Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1887 – 1898
© TÜBİTAK
doi:10.55730/1300-0098.3470

An extensive note on characteristic properties and possible implications of some operators designated by various type derivatives

Ömer Faruk KULALI[1,2], **Hüseyin IRMAK**[1,*]

[1]Department of Mathematics, Faculty of Science, Çankırı Karatekin University, Çankırı, Turkiye

[2]Çorum Provincial Directorate of National Education, Turkish Ministry of National Education, Çorum, Turkiye

Received: 08.11.2022 • **Accepted/Published Online:** 09.08.2023 • **Final Version:** 09.11.2023

Abstract: In this extensive note, various differential-type operators in certain domains of the complex plane will be first introduced, a number of their comprehensive characteristic properties will be next pointed out and an extensive theorem dealing with some argument properties for several multivalent(ly) analytic functions will be also presented. In addition, numerous implications and suggestions, which can be obtained with the help of general result, will be determined.

Key words: Complex plane, multivalent(ly) analytic function, operators, fractional-order calculus

1. Presentation of introduction and preliminary information

First of all, let us present some information about multivalent functions with complex variable defined in any domain of the familiar complex plane, which will be very important for our scientific notes. As it is well-known, those complex functions have various different-interesting roles in the theory of complex functions. This notion also is a natural generalization for the theory of univalent function. For $n = 1, 2, 3, \cdots$, an analytic function (or meromorphically analytic function) like $\xi(z)$ described in any domain $\mathfrak{S}$ of the z-plane is called as n-valent in $\mathfrak{S}$ if it takes each of its values at most n times there. In other words, for any complex number ω, the number of roots of any equation $\xi(z) = \omega$ in the set $\mathfrak{S}$ is unable to exceed n. Right along side this most simple family of n-valent functions, an essential role in the theory of the complex functions is also played by functions which are n-valent in certain generalized sense. In special, one of the most active roles is that studies involving various analytical-geometrical properties of those (multivalent) functions are still interesting for many researchers. For more information about those functions and some of their comprehensive implications, it can be also concentrated on the fundamental studies presented in [4, 14, 14, 21, 26, 32] in the references of this investigation.

Moreover, as more general expression of the functions emphasized just above, in the same time, for both our scientific investigation and the accentuated family of the multivalent functions which are analytic in the open unit disk:

$$\mathbb{U} := \left\{ z : z \in \mathbb{C} \text{ and } |z| < 1 \right\},$$

*Correspondence: hirmak@karatekin.edu.tr (*or* hisimya@yahoo.com)

2010 *AMS Mathematics Subject Classification:* 26A33, 30A10, 30B10, 30C55, 26D10.

let us also consider the series expansions of the pointed analytic functions like $\xi := \xi(z)$ being of the complex-series forms given by

$$\xi(z) = z^n + \kappa_{n+1} z^{n+1} + \kappa_{n+2} z^{n+2} + \kappa_{n+3} z^{n+3} \ \cdots \ , \tag{1.1}$$

where

$$\kappa_{n+j} \in \mathbb{C} \quad , \quad j := 1, 2, 3, \cdots \quad \text{and} \quad n \in \mathbb{N} \, .$$

In especial, for respective researchers, as a great variety of investigations appertaining to certain multivalent(ly) analytic functions, it may be also focused on the earlier-scientific investigations given by the references in [1–3, 5, 6, 8–10, 13, 15, 19, 20, 22–24, 34, 36, 39], which also relate to the extensive family consisting of the n-valent(ly) analytic functions created by the series forms given in (1.1).

