Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1873 – 1886
© TÜBİTAK
doi:10.55730/1300-0098.3469

Best proximity for proximal operators on b-metric spaces

Ariana PITEA*, **Monica STANCIU**
Department of Mathematics and Informatics, University Politehnica of Bucharest, Romania

Received: 18.09.2022 • **Accepted/Published Online:** 09.08.2023 • **Final Version:** 09.11.2023

Abstract: The paper presents existence results of (φ, φ) best proximity points for operators that fulfill implicit type inequalities. Classes of mappings endowed with continuity, monotone or monotone-type properties, and which additionally satisfy some adequate inequalities are studied from this point of view. Applications of our results are given with regard to fixed point theory.

Key words: Best proximity point, generalized proximal contraction, b-metric space

1. Introduction

Best proximity point theory has been developed as a response to those problems that cannot be solved by means of fixed point theory, since equations of the form $x = Tx$, where T is a given operator, do not necessarily have solutions. Therefore, the focus moved on determining points which satisfy a property related to that of fixed points, namely determining points for which the distance to their image through a given mapping is precisely that between two given sets properly related. Starting with the work of Fan [9], this direction has been studied by many researchers. From this point of view, Basha [4] and Eldred and Veeramani [8] obtained generalizations of the Banach principle for various proximal contractions. In [17], Petric used weak cyclic Kannan type mappings to state existence and uniqueness results on best proximity. Fernandéz-Léon [10] studied this subject using completeness instead of the compactness assumption. Gabeleh and Shahzad [11] generalized the Chatterjea operators to develop best proximity results. Suparatulatorn and Suantai [19] focused on designing algorithms to approximate best proximity points. Norm convergence has been used to approximate best proximity points in Jacob et al. [13].

A fruitful and interesting direction in the research of best proximity is represented by the use of diverse frameworks. Choudhury et al. [6] used partially ordered metric spaces to study best proximity, while Samreen et al. [18] moved towards extended metric spaces. An interesting opening was made by Czerwik [7] and Bakhtin [3], who modified the triangle inequality, introducing in this fashion the b-metric spaces. Ali et al. [1] used this setting to study solutions of Volterra integral inclusions. Joseph et al. [14] considered this framework to prove results regarding cyclic b contractive type mappings. Kamran et al. [15] studied Feng and Liu type mappings applied to solve integral equations. Nonself mappings were studied with respect to the existence of best proximity points in b-metric spaces by Batul et al. [5].

*Correspondence: arianapitea@yahoo.com, ariana.pitea@upb.ro
2010 *AMS Mathematics Subject Classification:* 47H10, 54H25

In this work, we extend the study of (φ, φ)-best proximity points initiated by Ali et al. [2], to the setting of b-metric spaces. In order to pass from classic metric spaces to b-metric spaces, we need an additional condition, inspired from [16], which ensures the good behaviour of the sequences that will converge to the best proximity point which is a zero of mappings with adequate properties. As operators, we have in view the accomplishment of Wardowski-type properties [20], and of some conditions compatible with those in [2]. Applications of these results are presented, concerning fixed point theory. The paper is organized as follows. In Section 2 we provide some preliminaries on b-metrics and the concept of best proximity point. In Section 3 we study the existence of best proximity points of operators with adequate properties of monotone or sum-monotone, continuity, asymptotic behavior of Wardowski type, or specific other inequalities. The last section is dedicated to consequences related to fixed points with respect to the proximal contractions introduced in the previous sections.

2. Preliminaries

The underlying space chosen to develop our original results is that of b-metric spaces, introduced by Bakhtin [3] and Czerwik [7] by weakening the classic triangle inequality from the definition of metric spaces.

Definition 2.1 *Let X be a non-empty set and $s \geq 1$ a real number. A function $d\colon X \times X \to [0, \infty)$ is called a b-metric if the following conditions are satisfied, for every x, y, $z \in X$:*

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$;*
3. *$d(x, y) \leq s\,[d(x, z) + d(z, y)]$.*

In this case, (X, d) is called a b-metric space with a constant $s \geq 1$.

Clearly, the class of b-metrics properly contains that of classic metric spaces, but the converse is not true, as the next example shows.

Example 1 *Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$. It can be easily checked that (X, ρ) is a b-metric space, but (X, ρ) is not a metric space.*

Regarding the behaviour of sequences in this framework, we recall the notions of convergence, Cauchy sequence, and completeness.

Definition 2.2 *Let $\{x_n\}$ be a sequence from a b-metric space (X, d). We say that $\{x_n\}$ is convergent to $x \in X$ if and only if $d(x_n, x) \to 0$, when $n \to \infty$.*

Definition 2.3 *A sequence $\{x_n\}$ is called a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, when m, $n \to \infty$.*

Remark 1 *A b-metric space is said to be complete if and only if each Cauchy sequence in this space is convergent.*

Proposition 1 *In a b-metric space (X, d), the following axioms hold:*

1. *Any convergent sequence has a unique limit;*

2. *Each convergent sequence is a Cauchy sequence;*

Continuity is not, in general, a feature of b-metrics, as the next example proves.

Example 2 ([12]) *Let $X = \mathbb{N} \cup \{\infty\}$ and let $d\colon X \times X \to \mathbb{R}$,*

$$d(x,y) = \begin{cases} 0, \text{ if } x = y; \\ \left|\dfrac{1}{x} - \dfrac{1}{y}\right|, \text{ if } x,\ y \text{ are even or } xy = \infty; \\ 5, \text{ if } x,\ y \text{ are odd and} x \neq y; \\ 2, \text{ otherwise.} \end{cases}$$

Then (X,d) is a b-metric space with $s = 3$, but d is not continuous. Considering $x_n = 2n$, for all $n \in \mathbb{N}$, we get $d(x_n, 1) = 2 \neq d(\infty, 1) = 1$, when $n \to \infty$, which completes the proof.

This setting will be used in order to obtain existence results on best proximity points. Therefore, we recall some facts related to this concept.

Definition 2.4 *Let A and B be two non-empty subsets of a b-metric space (X,d). The distance between A and B is defined by*

$$d(A,B) = \inf d(a,b),$$

where $a \in A$, $b \in B$.

