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Invariant subspaces of operators via Berezin symbols and Duhamel product

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Abstract: The Berezin symbol $\tilde{A}$ of an operator A on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel k_λ is defined by

$$\tilde{A}(\lambda) = \left\langle A\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|} \right\rangle, \ \lambda \in \Omega.$$

We study the existence of invariant subspaces for Bergman space operators in terms of Berezin symbols.

Key words: Reproducing kernel Hilbert space, Berezin symbol, Bergman space, invariant subspace, Duhamel product

1. Introduction

The famous Invariant Subspace Problem addresses the question of whether every bounded linear operator on an infinite-dimensional separable Hilbert space possesses a nontrivial invariant subspace.

For historical insights into the invariant subspace problem, readers can turn to the book [7].

In this article, we explore the existence of invariant subspaces for operators within the Bergman Hilbert space $L_a^2(\mathbb{D})$ using Berezin symbols.

Recall that a reproducing kernel Hilbert space (shortly, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Omega$, are continuous on $\mathcal{H}$ and for every $\lambda \in \Omega$ there exists a function $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$ or, equivalently there is no $\lambda_0 \in \Omega$ such that $f(\lambda_0) = 0$ for all $f \in \mathcal{H}$. Then by the Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The family $\{\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|} : \lambda \in \Omega\}$ is called the normalized reproducing kernel of the space $\mathcal{H}$. The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, the Bergman space $L_a^2(\mathbb{D})$, the Dirichlet space $D^2(\mathbb{D})$ and the Fock space $F(\mathbb{C})$. A detailed presentation of the theory of RKHSs and reproducing kernels is given, for instance, in Aronzajn [1], Bergman [5], Malyshev [16], Halmos [11], and Saitoh and Sawano [18].

For a bounded linear operator A on $\mathcal{H}$ (i.e., for $A \in \mathcal{B}(\mathcal{H})$, the Banach algebra of all bounded linear operators on $\mathcal{H}$), its Berezin symbol (also called Berezin transform) $\tilde{A}$ is defined on Ω as follows (see Berezin

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[3, 4] and also Engliś [8]):

$$\tilde{A}(\lambda) := \left\langle A\widehat{k}_{\lambda}(z), \widehat{k}_{\lambda}(z) \right\rangle,$$

where inner product $< .,. >$ is taken in the space $\mathcal{H}$. It is obvious from Cauchy-Schwarz inequality that the Berezin symbol $\tilde{A}$ is a bounded function and $\sup_{\lambda\in\Omega}\left|\tilde{A}(\lambda)\right|$ does not exceed $\|A\|$, i.e.,

$$\sup_{\lambda\in\Omega}\left|\tilde{A}(\lambda)\right| \leq \|A\|.$$

It is also clear from the definition of Berezin symbol that the range of the Berezin symbol $\tilde{A}$ lies in the numerical range $W(A)$ of operator A, i.e.,

$$\text{range}(\tilde{A}) = \{\tilde{A}(\lambda) : \lambda \in \Omega\} \subset W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}.$$

In the present paper, we apply the Berezin symbols technique in the study of invariant subspaces of operators.

For other applications of Berezin symbols and Duhamel products, the reader may refer to [6, 13, 19, 21] and references therein.

2. Existence of invariant subspace in Bergman space

We use reproducing kernels, Berezin symbols and Duhamel algebras methods to prove the existence of invariant subspaces for some operators acting on the Bergman space $L_a^2 = L_a^2(\mathbb{D})$. Let $\text{Hol}(\mathbb{D})$ denote the space of all holomorphic functions on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ with compact convergence. In $\text{Hol}(\mathbb{D})$, the Duhamel product is defined (see, for instance, [22]) by the formula

$$(f \circledast g)(z) := \frac{d}{dz}\int_0^z f(z-t)\,g(t)\,dt = \int_0^z f^{'}(z-t)\,g(t)\,dt + f(0)\,g(z), \tag{1}$$

where $f(z) = \sum_{n=0}^{\infty}\widehat{f}(n)\,z^n$, $g(z) = \sum_{n=0}^{\infty}\widehat{g}(n)\,z^n$, $\widehat{f}(n) = \dfrac{f^{(n)}(0)}{n!}$ denotes the n th Taylor coefficient of the function f. With this product (multiplication) $\text{Hol}(\mathbb{D})$ becomes a commutative algebra with unit $f(z) \equiv 1$.