In addition, let us now introduce (or remind) a number of operators consisting of various different-type derivatives. For those operators, let us denote them as the following forms:

$$\xi^{(s)}(z) \quad , \quad \mathcal{D}^s_{\vartheta}[\xi(z)] \quad , \quad \mathbf{D}^{\lambda}_z[\xi(z)] \quad \text{and} \quad \mathfrak{D}^{\lambda}_{s,\vartheta}[\xi(z)] \, , \tag{1.2}$$

and, for any n-valent(ly) analytic function $\xi := \xi(z)$ being of the series form given by (1.1), respectively, we also express them in the following definitions:

$$\xi^{(s)}(z) := \frac{d^s}{dz^s}\Big(\xi(z)\Big) \, , \tag{1.3}$$

$$\mathcal{D}^s_{\vartheta}\big[\xi(z)\big] := \vartheta \xi^{(s)}(z) + (1-\vartheta) z \xi^{(s+1)}(z) \, , \tag{1.4}$$

$$\mathbf{D}^{\lambda}_z\big[\xi(z)\big] := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{\xi(u)}{(z-u)^{\lambda}} du \, , \tag{1.5}$$

$$\mathbf{D}^{n+\lambda}_z\big[\xi(z)\big] := \frac{d^n}{dz^n}\Big(\mathbf{D}^{\lambda}_z\big[\xi(z)\big]\Big) \tag{1.6}$$

and

$$\mathfrak{D}^{s+\lambda}_{\vartheta}\big[\xi(z)\big] := \mathbf{D}^{\lambda}_z\big[\mathcal{D}^s_{\vartheta}[\xi(z)]\big] \, , \tag{1.7}$$

where

$$s < n \quad , \quad 0 \le \vartheta \le 1 \quad , \quad 0 \le \lambda < 1 \quad , \quad n \in \mathbb{N} \quad \text{and} \quad s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1.8}$$

Particularly, we note that the notation Γ denotes the well-known Gamma function and it would be also appropriate to remind some extra information here. When focusing on the comprehensive operators designed between (1.3) and (1.7), it can be easily seen that they are various type operators, each of which includes different-type derivative(s) in certain domains of the z-plane. It is obvious that, for any complex function ξ, the operators given in (1.3) and (1.4) are the ordinary-differential operators and the operators given by (1.5) and (1.6) also denote the operator of fractional-order derivative(s). In addition, the last operator given by (1.7) is also a special operator defined by a combination of the mentioned operators in (1.3)-(1.6). When one centers on all derivative operators and/or chooses the relevant parameters appropriately there, an excessive number of characteristic properties as well as special relationships can be also obtained. Some of those properties (and/or relationships) will be emphasized in the second part of this paper as various specific consequences of our main results are to be determined. However, especially, for the interested researchers, it is suggested to look over the recent papers presented in [17, 18], which consist of both the various characteristics of those operators between (1.3)-(1.7) and for some interesting applications of the respective operators.

In addition, for the operator of certain fractional order, one may refer to the earlier works given by [6, 30, 34, 35]. All in the same breath, for some of various applications of the fractional-order operators, one can also center on the earlier papers given by [7, 11, 25, 27, 30, 35]. We also note here that the relevant function $\xi(z)$, which is situated in the definition of the fractional-order derivative operator in (1.5) (and also in (1.6)), is specifically an analytic function in a simply-connected region of the complex plane covering the origin, and the multiplicity of the term $(z-u)^{-\lambda}$ is eliminated by necessitating $log(z-u)$ to be real *when* $z-u>0$, as principally indicated as in [26].

More particularly, in establishing of our fundamental results, of course, under the similar conditions accentuated in (1.8), we also need to determine some computational results of an analytic function being of the following-complex power form:

$$\varsigma(z) := z^{\rho} \quad \left(\rho > -1; z \in \mathbb{U}\right).$$

Indeed, by considering this complex function just above and also making use of the mentioned operators introduced by (1.3)-(1.7), the following-extensive results can be easily calculated as some elementary examples in relation to the complex function just above.