Definition 2.5 *Let (X,d) be a complete b metric space, A and B be two non-empty subsets of X, and a mapping $T\colon A \to B$. A point $x \in A$ is called a best proximity point of T if*

$$d(A,B) = d(x,Tx).$$

If φ, $\varphi\colon X \to [0,\infty)$, and the best proximity point x satisfies $\varphi(x) = \varphi(x) = 0$, then x is a (φ,φ)-best proximity point of T.

The development of our original results needs some additional sets, with suitable properties, introduced in the following.

Definition 2.6 *Let A and B be two non-empty subsets of a b-metric space (X,d). Define the subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ by*

$$\begin{aligned} A_0 &= \{a \in A \,|\, \exists b \in B, d(a,b) = d(A,B)\}, \\ B_0 &= \{b \in B \,|\, \exists a \in A, d(a,b) = d(A,B)\}. \end{aligned}$$

3. Existence of (φ, φ)-best proximity points

In this section, we use the classes of functions introduced in [2] to obtain results on the existence of best proximity points on another environment, namely the b-metric spaces.

Definition 3.1 *Denote by $\mathfrak{L}$ the class of functions $L\colon (0,\infty) \to \mathbb{R}$ which fulfills the axioms:*

(L_1) *L is strictly increasing, that is, if $a_1 < a_2$ then $L(a_1) < L(a_2)$;*

(L_2) *for each sequence $\{c_n : c_n > 0\}$, we have $\lim_{n\to\infty} c_n = 0$ if and only if $\lim_{n\to\infty} L(c_n) = -\infty$;*

(L_3) *for each sequence $\{c_n : c_n > 0\}$ with $\lim_{n\to\infty} c_n = 0$, there exists $h \in (0,1)$ such that $\lim_{n\to\infty} c_n^h L(c_n) = 0$.*

By $\mathfrak{L}_w$ we designate the set of functions which satisfy conditions L_1 and L_3.

With respect to property L_2, in [16] it is proved that, if L is increasing, $\{a_n\}$ is a decreasing sequence of positive terms, and $\lim_{n\to\infty} L(a_n) = -\infty$, then $\{a_n\}$ converges to zero.

Definition 3.2 *Denote by $\mathfrak{R}$ the class of functions $W\colon [0,\infty)^3 \to [0,\infty)$ which satisfy the next hypotheses:*

(W_1) *$W(a,b,c) = 0$ if and only if $a = b = c = 0$;*

(W_2) *W is continuous;*

(W_3) *$\frac{1}{\beta} a \le W(a,b,c)$, for any a, b, $c > 0$, where $\beta \ge 1$.*

Consider A and B non-void subsets of X, where (X,d) is a b-metric space, and φ, $\varphi\colon A \to [0,\infty)$.

By means of functions from $\mathfrak{L}_w$ and $\mathfrak{R}$, we are able now to introduce the first weakly contractive operator.

Definition 3.3 *A mapping $\mathcal{T}\colon A \to B$ is called L^I-proximal contraction if there exist the functions $\alpha\colon A\times A \to [0,\infty)$, $L \in \mathfrak{L}_w$, $W \in \mathfrak{R}$, and a constant $k > 0$ such that for all ξ_1, ξ_2, ζ_1, $\zeta_2 \in A$, with $\alpha(\xi_1,\xi_2) \ge 1$ and $d(\zeta_1, \mathcal{T}\xi_1) = d(A,B) = d(\zeta_2, \mathcal{T}\xi_2)$, we get $\alpha(\zeta_1,\zeta_2) \ge 1$, and*

$$k + L\left(W\left(sd(\zeta_1,\zeta_2), \varphi(\zeta_1), \varphi(\zeta_2)\right)\right) \le L\left(W\left(d(\xi_1,\xi_2), \varphi(\xi_1), \varphi(\xi_2)\right)\right), \tag{3.1}$$

whenever

$$\min\{W\left(sd(\zeta_1,\zeta_2), \varphi(\zeta_1), \varphi(\zeta_2)\right), W\left(d(\xi_1,\xi_2), \varphi(\xi_1), \varphi(\xi_2)\right)\} > 0.$$

Taking advantage of Definition 2.7 from [16], we can give the following property.

Definition 3.4 *Let (X,d) be a b-metric space, and $k > 0$. We say that $L\colon (0,\infty) \to \mathbb{R}$ satisfies the property (S_{LW}) if the next implication holds*

(S_{LW}) *If $\{\xi_n\} \subset (0,\infty)$ is a sequence such that*

$$\begin{aligned} k + L\left(W\left(sd(\xi_n,\xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1})\right)\right) \\ \le L\left(W\left(d(\xi_{n-1},\xi_n), \varphi(\xi_{n-1}), \varphi(\xi_n)\right)\right), \end{aligned}$$

for all $n \in \mathbb{N}$ and for some $k > 0$, then

$$\begin{aligned} k + L\left(W\left(s^n d(\xi_n,\xi_{n+1}), \varphi(\xi_n), \varphi(\xi_n)\right)\right) \\ \le L\left(W\left(s^{n-1} d(\xi_{n-1},\xi_n), \varphi(\xi_{n-1}), \varphi(\xi_n)\right)\right). \end{aligned}$$

Theorem 1 *Let A and B be non-void subsets of X, and (X,d) be a complete b-metric space. Consider that A_0 is closed with respect to d, and that $\mathcal{T}\colon A\to B$ is a L^I-proximal contraction mapping, which fulfills the next conditions:*

(i) $TA_0 \subseteq B_0$;

(ii) *there are ξ_0, $\xi_1\in A_0$, so that $\alpha(\xi_0,\xi_1)\geq 1$ and $d(\xi_1,\mathcal{T}\xi_0)=d(A,B)$;*

(iii) *every sequence $\{\xi_n\}\subseteq A_0$ with $\xi_n\to\xi$ and $\alpha(\xi_n,\xi_{n+1})\geq 1$ for all $n\in\mathbb{N}$, necessarily satisfies the inequality $\alpha(\xi_n,\xi)\geq 1$, for all $n\in\mathbb{N}$;*

(iv) *The property (S_{LW}) is satisfied;*

(v) *the functions φ, $\varphi\colon A\to[0,\infty)$ are bounded lower semi continuous.*

Then $\mathcal{T}$ has a (φ,φ)-best proximity point.

