Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 2122 – 2138
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doi:10.55730/1300-0098.3484

On polynomially partial-A-isometric operators

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Received: 05.09.2023 • **Accepted/Published Online:** 01.11.2023 • **Final Version:** 09.11.2023

Abstract: This paper presents a generalization of the concepts of partial-A-isometry and left polynomially partial isometry. Our investigation is inspired by previous work in the field ([5, 30, 31]). By extending the definition of partial-A-isometry, we provide new insights into the properties and applications of these mathematical objects. In particular, we define the notion of left p-partial-A-isometry as a broader class of operators, including partial-A-isometry and left polynomially partial isometry. Some basic properties of a left p-partial-A-isometry are proven, as well as its relation with A-isometry. Several decompositions of a left p-partial-A-isometry are developed. We consider spectral properties and matrix representation of left p-partial-A-isometries. Additionally, we provide some applications of left p-partial-A-isometries.

Key words: Partial isometry, A-isometry, semi-inner product, spectrum

1. Introduction

Denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$ and by $\mathcal{B}(\mathcal{H})^+$ the cone of positive (semidefinite) operators on $\mathcal{H}$, i.e. $\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}); \langle Ax, x\rangle \geq 0, \forall x \in \mathcal{H}\}$. Let $\mathcal{B}(\mathcal{H})^{++}$ be the set of positive definite operators on $\mathcal{H}$, i.e. $\mathcal{B}(\mathcal{H})^{++} = \{A \in \mathcal{B}(\mathcal{H}); \langle Ax, x\rangle > 0, \forall x \in \mathcal{H}\backslash\{0\}\}$.

For $T \in \mathcal{B}(\mathcal{H})$, the notations T^*, $\mathsf{N}(T)$, and $\mathsf{R}(T)$ will be used to represent the adjoint operator, kernel, and range of T.

A closed subspace M of $\mathcal{H}$ is considered a reducing subspace for an operator $T \in \mathcal{B}(\mathcal{H})$ if both M and its orthogonal complement, denoted as $M^\perp$, are invariant subspaces under the action of T (or equivalently, if M is invariant for both T and T^*). It is important to note that if M is a reducing subspace for T, then $(T|_M)^* = T^*|_M$.

For any operator $A \in \mathcal{B}(\mathcal{H})^+$, we can define a semi-inner product as follows:

$$< x, y >_A = < Ax, y >, \qquad x, y \in \mathcal{H}.$$

Additionally, this definition gives rise to an induced seminorm $\|x\|_A = < x, x >_A^{\frac{1}{2}}$. An operator $S \in \mathcal{B}(\mathcal{H})$ is referred to as an A-adjoint of another operator $T \in \mathcal{B}(\mathcal{H})$ if the following condition holds:

$$< Tx, y >_A = < x, Sy >_A, \qquad x, y \in \mathcal{H}.$$

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2010 *AMS Mathematics Subject Classification:* 47B02, 46C05, 47A05, 47A30, 47A80

It is worth noting that operator T possesses an A-adjoint if and only if a solution exists for the equation $AX = T^*A$, a condition which, as per Douglas' theorem, is equivalent to $\mathsf{R}(T^*A) \subseteq \mathsf{R}(A)$. Given that not every operator $T \in \mathcal{B}(\mathcal{H})$ has an A-adjoint, we define the set $\mathcal{B}_A(\mathcal{H})$ as follows: $\mathcal{B}_A(\mathcal{H}) = T \in \mathcal{B}(\mathcal{H}) : T$ has an A-adjoint. For any $T \in \mathcal{B}_A(\mathcal{H})$, there exists a distinguished A-adjoint operator, denoted as $T^{\#}$, satisfying the properties: $AT^{\#} = T^*A$, $T^{\#} = A^{\dagger}T^*A$, $\mathsf{R}(T^{\#}) \subseteq \mathsf{R}(A)$ and $\mathsf{N}(T^{\#}) \subseteq \mathsf{N}(T^*A)$, where $A^{\dagger}$ represents the Moore–Penrose inverse of A.

One of the most intriguing classes of linear operators between Hilbert spaces is the class of partial isometric operators. Partial isometries represent a captivating class of operators with wide-ranging applications in mathematics and physics. They provide a comprehensive extension of the concept of isometries. Drawing inspiration from the pioneering work of Erd'elyi [13] and the significant contributions of Halmos and McLaughlin [21, 22], among others, this class of operators has played a pivotal role in operator theory. It has been particularly influential in the theory of the polar decomposition of operators and in the dimension theory of von Neumann algebras and Banach algebras [25, 32]. For those interested in exploring this topic further, references such as [2, 4, 15, 16, 18, 19, 23, 27–29] offer extensive information on partial isometries in the infinite-dimensional context, while [3, 17, 20] provide insights into the finite-dimensional case.

The concept of isometry in a Hilbert space is extended by employing a semi-inner product, as presented in [7]. Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$. We say that T is A-isometry if

$$T^*AT = A.$$

In order to extend the concept of partial isometry and A-isometry, the definition of partial-A-isometry is presented in [5].

Definition 1.1 *([5]) Let $A \in \mathcal{B}(\mathcal{H})^+$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a partial-A-isometry if*

$$TT^*AT = TA.$$

In the special case where $A = I$, a partial-A-isometry becomes a partial isometry.

We denote by $Poly$ the set of all complex polynomials in one variable. By taking the complex conjugate of the coefficients of a polynomial $p \in Poly$, we obtain $\overline{p} \in Poly$. More precisely, for any $z \in \mathbb{C}$, we define $\overline{p}(z) := \overline{p(\overline{z})}$.

As an extension of the concepts of partial isometries and nilpotent operators, Garbouj and Skhiri introduced a new class of operators called semi-generalized partial isometries in their work [18]. This is defined as follows: for $n \in \mathbb{N}$, an operator $T \in \mathcal{B}(\mathcal{H})$ is considered to be an n-left (or n-right) generalized partial isometry if it satisfies the condition $T^nT^*T = T^n$ ($TT^*T^n = T^n$).

Another class of operators called polynomially partial isometries is defined in [31], encompassing semi-generalized partial isometries, partial isometries, isometries, and coisometries.

Definition 1.2 *([31]) Let $T \in \mathcal{B}(\mathcal{H})$ and $p \in Poly$ be nontrivial. If*

(*i*) *$p(T)T^*T = p(T)$, then T is called left p-partial isometry;*

(*ii*) *$TT^*p(T) = p(T)$, then T is called right p-partial isometry;*

(*iii*) *T is both left and right p-partial isometry, then T is called p-partial isometry;*

(*iv*) *$S \in \mathcal{B}(\mathcal{H})$ is q-partial isometry for some $q \in Poly$, then S is called polynomially partial isometry.*

Recall that the concept of polynomially normal operators was introduced in [12], providing a generalization of the notion of normal and n-normal operators [1].