For any $f \in Hol(\mathbb{D})$, let D_f denote the Duhamel operator defined on $\text{Hol}(\mathbb{D})$ by $D_f g = f \circledast g$. Let $dA(z)$ be the normalized Lebesgue measure on $\mathbb{D}$. The Lebesgue space of 2-summable complex-valued functions is denoted by $L^2(\mathbb{D}, dA)$. Recall that the Bergman space $L_a^2(\mathbb{D})$ is the Hilbert subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions with the norm given by

$$\|f\|_2 := \left(\int_{\mathbb{D}} |f(z)|^2\,dA(z)\right)^{1/2}.$$

This is reproducing kernel Hilbert space on $\mathbb{D}$ with reproducing kernel function $k_\lambda(z) := \frac{1}{\left(1-\overline{\lambda}z\right)^2}$. The shift operator S on the space L_a^2 is the operator defined by

$$(Sf)(z) = zf(z) = \sum_{k\geq 1} a_{k-1}z^k$$

for $f \in L_a^2$ given by the Taylor series $f(z) = \sum_{k\geq 0} a_k z^k$, $z \in \mathbb{D}$. It is straightforward to see that for any integer $n \geq 1$, S^n is a bounded operator on L_a^2 which is injective and has closed range. Therefore, S^n is left invertible, the operator $(S^{n*}S^n)^{-1}$ exists in $L(L_a^2)$ and the operator $L_n := (S^{n*}S^n)^{-1} S^{n*}$ in $L(L_a^2)$ is the left inverse of S^n; that is, $L_n S^n = I$ (the identity operator on L_a^2) in $L(L_a^2)$.

Recall that the function $\widehat{k}_\lambda(z) := \frac{1-|\lambda|^2}{(1-\bar{\lambda}z)^2}$, $z, \lambda \in \mathbb{D}$, is the normalized reproducing kernel of the space L_a^2, and $\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$, $\lambda \in \mathbb{D}$, is the Berezin symbol of operator A in $L(L_a^2)$ (for more information, see [2], [8] and [14]). Let $k_g(0)$ denote the multiplicity of zero of the function $g \in L_a^2(\mathbb{D})$ at the point $z = 0$.

Certainly, the Bergman space is one of the most important spaces of holomorphic functions, and therefore endowing it with a Banach algebra structure should be of great importance for operator theory and for function theory as well. Namely, in what follows, we need the following two known results (see [10, Theorem 2.3 and Proposition 2.6]):

Lemma 2.1 *The Bergman space $L_a^2(\mathbb{D})$ is a unital (the unit here is the constant function $\mathbf{1}$, defined by $\mathbf{1}(z) = 1, \forall z \in \mathbb{D}$) commutative Banach algebra with respect to the Duhamel convolution product $\circledast$, which will be called the Bergman-Duhamel algebra.*

Lemma 2.2 *The function $f \in L_a^2(\mathbb{D})$ is $\circledast$-invertible if and only if $k_f(0) \neq 0$.*

It is easy to see that, in general, the Berezin symbol is not multiplicative; that is, $\widetilde{AB} \neq \widetilde{A}\widetilde{B}$ (for example, if S is a shift operator on H^2, then $\widetilde{S^*S}(\lambda) = 1 \neq |\lambda|^2 = \widetilde{S^*}(\lambda)\widetilde{S}(\lambda)$ for all $\lambda \in \mathbb{D}$). Kiliç showed that (see [14, Theorem 1]) $\widetilde{AB} = \widetilde{A}(\lambda)\widetilde{B}(\lambda)$ for all B in $L(\mathcal{H}(\Omega))$ if and only if A is a multiplication operator M_φ on a reproducing kernel Hilbert space $\mathcal{H}(\Omega)$, where φ is a multiplier with $\varphi = \widetilde{A}$. Kiliç also proved (see [12, Corollary 2]) that if A is a bounded operator on the Hardy space H^2, then $\widetilde{AB}(\lambda) = \widetilde{A}(\lambda)\widetilde{B}(\lambda)$ for all B in $L(H^2)$ if and only if A is an analytic Toeplitz operator T_φ, $\varphi \in H^\infty$; moreover, $\varphi = \widetilde{A}$. Some generalizations of these results are proved in [12].