Proof By hypothesis (ii), the existence of ξ_0, $\xi_1\in A_0$ with $\alpha(\xi_0,\xi_1)\geq 1$ and $d(\xi_1,\mathcal{T}\xi_0)=d(A,B)$ is warranted. We may consider that $\xi_0\neq\xi_1$. We know that $\mathcal{T}\xi_1\in B_0$, so there exists $\xi_2\in A_0$ with $d(\xi_2,\mathcal{T}\xi_1)=d(A,B)$. We suppose that $\xi_1\neq\xi_2$. Since

$$\min\{W(sd(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2)),W(d(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1))\}>0,$$

from relation (3.1) we have

$$k+L(W(sd(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))\leq L(W(d(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1))), \tag{3.2}$$

and $\alpha(\xi_1,\xi_2)\geq 1$.

The previous assumptions tell us that there exist ξ_1, $\xi_2\in A_0$ with $\alpha(\xi_1,\xi_2)\geq 1$ and $d(\xi_2,\mathcal{T}\xi_1)=d(A,B)$. Using hypothesis (i), we have $\mathcal{T}\xi_2\in B_0$, therefore there exists $\xi_3\in A_0$ such as $d(\xi_3,\mathcal{T}\xi_2)=d(A,B)$. Considering $\xi_2\neq\xi_3$, by relation (3.1) we obtain

$$k+L(W(sd(\xi_2,\xi_3),\varphi(\xi_2),\varphi(\xi_3)))\leq L(W(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))), \tag{3.3}$$

and $\alpha(\xi_2,\xi_3)\geq 1$. By using condition (S_{LW}) for (3.3), it follows that

$$k+L(W(s^2d(\xi_2,\xi_3),\varphi(\xi_2),\varphi(\xi_3)))\leq L(W(sd(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))). \tag{3.4}$$

From inequalities (3.2) and (3.4), we obtain

$$L(W(s^2d(\xi_2,\xi_3),\varphi(\xi_2),\varphi(\xi_3)))\leq L(W(d(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)))-2k.$$

Continuing this way we obtain a sequence $\{\xi_n\}\subseteq A_0$, so that $\{\mathcal{T}\xi_n\}\subseteq B_0$, $\alpha(\xi_n,\xi_{n+1})\geq 1$, $d(\xi_{n+1},\mathcal{T}\xi_n)=d(A,B)$ and for any $n\in\mathbb{N}$, we have

$$\begin{aligned}&L(W(s^nd(\xi_n,\xi_{n+1}),\varphi(\xi_n),\varphi(\xi_{n+1})))\\&\qquad\leq L(W(d(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)))-nk.\end{aligned} \tag{3.5}$$

Taking the limit when $n \to \infty$ in this inequality, it follows that

$$\lim_{n\to\infty} L(W(s^n d(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1}))) = -\infty.$$

Since L is increasing, and $\{W(s^n d(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1}))\}$ is a decreasing sequence, the last inequality leads to

$$\lim_{n\to\infty} W(s^n d(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1}))) = 0.$$

The properties of W allow us to conclude that

$$\lim_{n\to\infty} s^n d(\xi_n, \xi_{n+1}) = \lim_{n\to\infty} \varphi(\xi_n) = \lim_{n\to\infty} \varphi(\xi_{n+1}) = 0.$$

Denote by $W_n = W(s^n d_n, \varphi_n, \varphi_n)$, $d_n = d(\xi_n, \xi_{n+1})$, $\varphi_n = \varphi(\xi_n)$, and $\varphi_n = \varphi(\xi_{n+1})$. From axiom (L_3), there exists a constant $h \in (0,1)$ so that

$$\lim_{n\to\infty} W_n^h L(W_n) = 0.$$

Using relation (3.5), it follows that

$$W_n^h L(W_n) - W_n^h L(W_0) \leq -nkW_n^h \leq 0,$$

for all $n \in \mathbb{N}$. The previous relations lead us to $\lim_{n\to\infty} nW_n^h = 0$, and so there is $n_1 \in \mathbb{N}$, $n_1 > 1$, for which $nW_n^h \leq 1$, for each $n \geq n_1$. Next, we can write

$$W_n \leq \frac{1}{n^{1/h}}, \tag{3.6}$$

for all $n \geq n_1$. Taking advantage of hypothesis (W_3) and inequality (3.6), we have

$$\frac{1}{\beta} s^n d_n \leq W_n \leq \frac{1}{n^{1/h}}.$$

We obtain

$$d_n \leq \frac{\beta}{s^n n^{1/h}}, \tag{3.7}$$

for all $n \geq n_1$.

Now, we want to prove that $\{\xi_n\}$ is a Cauchy sequence in A_0. By combining the generalized triangle inequality with (3.7), we get, for n, $p \in \mathbb{N}$,

$$\begin{aligned}
d(\xi_n, \xi_{n+p}) &\leq s[d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+p})] \\
&\leq sd(\xi_n, \xi_{n+1}) + s^2 d(\xi_{n+1}, \xi_{n+2}) + \cdots + s^p d(\xi_{n+p-1}, \xi_{n+p}) \\
&= \frac{1}{s^{n-1}}[s^n d_n + s^{n+1} d_{n+1} + \cdots + s^{n+p-1} d_{n+p-1}] \\
&= \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} s^i d_i \leq \frac{\beta}{s^{n-1}} \sum_{i=n}^{n+p-1} \frac{1}{i^{1/h}} \leq \frac{\beta}{s^{n-1}} \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}.
\end{aligned}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/h}}$ is convergent to zero, we obtain that $\lim_{n\to\infty} d(\xi_n, \xi_{n+p}) = 0$ independent of p, therefore $\{\xi_n\}$ is a Cauchy sequence in A_0.