Consider an operator $T \in \mathcal{B}(\mathcal{H})$ and a nontrivial polynomial $p \in Poly$. If $p(T)T^* = T^*p(T)$, then we say that T is p-normal. According to [12], T is p-normal if and only if $p(T)$ is a normal operator.

Recent investigations into polynomially partial isometries, partial-A-isometries, and the significance of partial isometries in both theory and applications have spurred us to explore extensions of the concept of partial isometries. Our primary objective is to introduce the concept of left polynomially partial-A-isometries as a new class of operators, with special cases encompassing left p-partial isometries and partial-A-isometries. This broadens the scope of the class of operators under consideration.

This paper is organized as follows: Section 2 covers the fundamental properties of a left p-partial-A-isometry, its relationship with A-isometry, and various decomposition theorems. Spectral properties of left p-partial-A-isometries are presented in Section 3. Section 4 provides the matrix representation of a left p-partial-A-isometry, along with some applications.

2. Basic properties of left p-partial-A-isometries

As an extension of partial-A-isometries and polynomially partial isometries, we define a new class of operators called left polynomially partial-A-isometries.

Definition 2.1 *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $p \in Poly$ be nontrivial. An operator $T \in \mathcal{B}(\mathcal{H})$ will be called a left p-polynomially partial-A-isometry (or left p-partial-A-isometry) if there exists $p \in Poly$ such that*

$$p(T)T^*AT = p(T)A.$$

We also call $T \in \mathcal{B}(\mathcal{H})$ as a right p-polynomially partial-A-isometry if there exists $p \in Poly$ such that $T^*ATp(T) = Ap(T)$.

Remark 2.2

(1) *For $A = I$, a left polynomially partial-A-isometry corresponds to a left polynomially partial isometry.*

(2) *For $p(t) = t$, the notion of left p-polynomially partial-A-isometry coincides with that of partial-A-isometry.*

(3) *If $A \in \mathcal{B}(\mathcal{H})^{++}$ and T is a left polynomially partial-A-isometry such that $AT = TA$, then T is a left polynomially partial isometry.*

(4) *Every A-isometry is left polynomially partial-A-isometry.*

(5) *If T is left p-partial-A-isometry such that $p(T)$ is injective, then it turns to be an A-isometry.*

Some basic properties of left p-polynomially partial-A-isometry are given now.

Lemma 2.3 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $T \in \mathcal{B}(\mathcal{H})$ and $p, q \in Poly$.*

(1) *T is a left p-polynomially partial-A-isometry if and only if T^* is a right $\overline{p}$-polynomially partial-A-isometry.*

(2) *For an unitary operator $U \in \mathcal{B}(\mathcal{H})$, T is a left (or right) p-polynomially partial-A-isometry if and only if UTU^* is a left (right) p-polynomially partial-A-isometry.*

(3) *If T is a left (or right) p-polynomially partial-A-isometry, then T is a left (right) λp-polynomially partial-A-isometry.*

(4) *If T is both left p-polynomially partial-A-isometry and left q-polynomially partial-A-isometry, then T is a left $p+q$-polynomially partial-A-isometry.*

Remark 2.4 *The first statement in the previous lemma indicates that it is enough to focus on the class of left p-polynomially partial-A-isometries, as results pertaining to the class of right p-polynomially partial-A-isometries can be derived through duality.*

Proposition 2.5 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $p \in Poly$ and $T \in \mathcal{B}(\mathcal{H})$ be a left p-polynomially partial-A-isometry. Then*

(1) $A(\mathsf{N}(T)) \subseteq \mathsf{N}(p(T))$.

(2) $T^*AT(x) = A(x), \quad \forall x \in \mathsf{N}(p(T))^\perp$.

(3) $\|Tx\|_A = \|x\|_A, \quad \forall x \in \mathsf{N}(p(T))^\perp$.

Proof Let $T \in \mathcal{B}(\mathcal{H})$ be a left p-polynomially partial-A-isometry.
(1) Let $x \in \mathsf{N}(T)$. Then $0 = p(T)T^*AT(x) = p(T)Ax$, and so $Ax \in \mathsf{N}(p(T))$.
(2) By duality, we have $Ap(T)^* = T^*ATp(T)^*$. It follows that $A = T^*AT$ on $\mathsf{R}(p(T)^*)$. By continuity, we get the result.
(3) Using part (2), for $x \in \mathsf{N}(p(T))^\perp$, $\|Tx\|_A^2 =< ATx, Tx >=< T^*ATx, x >=< Ax, x >= \|x\|_A$. □

We consider additional assumptions under which statements (1)–(3) of Proposition 2.5 implies that T is a left p-partial-A-isometry.

Corollary 2.6 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $p \in Poly$, and $T \in \mathcal{B}(\mathcal{H})$. If $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT) \cap \mathsf{N}(p(T)A)$ (or $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T) \cap \mathsf{N}(A)$), then the following statements are equivalent:*

(1) *T is a left p-partial-A-isometry.*

(2) $T^*AT(x) = A(x), \quad \forall x \in \mathsf{N}(p(T))^\perp$.

Proof (1) $\Longrightarrow$ (2) follows from Proposition 2.5.
(2) $\Longrightarrow$ (1) Since $\mathcal{H} = \mathsf{N}((p(T))^\perp \oplus \mathsf{N}((p(T))$, for $z \in \mathcal{H}$, $z = x + y$, where $x \in \mathsf{N}(p(T))^\perp$ and $y \in \mathsf{N}((p(T))$. The assumptions $T^*AT = A$ on $\mathsf{N}(p(T))^\perp$ and $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT) \cap \mathsf{N}(p(T)A)$ give

$$\begin{aligned} p(T)T^*ATz &= p(T)T^*ATx + p(T)T^*ATy \\ &= p(T)Ax + 0 = p(T)Ax + p(T)Ay \\ &= p(T)Az. \end{aligned}$$

Corollary 2.7 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $p \in Poly$ and $T \in \mathcal{B}_A(\mathcal{H})$. If $TT^{\#}T = T$, then the following statements are equivalent:*

(1) *T is a left p-partial-A-isometry.*

(2) $\mathsf{N}(T) \subseteq \mathsf{N}(p(T)A)$.