Recall that the Berezin symbol is called asymptotically multiplicative on $L(\mathcal{H}(\Omega))$ if

$$\lim_{\lambda \to \partial\Omega} \left(\widetilde{AB}(\lambda) - \widetilde{A}(\lambda)\widetilde{B}(\lambda) \right) = 0.$$

Our next result shows that in some cases, the boundary behavior of $\frac{\widetilde{A}(\lambda)\widetilde{B}(\lambda)}{\widetilde{AB}(\lambda)}$-type functions is useful in proving existence of nontrivial invariant subspace, which is the main result of the present section. We set $L_\lambda := \left(S^{\beta_\lambda^*} S^{\beta_\lambda}\right)^{-1} S^{\beta_\lambda^*}$, where $\beta_\lambda := k_{B\widehat{k}_\lambda}(0)$ is the order of zero of the function $B\widehat{k}_\lambda$ at $z = 0$.

Theorem 2.3 *Let $T : L_a^2 \to L_a^2$ be a bounded linear operator and S be a shift operator on L_a^2. Suppose that there exists a nonzero operator $B \in \{T\}'$ such that:*

(i) $\lim_{\lambda \to \partial\mathbb{D}} \left| \frac{\widetilde{I - 2SS^* + S^2S^{*2}}(\lambda)\widetilde{L_\lambda B}(\lambda)}{(I-2SS^*+S^2S^{*2})L_\lambda B(\lambda)} \right| = 0$;

(ii) there exists a sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ *tending to a point* $\xi_0 \in \partial\mathbb{D}$ *such that*

$$D_{\widehat{k}_{\lambda_n} \circledast \left(L_{\lambda_n} B\hat{k}_{\lambda_n}\right)^{-1\circledast} - \left(\widehat{k}_{\lambda_n} \circledast \left(L_{\lambda_n} B\hat{k}_{\lambda_n}\right)\right)^{-1\circledast}(0)} L_{\lambda_n}$$

converges in uniform operator topology to some operator $\mathcal{K}$ *on* L_a^2*, where* $D_{\widehat{k}_{\lambda_n} \circledast \left(L_{\lambda_n} B\hat{k}_{\lambda_n}\right)^{-1\circledast} - \left(\widehat{k}_{\lambda_n} \circledast \left(L_{\lambda_n} B\hat{k}_{\lambda_n}\right)\right)^{-1\circledast}(0)}$ *is the Duhamel operator on the Bergman space* L_a^2*. Then* T *has a nontrivial invariant subspace.*

Proof Suppose that T has no nontrivial invariant subspace in L_a^2. Then $\ker A = \{0\}$ for any nonzero operator A in $\{T\}'$, and hence $Ah \neq 0$ for any nonzero function h in L_a^2. In particular, $A\widehat{k}_\lambda \neq 0$ for all $A \in \{T\}' \setminus \{0\}$ and $\lambda \in \mathbb{D}$. We define $f_{A,\lambda} := A\widehat{k}_\lambda$ for any $\lambda \in \mathbb{D}$. Then $f_{A,\lambda} = z^{\alpha_\lambda} g_{A,\lambda}$, where $\alpha_{\lambda:=} k_{f_{A,\lambda}}(0)$ is the multiplicity of zero of the function $f_{A,\lambda}$ at the point $z = 0$, $g_{A,\lambda} \in L_a^2$ with $g_{A,\lambda}(0) \neq 0$. Hence $A\widehat{k}_\lambda = z^{\alpha_\lambda} g_{A,\lambda}$, and so $L_\lambda A\widehat{k}_\lambda = g_{A,\lambda}$, where $L_\lambda := \left(S^{\alpha_\lambda^*} S^{\alpha_\lambda}\right)^{-1} S^{\alpha_\lambda^*}$, $\lambda \in \mathbb{D}$. Since $g_{A,\lambda}(0) \neq 0$, by Lemmas 2.1 and 2.2, there exists a function $G_{A,\lambda} \in \left(L_a^2, \circledast\right)$ (which is a $\circledast$-inverse of $L_\lambda A\widehat{k}_\lambda$, that is, $G_{A,\lambda} = \left(L_\lambda A\widehat{k}_\lambda\right)^{-1\circledast}$) such that $G_{A,\lambda} \circledast g_{A,\lambda} = \mathbf{1}$ for all $\lambda \in \mathbb{D}$. Then we obtain $\left(\widehat{k}_\lambda \circledast G_{A,\lambda}\right) \circledast L_\lambda A\widehat{k}_\lambda = \widehat{k}_\lambda \circledast \mathbf{1} = \widehat{k}_\lambda$, and hence $D_{\widehat{k}_\lambda \circledast G_{A,\lambda}} L_\lambda A\widehat{k}_\lambda = \widehat{k}_\lambda$, or by denoting $F_{A,\lambda} := \widehat{k}_\lambda \circledast G_{A,\lambda} = \widehat{k}_\lambda \circledast \left(L_\lambda A\hat{k}_\lambda\right)^{-1\circledast}$, we obtain from (1) that