Remark 1.1 *Let* $s < \rho$, $0 \leq \vartheta \leq 1$, $0 \leq \lambda < 1$ *and* $s \in \mathbb{N}_0$. *Then, the following-comprehensive expressions are true:*

$$\left(z^{\rho}\right)^{(s)} = \mathcal{V}_s(\rho) z^{\rho-s} \quad (\rho \in \mathbb{N}), \tag{1.9}$$

$$\mathcal{D}_{\vartheta}^{s}[z^{\rho}] = [\vartheta + (1-\vartheta)(\rho - s)]\mathcal{V}_s(\rho) z^{\rho-s} \quad (\rho \in \mathbb{N}), \tag{1.10}$$

$$\mathbf{D}_z^{\lambda}[z^{\rho}] = \mathcal{V}_{\lambda}(\rho) z^{\rho-\lambda} \quad \left(\rho > -1\right) \tag{1.11}$$

$$z\mathbf{D}_z^{1+\lambda}[z^{\rho}] = (\rho - \lambda)\mathcal{V}_{\lambda}(\rho) z^{\rho-\lambda} \quad \left(\rho > -1\right), \tag{1.12}$$

and

$$\begin{aligned} \mathfrak{D}_{\vartheta}^{s+\lambda}[z^{\rho}] &= \mathbf{D}_z^{\lambda}\left[\mathcal{D}_{\vartheta}^{s}[z^{\rho}]\right] \\ &= [\vartheta + (1-\vartheta)(\rho - s)]\mathcal{V}_{\rho}(\rho - s)\mathbf{D}_z^{\lambda}\left[z^{\rho-s}\right] \\ &= [\vartheta + (1-\vartheta)(\rho - s)]\mathcal{V}_{s+\lambda}(\rho) z^{\rho-s-\lambda}, \end{aligned} \tag{1.13}$$

where

$$\mathcal{V}_{\kappa}(\rho) := \frac{\Gamma(\rho+1)}{\Gamma(\rho-\kappa+1)} \quad \left(\rho - \kappa > -1; z \in \mathbb{U}\right). \tag{1.14}$$

Under the pertinent conditions presented by (1.8), by taking into account the definitions of the mentioned operators between (1.3)-(1.7) (*or,* in the light of the elementary results between (1.9)-(1.13) in Remark 1.1), the following-comprehensive propositions, *which* will be also encountered in the section 2, can be easily demonstrated.

Remark 1.2 *Let $s < n$, $0 \leq \vartheta \leq 1$, $0 \leq \lambda < 1$, $n \in \mathbb{N}$ and $s \in \mathbb{N}_0$. Then, for the function $\xi := \xi(z)$ being of the complex-series form given by (1.1), the following results are also true:*

$$z^s \xi^{(s)}(z) = \sum_{j=n}^{\infty} \kappa_j \mathcal{V}_s(j) z^j \,, \tag{1.15}$$

$$z^s \mathcal{D}_{\vartheta}^s[\xi(z)] = \sum_{j=n}^{\infty} \kappa_j [\vartheta + (1-\vartheta)(j-s)] \mathcal{V}_s(j) z^j \,, \tag{1.16}$$

$$z^{\lambda} \mathbf{D}_z^{\lambda}[\xi(z)] = \sum_{j=n}^{\infty} \kappa_j \mathcal{V}_{\lambda}(j) z^j \,, \tag{1.17}$$

$$z^{1+\lambda} \mathbf{D}_z^{1+\lambda}[\xi(z)] = \sum_{j=n}^{\infty} \kappa_j (j-\lambda) \mathcal{V}_{\lambda}(j) z^j \tag{1.18}$$

and

$$z^{s+\lambda} \mathfrak{D}_{\vartheta}^{s+\lambda}[\xi(z)] = \sum_{j=n}^{\infty} \kappa_j [\vartheta + (1-\vartheta)(j-s)] \mathcal{V}_{s+\lambda}(j) z^j \,, \tag{1.19}$$

where $\kappa_n := 1$ and the notation $\mathcal{V}_{\rho}(\kappa)$ also denotes the mathematical expression formulated by the function in (1.14).