Proof Since $T^*A = AT^{\#}$, we observe that

$$\begin{aligned} p(T)T^*AT = p(T)A \quad &\Leftrightarrow \quad p(T)AT^{\#}T = p(T)A \\ &\Leftrightarrow \quad p(T)A(I - T^{\#}T) = 0 \\ &\Leftrightarrow \quad \mathsf{R}(I - T^{\#}T) \subseteq \mathsf{N}(p(T)A) \\ &\Leftrightarrow \quad \mathsf{N}(T^{\#}T) \subseteq \mathsf{N}(p(T)A) \\ &\Leftrightarrow \quad \mathsf{N}(T) \subseteq \mathsf{N}(p(T)A). \end{aligned}$$

□

For $T \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})^+$, we take $p(t) = t^q$ $(q \geq 1)$, and we put

$$\mathcal{Q}_q\big(T, A\big) = T^q(T^*AT - A).$$

Proposition 2.8 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $p \in Poly$, and $T \in \mathcal{B}(\mathcal{H})$ be a left p-polynomially partial-A-isometry for $q \geq 2$ such that $\mathsf{N}(T) = \mathsf{N}(T^2)$, then T is a partial-A-isometry.*

Proof Since T and T^2 have the same null space it follows that T and T^q have the same null space for all $q \geq 2$. Consequently, if $\mathcal{Q}_q\big(T, A\big) = 0$ then $\mathcal{Q}_1\big(T, A\big) = 0$. The assertion of the proposition is an immediate consequence of this fact. □

In his work [5, Proposition 2.17], Aouichaoui established a straightforward yet important result: If T stands as a partial-A-isometry, then the kernel of T, denoted by $\mathsf{N}(T)$, acts as a reducing subspace of A. This naturally prompts the question: Does a similar outcome hold true for the kernel of $p(T)$ when T takes on the role of a left p-partial-A-isometry, with the condition that $p(t) \neq t$? Exploring this, we inquire whether $\mathsf{N}(p(T))$ remains a reducing subspace of A. However, the subsequent example provides a counterpoint, demonstrating that the answer to this inquiry is negative.

Example 2.9 *We take $p(t) = t^2$. By writing down a few immediate necessary conditions, we observe that*

$$T := \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} 1 & -1 & 0 \\ -1 & \frac{7}{4} & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

are suitable. Indeed, we have $A > 0$ and $T^*AT - A$ has its last row equal to zero. Since $\mathsf{N}(p(T)) =$ $\mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$ and $\mathsf{R}(T^*AT - A) \subset \mathsf{N}(p(T))$, we indeed have

$$p(T)(T^*AT - A) = 0,$$

and so T is a left p-polynomially partial-A-isometry (T is not a partial-A-isometry). However, $A\begin{pmatrix}0\\1\\0\end{pmatrix} \notin \mathsf{N}(p(T))$, *which implies that* $\mathsf{N}(p(T))$ *is not a reducing subspace of* A.

Lemma 2.10 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $p \in Poly$, and $T \in \mathcal{B}(\mathcal{H})$ be a left p-polynomially partial-A-isometry. If $\mathsf{N}(p(T))$ is a reducing subspace for T and A, then $T^*AT(\mathsf{N}(p(T))) \subseteq \mathsf{N}(p(T))$.*

Proposition 2.11 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $p \in Poly$, and $T \in \mathcal{B}(\mathcal{H})$. If $\mathsf{N}(p(T))$ is a reducing subspace for T and A, then the following statements are equivalent:*

(1) *T is a left p-polynomially partial-A-isometry.*

(2) *$T_{|\mathsf{N}(\mathsf{p}(\mathsf{T}))^\perp}$ is an A-isometry.*

Proof (1) $\Longrightarrow$ (2) follows from Proposition 2.5.
(2) $\Longrightarrow$ (1) We have $T^*AT = A$ on $\mathsf{N}(p(T))^\perp$. This implies $T^*ATp(T)^* = Ap(T)^*$. The result follows by duality. □

In the following, we give a new characterization of a left p-partial-A-isometry.

Proposition 2.12 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $T \in \mathcal{B}(\mathcal{H})$ and $p \in Poly$. Then the following statements are equivalent.*

(1) *T is a left p-partial-A-isometry.*

(2) *$A(\mathsf{N}(T)) \subset \mathsf{N}(p(T))$ and*

$$p(T)T^*ATT^* = p(T)AT^*.$$

Proof (1) $\Longrightarrow$ (2) Trivial.

(2) $\Longrightarrow$ (1) Suppose that

$$p(T)T^*ATT^* = p(T)AT^*.$$

Put $\mathcal{H} = \mathsf{N}(T) \oplus \overline{\mathsf{R}(T^*)}$ and for $x \in \mathcal{H}, x = x_1 + x_2$ with $x_1 \in \mathsf{N}(T)$ and $x_2 \in \overline{\mathsf{R}(T^*)}$, we can write $x_2 = \lim_{n\to+\infty} T^*y_n$, for some $(y_n)_n \subseteq \mathcal{H}$ and we have

$$\begin{aligned}
&(p(T)T^*AT - p(T)A)x \\
&= (p(T)T^*AT - p(T)A)x_1 + (p(T)T^*AT - p(T)A)x_2 \\
&= (p(T)T^*AT - p(T)A)x_1 + \underbrace{\lim_{n\to+\infty}(p(T)T^*AT - p(T)A)T^*y_n}_{=0} \\
&= (p(T)T^*AT - p(T)A)x_1 \\
&= -p(T)Ax_1.
\end{aligned}$$

Since $A(\mathsf{N}(T)) \subset \mathsf{N}(p(T))$, we obtain $p(T)Ax_1 = 0$. Then T is a left p-partial-A-isometry. □

Proposition 2.13 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $T \in \mathcal{B}(\mathcal{H})$, and $p \in Poly$. If T is a left p-polynomially partial isometry and $\mathsf{N}(p(T)) \subseteq \mathsf{N}(A)$, then*

$$T(\mathsf{N}(p(T))^\perp) \perp_A T(\mathsf{N}(p(T))).$$

Proof Applying Proposition 2.5, $T^*AT(x) = A(x)$, $x \in \mathsf{N}(p(T))^\perp$. For $x \in \mathsf{N}(p(T))^\perp$ and $y \in \mathsf{N}(p(T)) \subseteq \mathsf{N}(A)$,

$$< Tx, Ty >_A = < T^*ATx, y > = < Ax, y > = < x, Ay > = 0$$

yields $T(\mathsf{N}(p(T))^\perp) \perp_A T(\mathsf{N}(p(T)))$. □

Now, we present a decomposition for a left p-partial-A-isometry.