$$\begin{aligned} F_{A,\lambda}(0) &= \left(\widehat{k}_\lambda \circledast \left(L_\lambda A\widehat{k}_\lambda\right)^{-1\circledast}\right)(0) = \widehat{k}_\lambda(0) \left(\left(L_\lambda A\widehat{k}_\lambda\right)^{-1\circledast}\right)(0) \\ &= \left(1 - |\lambda|^2\right) \left(L_\lambda A\widehat{k}_\lambda\right)^{-1\circledast}(0) \neq 0 \end{aligned}$$

and

$$D_{F_{A,\lambda}} L_\lambda A\widehat{k}_\lambda = \widehat{k}_\lambda \tag{2}$$

for all $A \in \{T\}' \setminus \{0\}$ and $\lambda \in \mathbb{D}$; here $D_{F_{A,\lambda}}$ is the Duhamel operator on L_a^2 defined by $D_{F_{A,\lambda}} h = F_{A,\lambda} \circledast h$. From (2), we have

$$\widehat{k}_\lambda - F_{A,\lambda}(0) L_\lambda A\widehat{k}_\lambda = D_{F_{A,\lambda} - F_{A,\lambda}(0)} L_\lambda A\widehat{k}_\lambda \tag{3}$$

for any $A \in \{T\}' \setminus \{0\}$ and any $\lambda \in \mathbb{D}$. It is not difficult to show that $D_{F_{A,\lambda} - F_{A,\lambda}(0)}$ is a compact operator on L_a^2 (see the proof of Proposition 2.6 in [10]). In particular, for $A = B$, where $B \in \{T\}'$ is an operator satisfying the conditions of the theorem, we have from (3) that

$$\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0) L_{\lambda_n} B\widehat{k}_{\lambda_n} = D_{F_{B,\lambda_n} - F_{B,\lambda_n}(0)} L_{\lambda_n} B\widehat{k}_{\lambda_n} \tag{4}$$

for all $n \geq 1$, where (λ_n) is a sequence satisfying condition (ii), $L_{\lambda_n} := \left(S^{\beta_{\lambda_n}^*} S^{\beta_{\lambda_n}}\right)^{-1} S^{\beta_{\lambda_n}^*}$ and $\beta_{\lambda_n} := k_{B\widehat{k}_{\lambda_n}}(0)$, $n \geq 1$.

By virtue of condition (ii), $\lim\limits_{n\to\infty} \left\| D_{F_{B,\lambda_n} - F_{B,\lambda_n}(0)} L_{\lambda_n} - \mathcal{K} \right\|_{L(L_a^2)} = 0$ for some operator $\mathcal{K}$ on L_a^2. Obviously, $\mathcal{K}$ is compact (because $D_{F_{B,\lambda_n} - F_{B,\lambda_n}(0)}$ is compact for any $n \geq 1$). Then by considering condition