Because A_0 is closed, there exists $\xi \in A_0$ such that $\xi_n \to \xi$. From hypothesis (iii), it follows that $\alpha(\xi_n, \xi) \geq 1$, $n \in \mathbb{N}$. Because $\mathcal{T}\xi \in B_0$, there exists $\xi^* \in A_0$ such that $d(\xi^*, \mathcal{T}\xi) = d(A, B)$. Hence, we have obtained $\alpha(\xi_n, \xi) \geq 1$, $d(\xi_{n+1}, \mathcal{T}\xi_n) = d(A, B)$ and $d(\xi^*, \mathcal{T}\xi) = d(A, B)$. Presume, without loss of generality, that $\xi_n \neq \xi^*$ and $\xi_n \neq \xi$, for all $n \in \mathbb{N}$, n large enough. Using relation (3.1), we have

$$k + L(W(sd(\xi_{n+1}, \xi^*), \varphi(\xi_{n+1}), \varphi(\xi^*))) \leq L(W(d(\xi_n, \xi), \varphi(\xi_n), \varphi(\xi))),$$

and so it follows that

$$\frac{1}{\beta} sd(\xi_{n+1}, \xi^*) \leq W(sd(\xi_{n+1}, \xi^*), \varphi(\xi_{n+1}), \varphi(\xi^*)) < W(d(\xi_n, \xi), \varphi(\xi_n), \varphi(\xi)).$$

Applying the limit when $n \to \infty$ and by axiom (W_2) we obtain

$$\limsup_{n\to\infty} \frac{1}{\beta} sd(\xi_{n+1}, \xi^*) \leq W(d(\xi, \xi), 0, \varphi(\xi)). \tag{3.8}$$

Because φ, φ are lower semicontinuous functions and

$$\lim_{n\to\infty} \varphi(\xi_n) = \lim_{n\to\infty} \varphi(\xi_n) = 0,$$

we have $\varphi(\xi) = \varphi(\xi) = 0$. From axiom (W_1) and inequality (3.8), we have

$$\limsup_{n\to\infty} d(\xi_{n+1}, \xi^*) \leq 0,$$

and it follows that $\xi = \xi^*$.

In conclusion, we proved that ξ is a (φ, φ)-best proximity point of $\mathcal{T}$. □

By considering the case of a classic metric space and $\beta = 1$, the previous theorem becomes an existing result of best proximity which can be found in [2].

Corollary 1 *Let A and B be non-void subsets of X, and (X, d) be a complete metric space. Consider that A_0 is closed with respect to d and $\mathcal{T}\colon A \to B$ is a mapping for which there exists $\alpha\colon A \times A \to [0, \infty)$, $L \in \mathfrak{L}_w$, $W \in \mathfrak{R}$, a constant $k > 0$ such that for all ξ_1, ξ_2, ζ_1, $\zeta_2 \in A$, with $\alpha(\xi_1, \xi_2) \geq 1$ and $d(\zeta_1, \mathcal{T}\xi_1) = d(A, B) = d(\zeta_2, \mathcal{T}\xi_2)$, we get $\alpha(\zeta_1, \zeta_2) \geq 1$, and*

$$k + L\left(W\left(d(\zeta_1, \zeta_2), \varphi(\zeta_1), \varphi(\zeta_2)\right)\right) \leq L\left(W\left(d(\xi_1, \xi_2), \varphi(\xi_1), \varphi(\xi_2)\right)\right),$$

whenever

$$\min\{W\left(d(\zeta_1, \zeta_2), \varphi(\zeta_1), \varphi(\zeta_2)\right), W\left(d(\xi_1, \xi_2), \varphi(\xi_1), \varphi(\xi_2)\right)\} > 0.$$

Moreover, the next conditions are fulfilled:

(i) *$TA_0 \subseteq B_0$;*

(ii) *there are ξ_1, $\xi_2 \in A_0$, so that $\alpha(\xi_1, \xi_2) \geq 1$ and $d(\xi_2, \mathcal{T}\xi_1) = d(A, B)$;*

(iii) *every sequence* $\{\xi_n\} \subseteq A_0$ *with* $\xi_n \to \xi$ *and* $\alpha(\xi_n, \xi_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *necessarily satisfies the inequality* $\alpha(\xi_n, \xi) \geq 1$, *for all* $n \in \mathbb{N}$;

(iv) *the functions* φ, φ *are bounded lower semi continuous.*

Then $\mathcal{T}$ *has a* (φ, φ)*-best proximity point.*

Next, using the class of functions used in [2] we are able to introduce another type of generalized contraction on b-metric spaces, in the following.

Definition 3.5 Denote by $\mathfrak{M}$ the class of functions $L\colon \mathbb{R}_+^3 \setminus \{(0,0,0)\} \to \mathbb{R}$ which fulfill the below hypotheses:

(M_1) $L(c_1, d_1, b_1) \leq L(c_2, d_2, b_2)$ if and only if $c_1 + d_1 + b_1 \leq c_2 + d_2 + b_2$;

(M_2) $L(c_1, d_1, b_1) < L(c_2, d_2, b_2)$ if and only if $c_1 + d_1 + b_1 < c_2 + d_2 + b_2$;

(M_3) for each $\{c_n : c_n \geq 0\}$, $\{d_n : d_n \geq 0\}$, $\{b_n : b_n \geq 0\}$, we have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = \lim_{n\to\infty} b_n = 0$$

if and only if

$$\lim_{n\to\infty} L(c_n, d_n, b_n) = -\infty;$$

(M_4) for each $\{c_n : c_n \geq 0\}$, $\{d_n : d_n \geq 0\}$, $\{b_n : b_n \geq 0\}$, with $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = \lim_{n\to\infty} b_n = 0$, there exists $h \in (0,1)$ so that $\lim_{n\to\infty} c_n^h L(c_n, d_n,\ b_n) = 0$.

