cl}(\{y\}) \subset \mathrm{Cl}(V_y)$, where V_y is an arbitrary neighbourhood of y. This implies that the closure of arbitrary neighbourhood of y intersects $\mathrm{Cl}_{\tau_\theta}(f[A])$, implying $y \in \mathrm{Cl}_{\tau_\theta}(f[A])$, which is impossible. So, as a consequence, we have that $\tau_Y(y_2)$ can not be finite (since the finite intersection of open sets is always open), which implies infinity of τ_Y, and, therefore infinity of Y.

Let us suppose that there exists $x_0 \in U_1$ such that $x_0 \in U$ for each $U \in \tau_X(x_2)$. Let $y_0 = f(x_0)$. Since $x_0 \in U_1$, we have $y_0 \in \mathrm{Cl}(V_1)$. Suppose that there exists $V_0 \in \tau_Y(y_0)$ and $V_2' \in \tau_Y(y_2)$ such that $V_0 \cap V_2' = \emptyset$, which implies

$$V_0 \cap \mathrm{Cl}(V_2') = \emptyset. \tag{5.4}$$

Then there exists $U_2' \in \tau_X(x_2)$ such that $f[U_2'] \subseteq \mathrm{Cl}(V_2')$, but this is impossible, since $x_0 \in U_2'$, so, $y_0 \in f[U_2']$, which is in contradiction with (5.4).

Therefore, for each $x \in U_1$ exists $U_x \in \tau_X(x_2)$ such that $x \notin U$. So, if $U1$ is finite, then the intersection of all such U_x is an open set which does not intersect U_1. Therefore, $U1$ has to be infinite, and there are infinitely many different open sets U_x, so X and τ_X have to be infinite.

□

Corollary 5.6 *If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is weakly continuous and if X or Y is finite, then f is τ_θ-continuous.*

The proof of Theorem 5.5 yielded an example of a weakly continuous function which is not τ_θ-continuous.

Example 5.7 *Let $X = \{x_0, x_1\} \cup \omega$ and $Y = \{y_0, y_1\} \cup \omega \times \{0, 1\}$. Let us define $f(x_0) = y_0$, $f(x_1) = y_1$, and $f(n) = \langle n, 1\rangle$, for $n \in \omega$. Let τ_X be defined by the neighbourhood base system*

$$\mathcal{B}_X(x_i) = \{\{x_i\} \cup \omega \setminus K : |K| < \aleph_0\}, \text{ for } i \in \{0, 1\}, \text{ and } \mathcal{B}_X(n) = \{n\}.$$

and τ_Y by the neighbourhood base system

$$\begin{aligned}
\mathcal{B}_Y(y_0) &= \{\{y_0\} \cup \{\langle k, 0\rangle : k \geq n\} : n \in \omega\},\\
\mathcal{B}_Y(y_1) &= \{\{y_1\} \cup ((\omega \times \{1\}) \setminus K) \cup \{\langle n, 0\rangle\} : |K| < \aleph_0, n \in \omega\},\\
\mathcal{B}_Y(\langle n, 0\rangle) &= \{\langle n, 0\rangle\},\\
\mathcal{B}_Y(\langle n, 1\rangle) &= \{\{y_1\} \cup ((\omega \times \{1\}) \setminus K) \cup \{\langle n, 0\rangle, \langle n, 1\rangle\} : |K| < \aleph_0, n \in \omega\}.
\end{aligned}$$

Let us prove that f is weakly continuous. We distinguish three cases.

1° $x = x_1$, $f(x_1) = y_1$: For an arbitrary neighbourhood $V_1 = \{y_1\} \cup (\omega \times \{1\}) \setminus K \cup \{\langle n, 0\rangle\}$, we have $f^{-1}[V_1] = \{x_0\} \cup \omega \setminus K = U_1$, which is open, so $f[U_1] = V_1 \subseteq \mathrm{Cl}(V_1)$.

2° $x = n$, $f(n) = \langle n, 1\rangle$: This case is trivial, since $\{n\}$ is an open singleton.

3° $x = x_0$, $f(x_0) = y_0$: For an arbitrary neighbourhood $V_0 = \{y_0\} \cup \{\langle k, 0\rangle : k \geq n\}$, let us notice that $\langle k, 1\rangle \in \mathrm{Cl}(V_0)$, for each $k \geq n$, since for $V_1 = \{y_1\} \cup ((\omega \times \{1\}) \setminus K) \cup \{\langle k, 0\rangle, \langle k, 1\rangle\}$, a base neighbourhood of $\langle k, 1\rangle$, we have $V_0 \cap V_1 = \{\langle k, 0\rangle\}$, i.e. it is not empty. So, for $U_0 = \{x_0\} \cup \{k : k \geq n\}$ which is an open neighbourhood of x_0 in X, we have $f[U_0] = \{y_0\} \cup \{\langle k, 1\rangle : k \geq n\} \subseteq \mathrm{Cl}(V_0)$.

Finally, let us notice that $\{y_0\}$ is a θ-closed set in τ_Y, since for each other point $y \in Y$, there exist open sets U_{y_0} and U_y such that $y_0 \in U_{y_0}$, $y \in U_y$ and $U_{y_0} \cap U_y = \emptyset$. On the other hand $f^{-1}[\{y_0\}] = \{x_0\}$ since each neighbourhood of x_1 intersects each neighbourhood of x_0, implying x_0 is in the closure of each open neighbourhood of x_1, so $x_1 \in \mathrm{Cl}_\theta(\{x_0\})$. Therefore, preimage of θ-closed set $\{y_0\}$ is not closed in θ-topology, since $\{x_0\} \neq \mathrm{Cl}_\theta(\{x_0\})$.

So, finally, this example completes our diagram, presented in Figure 2, and we conclude that there do not exist other implications between those five types of continuity.

Problem 5.8 *Is there a nice preserving theorem in ideal topological space, like Theorems 4.3 and 5.2, for faintly continuous functions?*

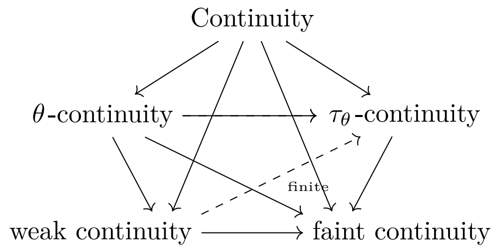

Figure 2. Relations between various types of continuity

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