(ii) and equality (3), we obtain

$$\left\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0) L_{\lambda_n} B\widehat{k}_{\lambda_n}\right\| = \left\|\left(D_{F_{B,\lambda_n}-F_{B,\lambda_n}(0)} L_{\lambda_n} - \mathcal{K}\right) B\widehat{k}_{\lambda_n} + \mathcal{K}B\widehat{k}_{\lambda_n}\right\|$$
$$\leq \left\|D_{F_{B,\lambda_n}-F_{B,\lambda_n}(0)} L_{\lambda_n} - \mathcal{K}\right\| \|B\| + \left\|\mathcal{K}B\widehat{k}_{\lambda_n}\right\| \to 0 \quad (n \to \infty),$$

because $\left(\widehat{k}_{\lambda_n}\right)_{n\geq 1}$ is a weak null sequence in L_a^2 and $\mathcal{K}B$ is a compact operator on L_a^2. Thus,

$$\lim_{n\to\infty} \left\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0) L_{\lambda_n} B\widehat{k}_{\lambda_n}\right\| = 0. \tag{5}$$

On the other hand, by $\left\|L_{\lambda_n} B\widehat{k}_{\lambda_n}\right\| \geq \left|\widetilde{L_{\lambda_n} B}(\lambda_n)\right|$, we have:

$$\left\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0) L_{\lambda_n} B\widehat{k}_{\lambda_n}\right\|^2 = 1 - 2\operatorname{Re}\left[F_{B,\lambda_n}(0)\widetilde{L_{\lambda_n} B}(\lambda_n)\right] + \left\|F_{B,\lambda_n}(0) L_{\lambda_n} B\widehat{k}_{\lambda_n}\right\|^2$$
$$\geq 1 - 2\left|F_{B,\lambda_n}(0)\widetilde{L_{\lambda_n} B}(\lambda_n)\right| + \left|F_{B,\lambda_n}(0)\widetilde{L_{\lambda_n} B}(\lambda_n)\right|^2$$
$$= 1 - 2\left|\frac{\widetilde{L_{\lambda_n} B}(\lambda_n)}{(L_n B k_{\lambda_n})(0)}\right| + \left|\frac{\widetilde{L_{\lambda_n} B}(\lambda_n)}{(L_n B k_{\lambda_n})(0)}\right|^2. \tag{6}$$

Note that the wandering subspace for the shift operator $S \in \mathcal{B}\left(L_a^2\right)$ is the subspace $\mathcal{E} = L_a^2 \ominus SL_a^2 = \ker(S^*)$ of L_a^2. It is well-known (and easy to verify; see, for instance, Olofsson [17], Giselsson and Olofsson [9] and Shimorin [20]) that the operator $L_1 := (S^*S)^{-1} S^*$ in $\mathcal{B}\left(L_a^2\right)$ is the left inverse of S with kernel $\mathcal{E}$:

$$L_1 S = I \text{ in } \mathcal{B}\left(L_a^2\right) \text{ and } \ker(L_1) = \ker(S^*) = \mathcal{E}.$$

The operator $P = I - SL_1$ in $\mathcal{B}\left(L_a^2\right)$ is the orthogonal projection of L_a^2 onto $\mathcal{E}$. Indeed, the operator $SL_1 = S(S^*S)^{-1} S^*$ is self-adjoint, idempotent and has range equal to SL_a^2. An interesting fact is that $(S^*S)^{-1} = 2I - SS^*$ (see [17, Formula (0.1)] and also [15]). A simple computation shows that the operator $L_1 = (S^*S)^{-1} S^*$ on L_a^2 acts as

$$(L_1 f)(z) = \frac{f(z) - f(0)}{z} = \sum_{k\geq 0} a_{k+1} z^k,\ z \in \mathbb{D},$$

where $f(z) = \sum_{k\geq 0} a_k z^k \in L_a^2$. Since $\dim \mathcal{E} = L_a^2 \ominus SL_a^2 = 1$, $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^2} = \sum_{n=0}^{\infty} (n+1)\overline{\lambda}^n z^n$ and $SL_1\mathbf{1} = S(S^*S)^{-1} S^*\mathbf{1} = 0$, we have