Definition 3.6 *A mapping* $\mathcal{T}\colon A \to B$ *is called* L^{II}*-proximal contraction if there exist the functions* $\alpha\colon A \times A \to [0, \infty)$, $L \in \mathfrak{M}$, *and a constant* $k > 0$ *such that for all* ξ_1, ξ_2, ζ_1, $\zeta_2 \in A$ *with* $\alpha(\xi_1, \xi_2) \geq 1$ *and* $d(\zeta_1, \mathcal{T}\xi_1) = d(A, B) = d(\zeta_2, \mathcal{T}\xi_2)$, *we get* $\alpha(\zeta_1, \zeta_2) \geq 1$ *and*

$$k + L(sd(\zeta_1, \zeta_2), \varphi(\zeta_1), \varphi(\zeta_2)) \leq L(d(\xi_1, \xi_2), \varphi(\xi_1), \varphi(\xi_2)), \tag{3.9}$$

whenever

$$\min\{sd(\zeta_1, \zeta_2) + \varphi(\zeta_1) + \varphi(\zeta_2), d(\xi_1, \xi_2) + \varphi(\xi_1) + \varphi(\xi_2)\} > 0.$$

To state a new theorem in this setting, we shall introduce an additional property, denoted by (S_L).

Definition 3.7 *Let* (X, d) *be a* b*-metric space and* $k > 0$. *A sequence in* X *is said to fulfill the property* (S_L) *if the next implication holds.*

(S_L) *If* $\{\xi_n\} \subset (0, \infty)$ *is a sequence such that*

$$\begin{aligned} k + L\left(sd(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1})\right) \\ \leq L\left(d(\xi_{n-1}, \xi_n), \varphi(\xi_{n-1}), \varphi(\xi_n)\right), \end{aligned}$$

for all $n \in \mathbb{N}$ *and for some* $k > 0$, *then*

$$\begin{aligned} k + L\left(s^n d(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1})\right) \\ \leq L\left(s^{n-1} d(\xi_{n-1}, \xi_n), \varphi(\xi_{n-1}), \varphi(\xi_n)\right). \end{aligned}$$

Theorem 2 *Let A and B be non-void subsets of X, and (X,d) be a complete b-metric space. Consider that A_0 is closed with respect to d and $\mathcal{T}\colon A \to B$ is a L^{II}-proximal contraction mapping, which fulfills the next conditions:*

(i) $TA_0 \subseteq B_0$*;*

(ii) *there are ξ_0, $\xi_1 \in A_0$, so that $\alpha(\xi_0,\xi_1) \geq 1$ and $d(\xi_1,\mathcal{T}\xi_0) = d(A,B)$;*

(iii) *every sequence $\{\xi_n\} \subseteq A_0$ with $\xi_n \to \xi$ and $\alpha(\xi_n,\xi_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, necessarily satisfies the inequality $\alpha(\xi_n,\xi) \geq 1$, for all $n \in \mathbb{N}$;*

(iv) *the property (S_L) is fulfilled;*

(v) *φ and φ are lower semi continuous.*

Then $\mathcal{T}$ has a (φ,φ)-best proximity point.

Proof From hypothesis (ii) we have ξ_0, $\xi_1 \in A_0$ such that $\alpha(\xi_0,\xi_1) \geq 1$ and $d(\xi_1,\mathcal{T}\xi_0) = d(A,B)$. We suppose that $\xi_0 \neq \xi_1$. We know that $\mathcal{T}\xi_1 \in B_0$, so there exists $\xi_2 \in A_0$ such that $d(\xi_2,\mathcal{T}\xi_1) = d(A,B)$. We may assume again that $\xi_1 \neq \xi_2$ and using inequality (3.9), we have

$$k + L(sd(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2)) \leq L(d(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)) \tag{3.10}$$

and $\alpha(\xi_1,\xi_2) \geq 1$. This leads to the existence of ξ_1, $\xi_2 \in A_0$, with $\alpha(\xi_1,\xi_2) \geq 1$ and $d(\xi_2,\mathcal{T}\xi_1) = d(A,B)$. Next, from $\mathcal{T}\xi_2 \in B_0$ we can obtain that $\xi_3 \in A_0$, with $d(\xi_3,\mathcal{T}\xi_2) = d(A,B)$. We assume that $\xi_2 \neq \xi_3$ and from the contractive condition, we get

$$k + L(sd(\xi_2,\xi_3),\varphi(\xi_2),\varphi(\xi_3)) \leq L(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))$$

and $\alpha(\xi_2,\xi_3) \geq 1$. By using the inequality from Definition 3.7 it follows that

$$k + L(s^2 d(\xi_2,\xi_3),\varphi(\xi_2),\varphi(\xi_3)) \leq L(sd(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2)). \tag{3.11}$$

From relations (3.10) and (3.11) we obtain

$$L(s^2 d(\xi_2,\xi_3),\varphi(\xi_2),\varphi(\xi_3)) \leq L(d(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)) - 2k.$$

By continuing this process, we have a sequence $\{\xi_n\} \subseteq A_0$ so that $\{\mathcal{T}\xi_n\} \subseteq B_0$, $\alpha(\xi_n,\xi_{n+1}) \geq 1$, $d(\xi_{n+1},\mathcal{T}\xi_n) = d(A,B)$ with

$$\begin{aligned} &L(s^n d(\xi_n,\xi_{n+1}),\varphi(\xi_n),\varphi(\xi_{n+1})) \\ &\qquad \leq L(d(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)) - nk, \end{aligned} \tag{3.12}$$

for any $n \in \mathbb{N}$.