Theorem 2.14 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $T \in \mathcal{B}(\mathcal{H})$ and $p \in Poly$. If $\mathsf{N}(p(T))$ is a reducing subspace for T and A, then the following statements are equivalent:*

(1) *T is a left p-partial-A-isometry;*

(2) *there exist $B, C \in \mathcal{B}(\mathcal{H})$ such that $C^*AC = AP_{\mathsf{N}(p(T))^\perp}$,*

$$T = C + B, \qquad CB = C^*B = BC^* = B^*AC = 0, \qquad p(B)B^*AB = 0,$$

where $P_{\mathsf{N}(p(T))^\perp}$ denotes the orthogonal projection onto $\mathsf{N}(p(T))^\perp$.

Proof (1) $\Longrightarrow$ (2) Assume that T is a left p-partial-A-isometry. Let $Q = P_{\mathsf{N}(p(T))^\perp}$, $C = TQ$, and $B = T(I - Q)$. Since $\mathsf{N}(p(T))$ is a reducing subspace for T and A, we verify that $CB = C^*B = BC^* = C^*AB = B^*AC = 0$, $(I-Q)T(I-Q) = T(I-Q)$ and $B^n = T^n(I-Q)$, for $n \in \mathbb{N}$. Also, $C^*AC = QT^*ATQ = QAQ = AQ$ and $p(B)B^*AB = (p(T)(I-Q) + p(0)Q)B^*AB = p(0)QB^*AB = p(0)Q(I-Q)T^*AB = 0$.

(2) $\Longrightarrow$ (1) Set $p(z) = \sum\limits_{i=0}^{n} a_i z^i$, where $a_0, a_1, \dots, a_n \in \mathbb{C}$. Notice that, by $CB = C^*B = BC^* = C^*AB = B^*AC = 0$, we have

$$p(T) = a_n \sum_{i=0}^{n} B^i C^{n-i} + a_{n-1} \sum_{i=0}^{n-1} B^i C^{n-1-i} + \cdots + a_1(C + B) + a_0 I$$

and $T^*AT = C^*AC + B^*AB$. Using $p(B)B^*AB = 0$, we get

$$\begin{aligned} p(T)B^*AB &= a_n B^n B^*AB + a_{n-1}B^{n-1}B^*AB + \cdots + a_1 BB^*AB + a_0 B^*AB \\ &= p(B)B^*AB = 0. \end{aligned}$$

If $z \in \mathcal{H}$, then $z = x + y$, where $x \in \mathsf{N}(p(T))^\perp$ and $y \in \mathsf{N}((p(T))$. Since $C^*AC = AP_{\mathsf{N}(p(T))^\perp}$, then

$$\begin{aligned} p(T)C^*ACz &= p(T)C^*ACx + p(T)C^*ACy \\ &= p(T)Ax + 0 = p(T)Ax + p(T)Ay \\ &= p(T)Az. \end{aligned}$$

Hence,

$$p(T)T^*AT = p(T)(C^*AC + B^*AB) = p(T)A.$$

□

Theorem 2.15 *Let $A \in \mathcal{B}(\mathcal{H})^+$, $T \in \mathcal{B}(\mathcal{H})$ and $p \in Poly$. If $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T) \cap \mathsf{N}(A)$, then the following statements are equivalent:*

(1) *T is a left p-partial-A-isometry,*

(2) *there exist two closed subspaces M and N of $\mathcal{H}$ such that $\mathcal{H} = M \oplus N$, $T|_M$ is an A-isometry and $N \subseteq \mathsf{N}(p(T))$.*

Proof (1) $\Longrightarrow$ (2) If T is a left p-partial-A-isometry, for $M = \mathsf{N}(p(T))^\perp$ and $N = \mathsf{N}(p(T))$, the rest is clear.

(2) $\Longrightarrow$ (1) For $x \in \mathcal{H}$, set $x = x_1 + x_2 \in \mathcal{H}$, $x_1 \in M$ and $x_2 \in N$. Then, by $N \subseteq \mathsf{N}(p(T)) \subseteq \mathsf{N}(T) \cap \mathsf{N}(A)$,

$$p(T)T^*ATx = p(T)T^*ATx_1 + p(T)T^*ATx_2 = p(T)Ax_1 + 0 = p(T)Ax.$$

□

We also investigate the decomposition of an operator that exhibits both p-normality and is a left p-partial-A-isometry.

Theorem 2.16 *Let $p \in Poly$, $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$ be p-normal. If T is a left p-partial-A-isometry, then T is decomposed by $\mathsf{N}(p(T))^\perp$ and $\mathsf{N}(p(T))$ in the direct sum $T = S \oplus C$, where S is an A-isometry and $p(C) = 0$.*

Proof Since $p(T)T^* = T^*p(T)$, we conclude that $\mathsf{N}(p(T))$ is a reducing subspace for T. It is well-known that $S = T|_{\mathsf{N}(p(T))^\perp}$ is an A-isometry. Also, $p(C) = 0$ for $C = T|_{\mathsf{N}(p(T))}$. □

The following condition for an operator $T \in \mathcal{B}(\mathcal{H})$ was introduced by Apostol [6]:

$$\lim_n \|T^*T^n - T^nT^*\|^{\frac{1}{n}} = 0. \tag{1}$$

Set

$$\mathcal{H}_0 = \{x \in \mathcal{H} : \lim_n \|T^n x\|^{\frac{1}{n}} = 0\}$$

and note that $\mathcal{H}_0$ is subspace of $\mathcal{H}$ and it is invariant under T.

Theorem 2.17 *Let $p \in Poly$, $A \in \mathcal{B}(\mathcal{H})^+$, $T \in \mathcal{B}(\mathcal{H})$ satisfy (1) and $\mathsf{N}(p(T)) \subseteq \overline{\mathcal{H}_0}$. If $\mathsf{N}(p(T))$ is a reducing subspace for T and A and $\overline{\mathcal{H}_0}$ is a reducing subspace for A, then T is a left p-partial-A-isometry if and only if there exist three subspaces $M_1, M_2, M_3 \subseteq \mathcal{H}$ such that*

(i) *M_1, M_2, M_3 are reducing subspaces of T^*AT;*

(ii) *$\mathcal{H} = M_1 \oplus M_2 \oplus M_3$;*

(iii) *M_1 is invariant under T, $M_1 \subseteq \mathsf{N}(p(T))$, $T|_{M_2}$ is isometry, M_3 reduce T and $T|_{M_3}$ is normal and A-isometry.*

Proof Assume that T is a left p-partial-A-isometry. Because $\mathsf{N}(p(T)) \subseteq \overline{\mathcal{H}_0}$, for $M_1 = \mathsf{N}(p(T))$, $M_2 = \mathsf{N}(p(T))^\perp \cap \overline{\mathcal{H}_0}$ and $M_3 = \overline{\mathcal{H}_0}^\perp$, note that $\mathsf{N}(p(T))^\perp = M_2 \oplus M_3$ and $\mathcal{H} = M_1 \oplus M_2 \oplus M_3$. Applying (or [14, Proposition 2]), we observe that M_1, M_2, M_3 are reducing subspaces of T^*AT and M_3 is a reducing subspace of T. Using [14, Proposition 2] and $\overline{\mathcal{H}_0}^{\perp} \subseteq \mathsf{N}(p(T))^{\perp}$, we deduce that $T|_{M_2}$ is A-isometry and $T|_{M_3}$ is normal and A-isometry.