$$Pk_\lambda = \left(I - S(S^*S)^{-1} S^*\right) k_\lambda = k_\lambda - S(S^*S)^{-1} S^* k_\lambda$$
$$= k_\lambda - \left(S(S^*S)^{-1} S^*\right)\left(\mathbf{1} + 2\overline{\lambda}z + 3\overline{\lambda}^2 z^2 + ...\right)$$
$$= k_\lambda - \left(S(S^*S)^{-1} S^*\right)\mathbf{1} - \left(S(S^*S)^{-1} S^*\right)\left(2\overline{\lambda}z + 3\overline{\lambda}^2 z^2 + ...\right)$$
$$= k_\lambda - \left(2\overline{\lambda}z + 3\overline{\lambda}^2 z^2 + ...\right) = \mathbf{1},$$

and hence $\left(I - S\left(S^*S\right)^{-1} S^*\right) k_\lambda = \mathbf{1}$. Since $\left(S^*S\right)^{-1} = 2I - SS^*$, we have that $I - S\left(S^*S\right)^{-1} S^* = I - 2SS^* + S^2 S^{*2}$, and hence

$$\left(I - 2SS^* + S^2 S^{*2}\right) k_\lambda = \mathbf{1}. \tag{7}$$

Then by using (7), we obtain:

$$\begin{aligned}\frac{1}{\left(L_{\lambda_n} B k_{\lambda_n}\right)(0)} = \frac{1}{\langle L_{\lambda_n} B k_{\lambda_n}, \mathbf{1}\rangle} &= \frac{1}{\langle L_{\lambda_n} B k_{\lambda_n}, \left(I - 2SS^* + S^2S^{*2}\right) k_{\lambda_n}\rangle} \\ &= \frac{\left(1 - |\lambda_n|^2\right)^2}{\left\langle \left(I - 2SS^* + S^2S^{*2}\right) L_{\lambda_n} B \widehat{k}_{\lambda_n}, \widehat{k}_{\lambda_n}\right\rangle} \\ &= \frac{\left(1 - |\lambda_n|^2\right)^2}{\left[\left(I - 2SS^* + S^2S^{*2}\right) L_{\lambda_n} B\right]^{\sim}(\lambda_n)},\end{aligned}$$

and hence, by using (6), we have

$$\begin{aligned}\left\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0)\, L_{\lambda_n} B \widehat{k}_{\lambda_n}\right\|^2 \geq 1 - 2\left|\frac{\left(1 - |\lambda_n|^2\right)^2 \widetilde{L_{\lambda_n} B}(\lambda_n)}{\left[\left(I - 2SS^* + S^2S^{*2}\right) L_{\lambda_n} B\right]^{\sim}(\lambda_n)}\right| \\ + \left|\frac{\left(1 - |\lambda_n|^2\right)^2 \widetilde{L_n B}(\lambda_n)}{\left[\left(I - 2SS^* + S^2S^{*2}\right) L_{\lambda_n} B\right]^{\sim}(\lambda_n)}\right|^2.\end{aligned}$$

Now, since $\left(I - 2SS^* + S^2S^{*2}\right)^{\sim}(\lambda_n) = \left(1 - |\lambda_n|^2\right)^2$, using condition (i) of the theorem, we can deduce that

$$\lim_{n\to\infty} \left\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0)\, L_{\lambda_n} B \widehat{k}_{\lambda_n}\right\| \geq 1;$$

which contradicts (5). □

Corollary 2.3.1 *Let $T \in L\left(L_a^2\right)$ be an operator. Suppose that there exists a nonzero operator $B \in \{T\}'$ such that:*

(i) $\left(B\widehat{k}_\lambda\right)(0) \neq 0$ *for all* $\lambda \in \mathbb{D}$ *and* $\lim_{\lambda\to\xi\in\partial\mathbb{D}} \frac{\widetilde{B}(\lambda)}{(Bk_\lambda)(0)} = 0$;

(ii) there exits a sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ *converging to some point* $\xi_0 \in \partial\mathbb{D}$ *such that the functional sequence* $\widehat{k}_{\lambda_n} \circledast \left(B\widehat{k}_{\lambda_n}\right)^{-1\circledast}$ *is convergent in* L_a^2. *Then T has a nontrivial invariant subspace.*