Taking the limit when $n \to \infty$ in relation (3.12), we get

$$\lim_{n\to\infty} L(s^n d(\xi_n,\xi_{n+1}),\varphi(\xi_n),\varphi(\xi_{n+1})) = -\infty.$$

Using axiom (M_3) and the previous equality, we obtain $\lim_{n\to\infty} s^n d(\xi_n, \xi_{n+1}) = \lim_{n\to\infty} \varphi(\xi_n) = \lim_{n\to\infty} \varphi(\xi_{n+1}) = 0$. Denote by $d_n = d(\xi_n, \xi_{n+1})$, $\varphi_n = \varphi(\xi_n)$, $\varphi_n = \varphi(\xi_{n+1})$. From axiom (M_4) we know that there exists $h \in (0,1)$, so that

$$\lim_{n\to\infty} s^{nh} d_n^h L(s^n d_n, \varphi_n, \varphi_n) = 0.$$

Denote by $L_n = L(s^n d_n, \varphi_n, \varphi_n)$; the previous relation becomes $\lim_{n\to\infty} s^{nh} d_n L_n = 0$. Taking advantage of relation (3.12) we get

$$s^{nh} d_n^h L_n - s^{nh} d_n^h L_0 \leq -s^{nh} d_n^h nk \leq 0.$$

It follows that $\lim_{n\to\infty} s^{nh} d_n^h = 0$, which ensures us that there is $n_1 \in \mathbb{N}$, $n_1 > 1$, with $s^{nh} d_n^h \leq 1$, for all $n \geq n_1$. We obtain

$$d_n \leq \frac{1}{s^n n^{1/h}}.$$

Next, we want to prove that $\{\xi_n\}$ is a Cauchy sequence. In this respect, we consider

$$\begin{aligned}
d(\xi_n, \xi_{n+p}) &\leq s[d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+p})] \\
&\leq sd(\xi_n, \xi_{n+1}) + s^2 d(\xi_{n+1}, \xi_{n+2}) + \cdots + s^p d(\xi_{n+p-1}, \xi_{n+p}) \\
&= \frac{1}{s^{n-1}}[s^n d_n + s^{n+1} d_{n+1} + \cdots + s^{n+p-1} d_{n+p-1}] \\
&= \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} s^i d_i \leq \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} \frac{1}{i^{1/h}} \leq \frac{1}{s^{n-1}} \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}.
\end{aligned}$$

Because of the convergence to zero of the series $\sum_{i=n}^{\infty} \dfrac{1}{i^{1/h}}$, we obtain that $\lim_{n\to\infty} d(\xi_n, \xi_{n+p}) = 0$, independent of p. We proved that $\{\xi_n\}$ ia a Cauchy sequence in A_0 and hence $\xi_n \to \xi \in A_0$. Since φ, φ are lower semi continuous functions with $\lim_{n\to\infty} \varphi(\xi_n) = \lim_{n\to\infty} \varphi(\xi_n) = 0$ and $\xi_n \to \xi$, then $\varphi(\xi) = \varphi(\xi) = 0$.

From hypothesis (iii), we have $\alpha(\xi_n, \xi) \geq 1$, for all $n \in \mathbb{N}$. As $\mathcal{T}\xi \in B_0$, there is $\xi^* \in A_0$ such as $d(\xi^*, \mathcal{T}\xi) = d(A, B)$.

Without loss of generality, we may assume that $\xi_n \neq \xi$, and $\xi_n \neq \xi^*$, for n large enough. Considering that $\alpha(\xi_n, \xi) \geq 1$, $d(\xi_{n+1}, \xi_n) = d(A, B)$, $d(\xi^*, T\xi) = d(A, B)$ and using relation (3.9), we have

$$k + L(sd(\xi_{n+1}, \xi^*), \varphi(\xi_{n+1}), \varphi(\xi^*)) \leq L(d(\xi_n, \xi), \varphi(\xi_n), \varphi(\xi)),$$

for all $n \in \mathbb{N}$. This implies that

$$L(sd(\xi_{n+1}, \xi^*), \varphi(\xi_{n+1}), \varphi(\xi^*)) < L(d(\xi_n, \xi), \varphi(\xi_n), \varphi(\xi)),$$

for all $n \in \mathbb{N}$. Using property (M_2), it follows that

$$sd(\xi_{n+1}, \xi^*) + \varphi(\xi_{n+1}) + \varphi(\xi^*) < d(\xi_n, \xi) + \varphi(\xi_n).$$

By taking the limit when $n \to \infty$ in the relation below, we obtain $\xi = \xi^*$.

We proved that ξ is a (φ, φ)-best proximity point for $\mathcal{T}$. □

Considering the case of classic metric space, we get the next result known in the literature.

Corollary 2 *Let A and B be non-void subsets of X, and (X,d) be a complete metric space. Consider that A_0 is closed with respect to d and $\mathcal{T}\colon A \to B$ is a mapping for which there exists $\alpha\colon A\times A \to [0,\infty)$, $L \in \mathfrak{M}$, a constant $k>0$ such that for all ξ_1, ξ_2, ζ_1, $\zeta_2 \in A$, with $\alpha(\xi_1,\xi_2)\geq 1$ and $d(\zeta_1,\mathcal{T}\xi_1)=d(A,B)=d(\zeta_2,\mathcal{T}\xi_2)$, we get $\alpha(\zeta_1,\zeta_2)\geq 1$, and*

$$k + L(d(\zeta_1,\zeta_2),\varphi(\zeta_1),\varphi(\zeta_2)) \leq L(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2)),$$

whenever

$$\min\{d(\zeta_1,\zeta_2)+\varphi(\zeta_1)+\varphi(\zeta_2), d(\xi_1,\xi_2)+\varphi(\xi_1)+\varphi(\xi_2)\} > 0.$$

In plus, the next conditions are fulfilled:

(i) $TA_0 \subseteq B_0$;

(ii) *there are ξ_1, $\xi_2 \in A_0$, so that $\alpha(\xi_1,\xi_2)\geq 1$ and $d(\xi_2,\mathcal{T}\xi_1)=d(A,B)$;*

(iii) *every sequence $\{\xi_n\} \subseteq A_0$ with $\xi_n \to \xi$ and $\alpha(\xi_n,\xi_{n+1})\geq 1$ for all $n\in\mathbb{N}$, necessarily satisfies the inequality $\alpha(\xi_n,\xi)\geq 1$, for all $n\in\mathbb{N}$.*

(iv) *φ and φ are lower semi continuous.*

Then $\mathcal{T}$ has a (φ,φ)-best proximity point.

4. Applications in fixed point theory

In this section, using a self-mapping $\mathcal{T}\colon A \to A$ we obtain some theorems that ensure the existence of fixed points. These results are stated by taking $A = B = X$ in the previous theorems.

In the first case, we get the next result.