When one concentrates on the operators stated in (1.2), it can be easily recognized that each of the respective definitions adds different meaningful value to this scientific study in many ways. Especially, it is easily seen that each of the elementary results stated with Remark 1.1 and Remark 1.2 also include quite comprehensive-important results which either consist of various type derivatives or are produced by the mentioned operators in (1.3)-(1.7). Moreover, in terms of literature, we think that some extra information about both those derivative operators and their possible implications would be beneficial. As it has been pointed out, those operators are ordinary-type operators and both contain higher-order derivatives, and their effects on any complex function ξ (being of the form in (1.1)) are certain multivalent(ly)-type analytic functions. Namely, those complex functions, which are analytic in the open set $\mathbb{U}$, will either be in any form of the multivalent(ly)-analytic functions or the normalized-analytic functions. For such complex functions, one can center upon the comprehensive results (or relationships) in the recent studies given in [19, 20].

In addition, some operators designated by the operators given with (1.5) and (1.6) are fractional-type operators of various orders, but they are different types of fractional operators that can be also encountered as the Srivastava-Owa-type operator in the mathematical literature. They also have any one of the similar forms constituted by (1.11) and (1.12) (or (1.17) and (1.18)). For some of those applications, the earlier investigations given in [11, 18, 30, 37] can also be examined.

Lastly, the next formation also is an operator formed by various type derivatives which both include the operators given by (1.3) and (1.5) and also their implications as in (1.7), (1.13) and (1.19). It is a very comprehensive operator and it also includes so many special results. For this special operator and also some of its possible implications, it is strongly recommended to center on the recent results in the paper given by [19, 20, 38] again.

2. Presentation of an auxiliary theorem and related results together with their implications

In the light of the extensive information in the section 1, in this section, an auxiliary theorem, which is Lemma 2.1 (just below), will be necessary for both the stating of the main result and its demonstrating. Firstly, we shall need to introduce that essential theorem. For its details and also a number of its implications, one can concentrate on the main papers in [28, 29] and also see the earlier results given in [19, 24].

Lemma 2.1 *Let the function $v(z)$ with $v(0) = 1$ be analytic in $\mathbb{U}$ and also let r be any real number with $|r| > 1$. If there exists a point $0 \neq z_0 \in \mathbb{U}$ such that*

$$\Re e\big\{v(z)\big\} > 0 \quad when \quad |z| \leq |z_0| < 1 \tag{2.1}$$

and

$$\Re e\big\{v(z_0)\big\} = 0 \quad \big(v(z_0) \neq 0\big)\,, \tag{2.2}$$

then

$$z_0 v'(z_0) = irv(z_0)\,. \tag{2.3}$$

Next, for the sake of convenience in establishing the main result together with its implications, there is a need to describe an extensive definition. Under the conditions particularized as in (1.7) and for a function $\xi := \xi(z)$ possessing the complex-series form given in (1.1), let us now present its expression in the following form:

$$\mathbb{J}^{\lambda}_{s,\vartheta}\{\xi(z)\} := \frac{z\mathfrak{D}^{1+s+\lambda}_{\vartheta}[\xi(z)]}{\mathfrak{D}^{s+\lambda}_{\vartheta}[\xi(z)]} \tag{2.4}$$

where $0 \leq \lambda < 1$ and $z \in \mathbb{U}$.

In consideration of the definitions of the operators given by (1.2), we can now state our main result and then demonstrate it.

Theorem 2.2 *Let*

$$s < n \quad , \quad 0 \leq \lambda < 1 \quad , \quad 0 \leq \vartheta \leq 1 \quad , \quad n \in \mathbb{N} \quad and \quad s \in \mathbb{N}_0 \tag{2.5}$$

and also let the function $\xi(z)$ possess the series expansion given the form in (1.1). Then, the following implication is also satisfied:

$$Arg\left(z\frac{d}{dz}\left[\mathbb{J}^{\lambda}_{s,\vartheta}\{\xi(z)\}\right]\right)\notin\{0,\pi\} \tag{2.6}$$

$$\Longrightarrow\quad \Re e\Big(\mathbb{J}^{\lambda}_{s,\vartheta}\{\xi(z)\}\Big)>\beta(n-s-\lambda)\,, \tag{2.7}$$