Proof We observe that

$$\begin{aligned}
(Bk_\lambda)(0) &= \langle Bk_\lambda, k_0\rangle = \langle Bk_\lambda, \mathbf{1}\rangle = \langle Bk_\lambda, k_\lambda\rangle + \langle Bk_\lambda, \mathbf{1}-k_\lambda\rangle \\
&= \langle Bk_\lambda, k_\lambda\rangle + \left\langle Bk_\lambda, 1-\frac{1}{\left(1-\overline{\lambda}z\right)^2}\right\rangle \\
&= \langle Bk_\lambda, k_\lambda\rangle + \left\langle Bk_\lambda, \frac{\overline{\lambda}^2 z^2 - 2\overline{\lambda}z}{\left(1-\overline{\lambda}z\right)^2}\right\rangle \\
&= \langle Bk_\lambda, k_\lambda\rangle - \left\langle Bk_\lambda, \left(2SS^* - S^2S^{*2}\right)k_\lambda\right\rangle \\
&= \left\langle \left(I - 2SS^* + S^2S^{*2}\right)Bk_\lambda, k_\lambda\right\rangle \\
&= \|k_\lambda\|^2 \left\langle \left(I - 2SS^* + S^2S^{*2}\right)B\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&= \left(1-|\lambda|^2\right)^{-2}\left[\left(I - 2SS^* + S^2S^{*2}\right)B\right]^{\sim}(\lambda).
\end{aligned}$$

Hence,

$$(Bk_\lambda)(0) = \left(1-|\lambda|^2\right)^{-2}\left(I - 2\widetilde{SS^* + S^2S^{*2}}\right)B(\lambda) \tag{8}$$

for all $\lambda\in\mathbb{D}$. Since $(Bk_\lambda)(0)\neq 0$ if and only if $\left(B\widehat{k}_\lambda\right)(0)\neq 0$, we have that $k_{B\widehat{k}_{\lambda,a}}(0)=0$. On the other hand, let $f := \lim_{n\to\infty}\left(\widehat{k}_{\lambda_n}\circledast\left(B\widehat{k}_{\lambda_n}\right)^{-1\circledast}\right)$. Then $\widehat{k}_{\lambda_n}\circledast\left(B\widehat{k}_{\lambda_n}\right)^{-1\circledast} - \left(\widehat{k}_{\lambda_n}\circledast\left(B\widehat{k}_{\lambda_n}\right)^{-1\circledast}\right)(0)\to f - f(0)$ $(n\to\infty)$ in L_a^2. Since $\|D_\varphi\| = \|\varphi\|$ for any $\varphi\in L_a^2$, $D_{\widehat{k}_{\lambda_n}\circledast\left(B\widehat{k}_{\lambda_n}\right)^{-1\circledast}-\left(\widehat{k}_{\lambda_n}\circledast\left(B\widehat{k}_{\lambda_n}\right)^{-1\circledast}\right)(0)}$ uniformly converges to the Duhamel operator $D_{f-f(0)}$, which is compact on L_a^2. Now, in view of equation (8), the desired result is immediate from Theorem 2.3. □

It is immediate from Corollary 2.3.1 that an operator T has a nontrivial invariant subspace, whenever T self-satisfies the conditions of Corollary 2.3.1.

The following proposition gives the existence of invariant subspace of operator T on L_a^2 under a more strong condition. We denote by $\mathrm{Lat}T$ the lattice of all invariant subspaces of operator T on L_a^2.

Proposition 2.4 *If $T\in L\left(L_a^2\right)$ is an operator such that $(Tk_\lambda)(0)=\gamma$ for all $\lambda\in\mathbb{D}$ and some constant $\gamma\in\mathbb{C}$, then $TzL_a^2\subset zL_a^2$, i.e. $zL_a^2\in\mathrm{Lat}T$.*

Proof If $\gamma=0$, then by considering that the reproducing kernels k_λ, $\lambda\in\mathbb{D}$, span L_a^2, we have $(Tf)(0)=0$ for all $f\in L_a^2$, which implies that $TzL_a^2\subset zL_a^2$. If $\gamma\neq 0$, then we have that

$$\left(\left(I-2SS^*+S^2S^{*2}\right)T\right)^{\sim}(\lambda) = \gamma\left(1-|\lambda|^2\right)^2 = \gamma\left(I-2\widetilde{SS^*+S^2S^{*2}}\right)(\lambda)$$