Theorem 3 *Let (X,d) be a complete b- metric space, and $\mathcal{T}\colon X \to X$ be a mapping for which there exist $\alpha\colon X\times X \to [0,\infty)$, φ, $\varphi\colon X \to [0,\infty)$, $L \in \mathfrak{L}_w$, $W \in \mathfrak{R}$, a positive number $k>0$ such that for all ξ_1, $\xi_2 \in A$ with $\alpha(\xi_1,\xi_2)\geq 1$, we have $\alpha(\mathcal{T}\xi_1,\mathcal{T}\xi_2)\geq 1$ and*

$$\begin{aligned} k + L(W(sd(\mathcal{T}\xi_1,\mathcal{T}\xi_2),\varphi(\mathcal{T}\xi_1),\varphi(\mathcal{T}\xi_2))) \\ \leq L(W(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))), \end{aligned}$$

whenever

$$\begin{aligned} \min\{W(sd(\mathcal{T}\xi_1,\mathcal{T}\xi_2),\varphi(\mathcal{T}\xi_1),\varphi(\mathcal{T}\xi_2)), \\ W(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))\} > 0. \end{aligned}$$

Also, the next hypothesis are true:

(i) *there is a point $\xi_1 \in X$ so that $\alpha(\xi_1,\mathcal{T}\xi_1)\geq 1$;*

(ii) *every $\{\xi_n\}\subseteq X$ with $\xi_n\to\xi$ and $\alpha(\xi_n,\xi_{n+1})\geq 1$, for all $n\in\mathbb{N}$, fulfills the inequality $\alpha(\xi_n,\xi)\geq 1$, for all $n\in\mathbb{N}$;*

(iii) *φ, $\varphi\colon A \to [0,\infty)$ are bounded lower semi continuous functions.*

Then $\mathcal{T}$ has a (φ,φ)-fixed point in X.

For the particular case obtained in the situation of L^{II} proximal contractions, the next theorem holds true.

Theorem 4 *Let (X,d) be a complete b-metric space, and $\mathcal{T}\colon X\to X$ be a mapping for which there exist $\alpha\colon X\times X\to[0,\infty)$, $\varphi,\ \varphi\colon X\to[0,\infty)$, $L\in\mathfrak{M}$, a constant $k>0$, such that for all ξ_1, $\xi_2\in X$ with $\alpha(\xi_1,\xi_2)\geq 1$, we have $\alpha(\mathcal{T}\xi_1,\mathcal{T}\xi_2)\geq 1$ and*

$$k+L(sd(\mathcal{T}\xi_1,\mathcal{T}\xi_2),\varphi(\mathcal{T}\xi_1),\varphi(\mathcal{T}\xi_2))\leq L(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2)),$$

whenever $\min\{sd(\mathcal{T}\xi_1,\mathcal{T}\xi_2)+\varphi(\mathcal{T}\xi_1)+\varphi(\mathcal{T}\xi_2),d(\xi_1,\xi_2)+\varphi(\xi_1)+\varphi(\xi_2)\}>0$. Also, the next hypotheses are true:

(i) *there is a point $\xi_1\in X$ so that $\alpha(\xi_1,\mathcal{T}\xi_1)\geq 1$;*

(ii) *every $\{\xi_n\}\subseteq X$ with $\xi_n\to\xi$ and $\alpha(\xi_n,\xi_{n+1})\geq 1$, for all $n\in\mathbb{N}$, fulfills the inequality $\alpha(\xi_n,\xi)\geq 1$, for all $n\in\mathbb{N}$;*

(iii) *$\varphi,\ \varphi\colon X\to[0,\infty)$ are lower semi continuous functions.*

Then $\mathcal{T}$ has a (φ,φ)-fixed point in X.

If we consider $s=1$ and $\beta=1$ in the previous fixed point theorems, we obtain Theorem 5 and Theorem 7 from [2], that give us conditions for the existence of a fixed point in classic metric spaces.

Corollary 3 Let (X,d) be a complete metric space, and $\mathcal{T}\colon X\to X$ be a mapping for which there exist $\alpha\colon X\times X\to[0,\infty)$, $\varphi,\ \varphi\colon X\to[0,\infty)$, $L\in\mathfrak{L}_w$, $W\in\mathfrak{R}$ and a constant $k>0$ such that for all ξ_1, $\xi_2\in X$ with $\alpha(\xi_1,\xi_2)\geq 1$, we have $\alpha(\mathcal{T}\xi_1,\mathcal{T}\xi_2)\geq 1$ and

$$\begin{aligned}k+L(W(d(\mathcal{T}\xi_1,\mathcal{T}\xi_2),\varphi(\mathcal{T}\xi_1),\varphi(\mathcal{T}\xi_2)))&\\ \leq L(W(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))),&\end{aligned}$$

whenever

$$\begin{aligned}\min\{W(d(\mathcal{T}\xi_1,\mathcal{T}\xi_2),\varphi(\mathcal{T}\xi_1),\varphi(\mathcal{T}\xi_2)),&\\ W(d(\xi_1,\xi_2),\varphi(\xi_1),\varphi(\xi_2))\}>0.&\end{aligned}$$

Also, the next hypothesis are true:

(i) there is a point $\xi_1\in X$ so that $\alpha(\xi_1,\mathcal{T}\xi_1)\geq 1$;

(ii) every $\{\xi_n\}\subseteq X$ with $\xi_n\to\xi$ and $\alpha(\xi_n,\xi_{n+1})\geq 1$, for all $n\in\mathbb{N}$, fulfill the inequality $\alpha(\xi_n,\xi)\geq 1$, for all $n\in\mathbb{N}$;

(iii) $\varphi,\ \varphi\colon X\to[0,\infty)$ are bounded lower semi continuous functions.

Then $\mathcal{T}$ has a (φ,φ)-fixed point in X.