On the other hand, for $x = x_1 + x_2 + x_3 \in M_1 \oplus M_2 \oplus M_3$, by $(T^*AT)|_{M_k} = T^*|_{M_k} A|_{M_k} T|_{M_k}$, $k = 1, 2, 3$;

$$\begin{aligned} p(T)T^*ATx &= p(T)T^*|_{M_1}A|_{M_1}T|_{M_1}x_1 + p(T)T^*|_{M_2}A|_{M_2}T|_{M_2}x_2 + p(T)T^*|_{M_3}A|_{M_3}T|_{M_3}x_3 \\ &= p(T)A(x_2 + x_3) = p(T)Ax. \end{aligned}$$

Hence, T is a left p-partial-A-isometry. □

3. Spectral properties of left p-partial-A-isometries

For an operator $T \in \mathcal{B}(\mathcal{H})$, let $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$, and $\sigma_{su}(T)$ denote its spectrum, approximative point spectrum, point spectrum and surjective spectrum respectively. For a polynomial p, we denote $\mathfrak{R}(p)$ as the set of its roots.

Remark 3.1 *Note that for $p(t) = t$, if $T \in \mathcal{B}(\mathcal{H})$ is a left p-partial-A-isometry (or equivalently T is a partial-A-isometry), then $\mathsf{N}(p(T))$ is A-invariant and $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT)$.*

The following main theorem and its corollary generalize the results presented in [5, Theorem 4.3 & Corollary 4.5]. They provide crucial spectral properties of left p-partial-A-isometries T subject to appropriate conditions. These results represent a significant extension of the findings from the aforementioned reference and shed new light on the understanding of left p-partial-A-isometries.

Theorem 3.2 *Let $T \in \mathcal{B}(\mathcal{H})$ be a left p-partial-A-isometry such that $\mathsf{N}(p(T))$ is invariant by A, $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT)$ and $\mathsf{R}(Ap(T)) \subseteq \mathsf{R}(\bar{p}(T^*))$. If $0 \notin \sigma_{ap}(A)$ then*

(1) $\sigma_{ap}(T) \subseteq \partial\mathbb{D} \cup \mathfrak{R}(p)$. *In particular,* $\overline{\sigma_p(T)} \subseteq \partial\mathbb{D} \cup \mathfrak{R}(p)$.

(2) $\bar{\lambda} \in \sigma_{ap}(T^*) \setminus \mathfrak{R}(\bar{p})$ *whenever* $\lambda \in \sigma_{ap}(T) \setminus \mathfrak{R}(p)$.

(3) $\bar{\lambda} \in \sigma_p(T^*) \setminus \mathfrak{R}(\bar{p})$ *whenever* $\lambda \in \sigma_p(T) \setminus \mathfrak{R}(p)$.

(4) *If p is a (nontrivial) monomial, then the eigenspaces of T corresponding to distinct eigenvalues are mutually orthogonal in $(\mathcal{H}, \|\cdot\|_A)$.*

Proof Consider $T \in \mathcal{B}(\mathcal{H})$ be a left p-partial-A-isometry such that $\mathsf{R}(Ap(T)) \subseteq \mathsf{R}(\bar{p}(T^*))$ and suppose that $0 \notin \sigma_{ap}(A)$.

(1) Let $\lambda \in \sigma_{ap}(T)$, then there exists a sequence $(x_n) \subset \mathcal{H}$ such that for all $n \in \mathbb{N}, \|x_n\| = 1$, and $(T - \lambda)x_n \longrightarrow 0$ as $n \longrightarrow \infty$. For all $n \geq 1$, write $p(T)x_n = t_n + w_n \in \mathsf{N}(p(T))^{\perp} \oplus \mathsf{N}(p(T))$ and

$x_n = y_n + z_n \in \mathsf{N}(p(T))^\perp \oplus \mathsf{N}(p(T))$. Then

$$\begin{aligned}
&\langle (T^*AT - A)p(T)x_n, x_n\rangle \\
=& \langle (T^*AT - A)w_n, y_n + z_n\rangle \quad (\textit{because } T^*AT - A = 0 \textit{ on } \mathsf{N}(p(T))^\perp) \\
=& - \langle Aw_n, y_n + z_n\rangle \quad (\textit{because } \mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT)) \\
=& - \langle Aw_n, z_n\rangle \quad (\textit{because } \mathsf{N}(p(T)) \textit{ is invariant by } A) \\
=& \langle At_n - Ap(T)x_n, z_n\rangle \\
=& 0 \quad (\textit{because } \mathsf{N}(p(T))^\perp \textit{ is invariant by } A \textit{ and } \mathsf{R}(Ap(T)) \perp \mathsf{N}(p(T))).
\end{aligned}$$

This leads to

$$\begin{aligned}
0 =& \langle (T^*AT - A)p(T)x_n, x_n\rangle \\
=& \langle (T^*AT - A)(p(T) - p(\lambda))x_n, x_n\rangle + p(\lambda)(\|Tx_n\|_A^2 - \|x_n\|_A^2) \\
=& \langle (T^*AT - A)(p(T) - p(\lambda))x_n, x_n\rangle + p(\lambda)(\|(T-\lambda)x_n\|_A^2 \\
&+ 2\Re e\langle (T-\lambda)x_n, \lambda x_n\rangle_A + |\lambda|^2\|x_n\|_A^2 - \|x_n\|_A^2).
\end{aligned}$$

Since $(T-\lambda)\,x_n \longrightarrow 0$, by induction it follows that for each integer k, $\left(T^k - \lambda^k\right)x_n \longrightarrow 0$, and so $(p(T) - p(\lambda))\,x_n \longrightarrow 0$. This leads to

$$p(\lambda)\left(|\lambda|^2 - 1\right)\lim_{n\to\infty}\|x_n\|_A^2 = 0.$$

Moreover, since $0 \notin \sigma_{ap}(A)$ and A is positive, it follows that $\lim_{n\to\infty}\|x_n\|_A^2 \neq 0$. Then $p(\lambda) = 0$ or $|\lambda| = 1$.