for all $\lambda\in\mathbb{D}$. Since the Berezin symbols uniquely determines the operator on L_a^2, from the latter we obtain that $\left(I-2SS^*+S^2S^{*2}\right)T = \gamma\left(I-2SS^*+S^2S^{*2}\right)$, that is $PT=\gamma P$, where P is the orthogonal projection of L_a^2 onto $\mathcal{E}=L_a^2\ominus SL_a^2$. Hence, $PT(I-P)=0$, or equivalently, $(I-(I-P))T(I-P)=0$, which implies that $T\mathcal{E}^\perp\subset\mathcal{E}^\perp$, that is, $TSL_a^2\subset SL_a^2$, so, $zL_a^2\in\mathrm{Lat}T$, as desired. □

For an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ the associated composition operator C_φ is defined on L_a^2 by the formula $C_\varphi := f \circ \varphi = f(\varphi(z))$. Note that the concrete example of operators on L_a^2 satisfying the condition of Proposition 2.4 is the conjugated composition operator C_φ^* for an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$: $\left(C_\varphi^* k_\lambda\right)(0) = \left(k_{\varphi(\lambda)}\right)(0) = \left(\frac{1}{1-\overline{\varphi(\lambda)}z}\right)^2 |_{z=0} = 1 \ \forall \lambda \in \mathbb{D}$; of course, for $T = C_\varphi^*$, the conclusion of the above proposition is immediate from the inclusion $C_\varphi \{\mu \mathbf{1} : \mu \in \mathbb{C}\} \subset \{\mu \mathbf{1} : \mu \in \mathbb{C}\}$.

In the next result, an analytic automorphism of the unit disc $\mathbb{D}$ (i.e., the Möbius maps on $\mathbb{D}$) will play a key role, so we start with some properties of Möbius maps (see, for example, [2]). For $\lambda \in \mathbb{D}$, let ω_λ be the Möbius map on $\mathbb{D}$ defined by

$$\omega_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}. \tag{9}$$

Let us define the following unitary operator $U_\lambda : L_a^2 \to L_a^2$:

$$U_\lambda f = (f \circ \omega_\lambda) \omega_\lambda^{'}. \tag{10}$$

To show that U_λ is indeed unitary, first make a change of variables in the integral defining $\|U_\lambda f\|_2$ to show that U_λ is an isometry on L_a^2. Next, a simple computation shows that $U_\lambda^2 = I$; this holds because ω_λ is its own inverse under composition. Being an invertible isometry, U_λ must be unitary. Notice that $U_\lambda^* = U_\lambda^{-1} = U_\lambda$, so U_λ is a self-adjoint unitary operator. We will need the following simple property of U_λ :

$$U_\lambda \mathbf{1} = -\widehat{k}_\lambda,$$

which is immediate from (9) and (10).

Proposition 2.5 *Let $T \in \mathcal{B}\left(L_a^2\right)$ be an operator for which there exists a nonzero operator $B \in \{T\}^{'}$ such that:*

(i) $\left(B\widehat{k}_\lambda\right)(0) \neq 0$ for all $\lambda \in \mathbb{D}$ and $\left|\frac{\widetilde{B}(\lambda)}{\widetilde{U_\lambda B}(\lambda)}\right| = o\left(\frac{1}{1-|\lambda|^2}\right)$ as $\lambda \to \partial\mathbb{D}$;

(ii) there exists a sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ converging to some point $\xi_0 \in \partial\mathbb{D}$ such that the sequence $\widehat{k}_{\lambda_n} \circledast \left(B\widehat{k}_{\lambda_n}\right)^{-1\circledast}$ is convergent in L_a^2. Then T has a nontrivial invariant subspace.

Proof Since $U_\lambda^2 = I$ and $U_\lambda \mathbf{1} = -\widehat{k}_\lambda$, we have:

$$\left(B\widehat{k}_{\lambda_n}\right)(0) = \left\langle B\widehat{k}_{\lambda_n}, \mathbf{1}\right\rangle = \left\langle B\widehat{k}_{\lambda_n}, -U_{\lambda_n}\widehat{k}_{\lambda_n}\right\rangle = -\widetilde{U_{\lambda_n}B}(\lambda_n).$$

Hence, the required result follows alone the lines of the proof of Theorem 2.3. □

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