Corollary 4 *Let (X, d) be a complete metric space, and $\mathcal{T}\colon X \to X$ be a mapping for which there exist $\alpha\colon X \times X \to [0, \infty)$, φ, $\varphi\colon X \to [0, \infty)$, $L \in \mathfrak{M}$ and a constant $k > 0$ such that for all ξ_1, $\xi_2 \in X$ with $\alpha(\xi_1, \xi_2) \geq 1$, we have $\alpha(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \geq 1$ and*

$$k + L(d(\mathcal{T}\xi_1, \mathcal{T}\xi_2), \varphi(\mathcal{T}\xi_1), \varphi(\mathcal{T}\xi_2)) \leq L(d(\xi_1, \xi_2), \varphi(\xi_1), \varphi(\xi_2)),$$

whenever

$$\min\{d(\mathcal{T}\xi_1, \mathcal{T}\xi_2) + \varphi(\mathcal{T}\xi_1) + \varphi(\mathcal{T}\xi_2), d(\xi_1, \xi_2) + \varphi(\xi_1) + \varphi(\xi_2)\} > 0.$$

Also, the next hypothesis are true:

(i) *there is a point $\xi_1 \in X$ so that $\alpha(\xi_1, \mathcal{T}\xi_1) \geq 1$;*

(ii) *every $\{\xi_n\} \subseteq X$ with $\xi_n \to \xi$ and $\alpha(\xi_n, \xi_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, fulfill the inequality $\alpha(\xi_n, \xi) \geq 1$, for all $n \in \mathbb{N}$;*

(iii) *φ, $\varphi\colon X \to [0, \infty)$ are lower semi continuous functions.*

Then $\mathcal{T}$ has a (φ, φ)-fixed point in X.

5. Conclusions

In this paper, we underlined the importance that b-metrics have in the development of best proximity point theory. We defined two types of proximal contractions using b-metric spaces for functions that have adequate properties which refer, for example, to continuity or monotone. With these new contractive conditions we formulated theorems stating the existence of best proximity points. We emphasize that the shift from classic metrics to b-metrics requires additional properties with respect to a convergent sequence. Also, as applications, we gave some fixed point results for the case of self mappings. As a further development, we aim to use the background of extended b-metric spaces and other generalized metric spaces to study implicit type contractive conditions inspired by those from the current work.

References

[1] Ali MU, Kamran T, Postolache M. Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem. Nonlinear Analysis: Modelling and Control 2017; 22: 17-30. https://doi.org/10.15388/NA.2017.1.2

[2] Ali MU, Ionescu C, Stanciu M. On (φ, φ)-best proximity points for proximal type contraction mappings. University Politehnica of Buchrest Scientific Bulletin, Series A 2021; 83: 15-26.

[3] Bakhtin IA. The contraction mapping principle in almost metric spaces. Functional Analysis 1989; 30: 26-37.

[4] Basha S. Best proximity point theorems. Journal of Approximation Theory 2011; 163: 1772-1781. https://doi.org/10.1016/j.jat.2011.06.012

[5] Batul S, Sagheer D, Aydi H, Mukheimer A, Aiadi S. Best proximity point results for Presić type nonself operators in b-metric spaces. AIMS Mathematics 2022; 7: 10711-10730. https://doi.org/10.3934/math.2022598

[6] Choudhury BS, Metiya N, Postolache M, Konar P. A discussion on best proximity point and coupled best proximity point in partially ordered metric spaces. Fixed Point Theory and its Applications 2015; Art. No. 170. https://doi.org/10.1186/s13663-015-0423-1

[7] Czerwik S. Contraction mappings in b-metric spaces. Acta Mathematica and Informatica Universitatis Ostraviensis 1993; 1: 5-11.

[8] Eldred A, Veeramani P. Existence and convergence of best proximity points. Journal of Mathematical Analysis and Applications 2006; 323: 1001-1006. https://doi.org/10.1016/j.jmaa.2005.10.081

[9] Fan K. Extensions of two fixed point theorems of F.E. Browder. Mathematische Zeitschrift 1969; 112: 234-240. https://doi.org/10.1007/BF01110225

[10] Fernandéz-Léon A. Best proximity points for proximal contractions. Journal of Nonlinear and Convex Analysis. 2014; 15: 313-324. https://doi.org/10.48550/arXiv.1207.4349

[11] Gabaleh M, Shahzad N. Approximate optimal solutions and generalized contractions in the sense of Chatterjea. University Politehnica of Bucharest Scientific Bulletin Series A 2015; 77: 45-56.

[12] Hussain N, Dorić D, Kadelburg Z, Radenović S. Suzuki-type fixed point results in metric type spaces. Fixed Point Theory and its Applications 2012; Article ID 2012:126. https://doi.org/10.1186/1687-1812-2012-126

[13] Jacob JK, Postolache M, Marudai M, Raja V. Norm convergence iterations for best proximity points of non-self non-expansive mappings. University Politehnica of Bucharest Scientific Bulletin Series A 2017; 79: 49-56.

[14] Joseph JM, Beny J, Marudai M. Best proximity point theorems in b-metric spaces. Journal of Analysis 2019; 27: 859-866. https://doi.org/10.29055/jcms/978

[15] Kamran T, Postolache M, Ali MU, Kiran Q. Feng and Liu type F-contraction in b-metric spaces with application to integral equations. Journal of Mathematical Analysis 2016; 7: 18-27.

[16] Lucács A, Kajántó S. Fixed point theorems for various types of F-contractions in complete b-metric spaces. Fixed Point Theory 2018; 19: 321-334. https://doi.org/10.24193/fpt-ro.2018.1.25

[17] Petric M. Best proximity point theorems for weak cyclic Kannan contractions. Filomat 2011; 25: 145-154. https://doi.org/10.2298/FIL1101145P

[18] Samreen M, Kamran T, Postolache M. Extended b-metric space, extended b-comparison function and nonlinear contractions. University Politehnica of Bucharest Scientific Bulletin Series A 2018; 80: 21-28.

[19] Suparatulatorn R, Suantai S. A new hybrid algorithm for global minimization of best proximity points in Hilbert spaces. Carpathian Journal of Mathematics 2019; 35: 95-102.

[20] Wardowski D. Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory and its Applications 2012; Art. no. 2012:94. https://doi.org/10.1186/1687-1812-2012-94