(2) Let $\lambda \in \sigma_{ap}(T)\backslash\mathfrak{R}(p)$. Let $(x_n)_n$ be a sequence of unit vectors such that $(T-\lambda)x_n \longrightarrow 0$. Since T is a left p-partial-A-isometry, satisfying the following conditions: $\mathsf{N}(p(T))$ is invariant by A, $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT)$ and $\mathsf{R}(Ap(T)) \subseteq \mathsf{R}(\bar{p}(T^*))$, it follows that $(T^*AT - A)p(T)x_n \longrightarrow 0$. Since $(T-\lambda)x_n \longrightarrow 0$, then

$$p(\lambda)(-Ax_n + \lambda T^*Ax_n) \longrightarrow 0.$$

Using (1), we have $\sigma_{ap}(T) \subseteq \partial\mathbb{D} \cup \mathfrak{R}(p)$ then for $\lambda \notin \mathfrak{R}(p)$, we obtain

$$-Ax_n + \lambda T^*Ax_n = (\lambda T^* - I)\,Ax_n \longrightarrow 0,$$

so $\left(T^* - \frac{1}{\lambda}I\right)\frac{Ax_n}{\|Ax_n\|} \longrightarrow 0$. This leads to $\bar{\lambda} = \frac{1}{\lambda} \in \sigma_{ap}(T^*)\backslash\mathfrak{R}(\bar{p})$.

(3) Similarly proven.

(4) Consider α and β two distinct eigenvalues of T. Without loss of generality, we can assume that $\alpha \neq 0$. Take x and y two nonzero vectors such that $Tx = \alpha x$ and $Ty = \beta y$. By induction, it follows that for each integer k, $T^kx = \alpha^kx$, and so $p(T)x = p(\alpha)x$. Considering the left p-partial-A-isometry nature of operator T, and with the following conditions in place: $\mathsf{N}(p(T))$ being A-invariant, $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT)$, and $\mathsf{R}(Ap(T)) \subseteq \mathsf{R}(\bar{p}(T^*))$, we are poised to derive the following:

$$\begin{aligned}
0 =& \langle (T^*AT - A)p(T)x, y\rangle \\
=& p(\alpha)(\langle AT(x), Ty\rangle - \langle A(x), y\rangle) \\
=& p(\alpha)(\alpha\bar{\beta} - 1)\langle x, y\rangle_A.
\end{aligned}$$

As $\alpha \neq \beta$ and $|\alpha| = 1$, we infer that $\langle x, y\rangle_A = 0$ and, therefore, the proof is completed. □

Lemma 3.3 *Consider A as a compact set in $\mathbb{C}$ such that its boundary ∂A satisfies $\partial A \subseteq \partial\mathbb{D} \cup \{0\}$. Then, either $A \subseteq \partial\mathbb{D} \cup \{0\}$, or $A = \overline{\mathbb{D}}$.*

Proof First, we show that $A \subseteq \overline{\mathbb{D}}$. The mapping

$$A \longrightarrow \mathbb{R}$$

$$z \longmapsto |z|$$

is continuous; it attains its maximum at a point w. This point w is not an interior point of A, as otherwise, A would contain a certain ball $B(w, \varepsilon)$. However, in this case, the point $u = w + \varepsilon\frac{w}{|w|} \in A$ would have a norm greater than that of w, which contradicts the maximality of w. Consequently, w lies on the boundary of A and, therefore, $A \subseteq \overline{\mathbb{D}}$.

Now, we will demonstrate that either A is the closed unit disk or is contained within the unit circle union $\{0\}$. Set

$$U := \{z \in \mathbb{C} \,:\, 0 < |z| < 1\}.$$

One can see that the set $A \cap U$ is both a closed and an open subset of U. Since U is connected, there are two possible scenarios: either A contains U, in which case A is the closed unit disk, or A is contained within the complement of U. In the latter case, A is contained within the unit circle union $\{0\}$. This concludes the proof. □

Combining Theorem 3.2 and Lemma 3.3, we derive the next corollary.

Corollary 3.4 *Let p be a (nontrivial) monomial. Let $T \in \mathcal{B}(\mathcal{H})$ be a left p-partial-A-isometry such that $\mathsf{N}(p(T))$ is invariant by A, $\mathsf{N}(p(T)) \subseteq \mathsf{N}(T^*AT)$ and $\mathsf{R}(Ap(T)) \subseteq \mathsf{R}(\bar{p}(T^*))$. If $0 \notin \sigma_{ap}(A)$, then*

$$\sigma(T) \subseteq \partial\mathbb{D} \cup \{0\} \text{ or } \sigma(T) = \overline{\mathbb{D}}.$$

Proof Since we always have $\sigma(T)$ is compact and $\partial\sigma(T) \subseteq \sigma_{ap}(T)$, the result follows directly from Theorem 3.2 and Lemma 3.3.

4. Matrix representation of left p-partial-A-isometries and some applications

Examining the matrix representation of left p-partial-A-isometries, let us begin with this straightforward proposition.

Proposition 4.1 *Let $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ be a closed subspace which is T-invariant and A-invariant. If T is a left p-partial A-isometry, then $T_{|\mathcal{M}}$ is a left p-partial-$A_{|\mathcal{M}}$-isometry.*

Proof Since $\mathcal{M}$ is A-invariant and $A \geq 0$, it follows that $\mathcal{M}$ is a reducing subspace for A and therefore

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

Let us consider the matrix representation of T as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

We have

$$T^r = \begin{pmatrix} T_1^r & \sum_{i=0}^{r-1} T_1^i T_2 T_3^{r-1-i} \\ 0 & T_3^r \end{pmatrix} \text{, for all } r \geq 1.$$

Since T is a p-partial-A-isometry, we have

$$\begin{pmatrix} p(T_1) & \star \\ 0 & p(T_3) \end{pmatrix} \left(\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^* \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right) = 0$$

and it is easy to see that

$$p(T_1)\Big(T_1^* A_1 T_1 - A_1\Big) = 0.$$

Thus, T_1 is a left p-partial-$A_{|\mathcal{M}}$-isometry

Remark 4.2 *Let $p \in Poly$ be nontrivial. Since $\mathsf{N}(p(T))$ is T-invariant, then $\mathsf{N}(p(T))$ reduces T if and only if $\mathsf{N}(p(T))^{\perp}$ is T-invariant. In this case and relative to the decomposition $\mathcal{H} = \mathsf{N}(p(T))^{\perp} \oplus \mathsf{N}(p(T))$, T can be written as*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

Theorem 4.3 *Let $T \in \mathcal{B}(\mathcal{H})$ be such that $\mathsf{N}(p(T))^{\perp}$ is T-invariant and A-invariant. Then the following assertions are equivalent:*

(1) *T is a left p-partial-A-isometry;*

(2) *$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} = \mathsf{N}(p(T))^{\perp} \oplus \mathsf{N}(p(T))$, where T_1 is an $A_{|\mathsf{N}(p(T))^{\perp}}$-isometric operator and $p(T_2) = 0$.*

Proof (1)$\Longrightarrow$ (2). Since T is a left p-partial-A-isometry, by Proposition 2.5, it follows that

$$T^*AT - A = 0, \text{ on } \mathsf{N}(p(T))^{\perp}.$$

In particular, this implies that

$$T_1^* A_1 T_1 - A_1 = 0;$$

where $A_1 = A_{|\mathsf{N}(p(T))^{\perp}}$. Hence, T_1 is an A_1-isometric operator.

On the other hand, we have $p(T_2) = p(T)_{|\mathsf{N}(p(T))} = 0$.

(2) $\Longrightarrow$ (1). Conversely, assume that $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} = \mathsf{N}(p(T))^\perp \oplus \mathsf{N}(p(T))$, where T_1 is an $A_{|\mathsf{N}(p(T))^\perp}$-isometric operator and $p(T_2) = 0$. Then we have

$$p(T)T^*AT = \begin{pmatrix} p(T_1)T_1^*A_1T_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p(T_1)A_1 & 0 \\ 0 & 0 \end{pmatrix} = p(T)A.$$

□

Corollary 4.4 *Let $p \in Poly$ be a (nontrivial) monomial. Let $T \in \mathcal{B}(\mathcal{H})$ be such that $\mathsf{N}(p(T))^\perp$ is T-invariant and A-invariant. If T is a left p-partial-A-isometry, then T is a direct sum of an A-isometric operator and a nilpotent operator. In particular, if $0 \notin \sigma_{ap}(A)$, then $\sigma_{ap}(T) \subseteq \partial\mathbb{D} \cup \{0\}$ and $\sigma(T) \subseteq \overline{\mathbb{D}}$.*

Theorem 4.5 *Let q be a positive integer. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})^+$. If T_1 is a surjective A-isometry and T_3 is a nilpotent operator of order q, then T is similar to a left t^q-partial-$(A \oplus A)$-isometry.*

Proof Under the assumptions that T_1 is surjective and $T_3^q = 0$, we have $\sigma_{su}(T_1) \cap \sigma_{ap}(T_3) = \emptyset$. From the statement (c) in [26, Theorem 3.5.1], there exists an operator $S \in \mathcal{B}(\mathcal{H})$ for which $T_1S - ST_3 = T_2$. Therefore, we can write

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},$$

which implies that T is similar to $\mathbf{X} = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$.

Since T_1 is an A-isometry and $T_3^q = 0$, we obtain

$$\mathbf{X}^q\mathbf{X}^*(A \oplus A)\mathbf{X} = \begin{pmatrix} T_1^qT_1^*AT_1 & 0 \\ & \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1^qA & 0 \\ & \\ 0 & 0 \end{pmatrix} = \mathbf{X}^q(A \oplus A).$$

Therefore, T is similar to a left t^q-partial-$(A \oplus A)$-isometric operator. □

Theorem 4.5 establishes a significant similarity result, affirming that, under certain conditions, operators exhibit a structured similarity to a left t^q-partial-$(A\oplus A)$-isometry. It is crucial to emphasize that this similarity, as demonstrated by the theorem, does not necessarily imply equality. This distinction becomes particularly evident in the subsequent remark.

Remark 4.6 *We take $p(t) = t$ (i.e. $q = 1$). Let $I \in \mathcal{B}(\mathcal{H})$ denote the identity operator. Let $T_1 \in \mathcal{B}(\mathcal{H})$ be an unitary operator, $T_2 \in \mathcal{B}(\mathcal{H})$ an arbitrary nonzero operator and $T_3 = 0$. Then T_1 is a surjective I-isometry, but $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is not a left p-partial-$(I \oplus I)$-isometry.*

Let $\mathcal{HS}(\mathcal{H})$ refer to the class of Hilbert-Schmidt operators on $\mathcal{H}$. These operators have held a significant significance within operator theory. For more comprehensive insights, refer to [8, Problem 40], as well as [10] and [11].

For $S \in \mathcal{B}(\mathcal{H})$, let $L_S \in \mathcal{B}(\mathcal{HS}(\mathcal{H}))$ and $R_S \in \mathcal{B}(\mathcal{HS}(\mathcal{H}))$ denote the operators $L_S(T) := ST$ and $R_S(T) := TS$ of left and right multiplication by S.

The following observations are easily verified:

$$L_S^* = L_{S^*} \quad \text{and} \quad R_S^* = R_{S^*}.$$

Moreover, through induction, for any integer k, the relationships hold: $L_S^k = L_{S^k}$ and $R_S^k = R_{S^k}$. Consequently, for any polynomial p, the following are valid: $p(L_S) = L_{p(S)}$ and $p(R_S) = R_{p(S)}$. In the special case where S is positive, it follows that both L_S and R_S retain their positivity.

Theorem 4.7 *Let $p \in Poly$ be nontrivial. Assume that $S \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})^+$. Then the following statements are equivalent:*

(1) *L_S is a left p-partial-L_A-isometry on $\mathcal{HS}(\mathcal{H})$;*

(2) *S is a left p-partial-A-isometry on $\mathcal{H}$;*

(3) *R_S^* is a left $\bar{p}$-partial-R_A-isometry on $\mathcal{HS}(\mathcal{H})$.*

Proof Note that L_S is a left p-partial-L_A-isometry on $\mathcal{HS}(\mathcal{H})$ equivalently to

$$p(L_S)L_{S^*}L_A L_S = p(L_S)L_A.$$

That is, for any $T \in \mathcal{HS}(\mathcal{H})$,

$$p(S)S^*AST = p(S)AT.$$

Hence, the implication $(2) \Longrightarrow (1)$ is obvious. Since

$$\mathcal{H} = \bigcup_{T\in\mathcal{B}(\mathcal{H}),\ rank(T)=1} \mathsf{R}(T)$$

and the operators of rank one are Hilbert-Schmidt operators, we get

$$p(S)S^*AST = p(S)AT;\ \forall\, T \in \mathcal{HS}(\mathcal{H}) \Longrightarrow p(S)S^*AS = p(S)A.$$

As an operator T is Hilbert-Schmidt if and only if its adjoint is so, the equivalence $(2) \Longleftrightarrow (3)$ is obtained similarly. Therefore, the proof is completed. $\square$

Let S_1 and S_2 be operators in $\mathcal{B}(\mathcal{H})$. The tensor product $S_1 \otimes S_2^*$ may be identified with the elementary operator $\mathcal{E}_{S_1,S_2}$ which is defined on $\mathcal{HS}(\mathcal{H})$ by $\mathcal{E}_{S_1,S_2}(T) := S_1TS_2$, for $T \in \mathcal{HS}(\mathcal{H})$, that is

$$\mathcal{E}_{S_1,S_2} = L_{S_1}R_{S_2}.$$

Note that if S_1 and S_2 are positive, then one can check that $\mathcal{E}_{S_1,S_2}$ is so.

Theorem 4.8 *Let $p \in Poly$ be a (nontrivial) monomial. Let $S_1, S_2 \in \mathcal{B}(\mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H})^+$. Assume that S_1 is a left p-partial-A-isometry and S_2^* is a left $\bar{p}$-partial-B-isometry. Then $\mathcal{E}_{S_1,S_2}$ is a left p-partial-$\mathcal{E}_{A,B}$-isometry. If, furthermore, one of the following statements hold:*

(1) $\mathsf{N}(p(S_1))$ *is invariant by* A, $\mathsf{N}(p(S_1)) \subseteq \mathsf{N}(S_1^* A S_1)$, $\mathsf{R}(Ap(S_1)) \subseteq \mathsf{R}(\bar{p}(S_1^*))$, $\mathsf{R}(p(S_2))$ *is invariant by* B, $\mathsf{N}(\bar{p}(S_2^*)) \subseteq \mathsf{N}(S_2 B S_2^*)$, $\mathsf{R}(B\bar{p}(S_2^*)) \subseteq \mathsf{R}(p(S_2))$, *and* $0 \notin \sigma_{ap}(A) \cup \sigma_{ap}(B)$.

(2) $\mathsf{N}(\mathcal{E}_{p(S_1),p(S_2)})$ *is invariant by* $\mathcal{E}_{A,B}$, $\mathsf{N}(\mathcal{E}_{p(S_1),p(S_2)}) \subseteq \mathsf{N}(\mathcal{E}_{S_1^* A S_1, S_2 B S_2^*})$, $\mathsf{R}(\mathcal{E}_{Ap(S_1),p(S_2)B}) \subseteq \mathsf{R}(\mathcal{E}_{\bar{p}(S_1^*),\bar{p}(S_2^*)})$, *and* $0 \notin \sigma_{ap}(A) \cup \sigma_{ap}(B)$,

then

$$\sigma_{ap}(\mathcal{E}_{S_1,S_2}) \subseteq \partial\mathbb{D} \cup \{0\},$$

$$\overline{\sigma_p(\mathcal{E}_{S_1,S_2})} \subseteq \partial\mathbb{D} \cup \{0\},$$

and

$$\sigma(\mathcal{E}_{S_1,S_2}) \subseteq \overline{\mathbb{D}}.$$

Proof Without loss of generality, we may assume that p is a monic polynomial. Since S_1 is a left p-partial-A-isometry and S_2^* is a left $\bar{p}$-partial-B-isometry, it follows from Theorem 4.7 that L_{S_1} is a left p-partial-L_A-isometry and R_{S_2} is a left p-partial-R_B-isometry. Consequently,

$$\begin{aligned}
&p(\mathcal{E}_{S_1,S_2})\mathcal{E}_{S_1,S_2}^*\mathcal{E}_{A,B}\mathcal{E}_{S_1,S_2} \\
&= p(L_{S_1})p(R_{S_2})R_{S_2}^* L_{S_1}^* L_A R_B L_{S_1} R_{S_2} \\
&= p(L_{S_1})L_{S_1}^* L_A L_{S_1} p(R_{S_2}) R_{S_2}^* R_B R_{S_2} \\
&= p(L_{S_1}) L_A p(R_{S_2}) R_B \\
&= p(L_{S_1}) p(R_{S_2}) L_A R_B \\
&= p(\mathcal{E}_{S_1,S_2})\mathcal{E}_{A,B}
\end{aligned}.$$

This means that $\mathcal{E}_{S_1,S_2}$ is a left p-partial-$\mathcal{E}_{A,B}$-isometry.

Assume that (1) holds. Then by Theorem 3.2 and Corollary 3.4, we get

$$\sigma_{ap}(S_1) \cup \sigma_{ap}(S_2^*) \subseteq \partial\mathbb{D} \cup \{0\}$$

and

$$\sigma(S_1) \cup \sigma(S_2^*) \subseteq \overline{\mathbb{D}}.$$

So from [9, 24], we obtain

$$\sigma_{ap}(\mathcal{E}_{S_1,S_2}) = \sigma_{ap}(S_1 \otimes S_2^*) = \sigma_{ap}(S_1)\sigma_{ap}(S_2^*) \subseteq \partial\mathbb{D} \cup \{0\}$$

and

$$\sigma(\mathcal{E}_{S_1,S_2}) = \sigma(S_1 \otimes S_2^*) = \sigma(S_1)\sigma(S_2^*) \subseteq \overline{\mathbb{D}}.$$

Now, if (2) is satisfied, then $\mathsf{N}(p(\mathcal{E}_{S_1,S_2}))$ is invariant by $\mathcal{E}_{A,B}$, $\mathsf{N}(p(\mathcal{E}_{S_1,S_2})) \subseteq \mathsf{N}(\mathcal{E}_{S_1,S_2}^*\mathcal{E}_{A,B}\mathcal{E}_{S_1,S_2})$, $\mathsf{R}(\mathcal{E}_{A,B}p(\mathcal{E}_{S_1,S_2}) \subseteq \mathsf{R}(\bar{p}(\mathcal{E}_{S_1,S_2}^*))$ and $0 \notin \sigma_{ap}(\mathcal{E}_{A,B})$. Since $\mathcal{E}_{S_1,S_2}$ is a left p-partial-$\mathcal{E}_{A,B}$-isometry, the result follows directly from Theorem 3.2 and Corollary 3.4. This completes the proof. □

Acknowledgments

The authors wish to extend their heartfelt appreciation to the editor for his diligent assessment and insightful comments, which significantly improved the quality and clarity of this manuscript. The second author is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 451-03-47/2023-01/200124.

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