where

$$z\in\mathbb{U}\quad and\quad 0\leq\beta<\frac{1}{n-s-\lambda}\,. \tag{2.8}$$

Proof Firstly, let $z\in\mathbb{U}$ and also let the mentioned function $\xi(z)$ have the series form as in (1.1). Through the instrument of the information in relation to the definitions of the different-type-derivative operators between (1.3)-(1.7) and by considering the definition given by (2.4), one can easily get the following-comprehensive correlations between the operators given in (1.2):

$$\begin{aligned}\mathbb{J}^{\lambda}_{s,\vartheta}\{\xi(z)\}&:=\frac{z\mathfrak{D}^{1+s+\lambda}_{\vartheta}[\xi(z)]}{\mathfrak{D}^{s+\lambda}_{\vartheta}[\xi(z)]}\\&\equiv\frac{z\mathbf{D}^{1+\lambda}_{z}\left[\mathcal{D}^{s}_{\vartheta}[\xi(z)]\right]}{\mathbf{D}^{\lambda}_{z}\left[\mathcal{D}^{s}_{\vartheta}[\xi(z)]\right]}\\&\equiv\frac{z\mathbf{D}^{1+\lambda}_{z}\left[\vartheta\xi^{(s)}(z)+(1-\vartheta)z\xi^{(1+s)}(z)\right]}{\mathbf{D}^{\lambda}_{z}\left[\vartheta\xi^{(s)}(z)+(1-\vartheta)z\xi^{(s+1)}(z)\right]}\,,\end{aligned} \tag{2.9}$$

and so, in the light of the information given by (1.13)-(1.19) and (2.9), and, in particular, by making use of the complex-series expansion given by (1.1), the following-elementary results associating with the analytic function $\xi(z)$:

$$\begin{aligned}\mathbb{J}^{\lambda}_{s,\vartheta}\{\xi(z)\}&\equiv\frac{z\mathfrak{D}^{1+s+\lambda}_{\vartheta}[\xi(z)]}{\mathfrak{D}^{s+\lambda}_{\vartheta}[\xi(z)]}\\&=\cdots\\&=\frac{\sum_{j=n}^{\infty}\kappa_j[\vartheta+(1-\vartheta)(j-s)]\mathcal{V}_{1+s+\lambda}(j)z^j}{\sum_{j=n}^{\infty}\kappa_j[\vartheta+(1-\vartheta)(j-s)]\mathcal{V}_{s+\lambda}(j)z^j}\\&=\frac{n-s-\lambda+\sum_{j=1}^{\infty}\left((n+j-s-\lambda)\frac{\mathcal{W}^{\vartheta}_{s+\lambda}(n+j)}{\mathcal{W}^{\vartheta}_{s+\lambda}(n)}\right)\kappa_{n+j}z^j}{1+\sum_{j=1}^{\infty}\left(\frac{\mathcal{W}^{\vartheta}_{s+\lambda}(n+j)}{\mathcal{W}^{\vartheta}_{s+\lambda}(n)}\right)\kappa_{n+j}z^j}\end{aligned} \tag{2.10}$$

can be next calculated (by the help of the combining of the computational results given by (1.16), (1.19) and (2.10) *when* $\kappa_n:=1$), where

$$\mathcal{W}^{\vartheta}_{k}(j):=[\vartheta+(1-\vartheta)(j-s)]\frac{\Gamma(j+1)}{\Gamma(j-k-\lambda)}\,. \tag{2.11}$$

We want to use the mentioned lemma for the pending proof of Theorem 2.2. Additionally, for the related proof, we also want to consider Lemma 2.1 together with the extensive information in Remark 1.1 and Remark

1.2. For that, we have to define a complex function like the accentuated function in the lemma. For this, firstly, under the conditions of the definitions in (2.5), (2.8) and (2.11), and with the help of the information generated in (2.10), let us now define an analytic function like $v(z)$ in Lemma 2.1, which is given by the following-implicit form:

$$\begin{aligned}\mathbb{J}_{s,\vartheta}^{\lambda}\{\xi(z)\} &= \frac{n-s-\lambda+\sum_{j=1}^{\infty}\Big((n+j-s-\lambda)\frac{\mathcal{W}_{s+\lambda}^{\vartheta}(n+j)}{\mathcal{W}_{s+\lambda}^{\vartheta}(n)}\Big)\kappa_{n+j}z^j}{1+\sum_{j=1}^{\infty}\Big(\frac{\mathcal{W}_{s+\lambda}^{\vartheta}(n+j)}{\mathcal{W}_{s+\lambda}^{\vartheta}(n)}\Big)\kappa_{n+j}z^j} \\ &\equiv (n-s-\lambda)\big[1+c_1z+c_2z^2+\cdots\,\big] \\ &:= (n-s-\lambda)\big[\beta+(1-\beta)v(z)\big]\,, \end{aligned} \tag{2.12}$$

where $z\in\mathbb{U}$ and $\mathcal{W}_k^{\vartheta}(j)$ is also defined by (2.11).

Since the complex function $v(z)$, *which* is suitable for Lemma 2.1, is an analytic function in the complex domain $\mathbb{U}$ with $v(0)=1$, by differentiating of this function $v(z)$ (with respect to the complex variable z), it can easily stated by the following relation:

$$\frac{d}{dz}\Big(\mathbb{J}_{s,\vartheta}^{\lambda}\{\xi(z)\}\Big) = (1-\beta)(n-s-\lambda)v'(z) \quad \big(z\in\mathbb{U}\big)\,. \tag{2.13}$$

Now, let us assume that there is a point like $z_0\in\mathbb{U}$ (with $z_0\neq 0$) as underlined in the related lemma. Namely, there exists a point z_0 possessing the condition there, which is

$$\Re e\big(v(z_0)\big)=0 \quad \big(z_0\in\mathbb{U}-\{0\}\big). \tag{2.14}$$

In the same time, this assumption (2.14) also requires the condition $v(z_0)=i\omega$ with $\omega\neq 0$ as it was indicated in (2.2).

Therefore, for the related proof, by making allowances for the apparent result (2.3) given by Lemma **??**, the following relations:

$$\begin{aligned} &\mathrm{Arg}\Big\{z\frac{d}{dz}\Big(\mathbb{J}_{s,\vartheta}^{\lambda}\{\xi(z)\}\Big)\Big|_{z:=z_0}\Big\} \\ &\qquad = \mathrm{Arg}\Big((1-\beta)(n-s-\lambda)zv'(z)\big|_{z:=z_0}\Big) \\ &\qquad = \mathrm{Arg}\Big((1-\beta)(n-s-\lambda)irv(z_0)\Big) \\ &\qquad = \mathrm{Arg}\Big(-(1-\beta)(n-s-\lambda)rw\Big) \end{aligned} \tag{2.15}$$

can be easily created through the instrument of the relation given by (2.13) under the bordered conditions indicated as before. When we focus on the presented conditions of all the parameters in the relationships in (2.15), that is, when the conditions given in (2.5) and (2.8) are taken into account there, first of all, we have to consider the conditions given by

$$n-s-\lambda>0 \quad \text{and} \quad 0\leq\beta<1/[n-s-\lambda].$$

Particularly, by having regard to the special information just above, when the concerned conditions of the real parameters w and r are considered, the following assertions:

$$[\,r > 1 \text{ and } w > 0\,] \;\Rightarrow\; \mathrm{Arg}\big(-(1-\beta)(n-s-\lambda)rw\big) = \pi,$$

$$[\,r > 1 \text{ and } w < 0\,] \;\Rightarrow\; \mathrm{Arg}\big(-(1-\beta)(n-s-\lambda)rw\big) = 0,$$

$$[\,r < -1 \text{ and } w > 0\,] \;\Rightarrow\; \mathrm{Arg}\big(-(1-\beta)(n-s-\lambda)rw\big) = 0$$

and

$$[\,r < -1 \text{ and } w < 0\,] \;\Rightarrow\; \mathrm{Arg}\big(-(1-\beta)(n-s-\lambda)rw\big) = \pi$$

are easily determined. However, these argument values (just above) are contradictions with the earlier argument values presented by (2.6) in the theorem. This situation also tells us that there is no point z_0 involved by the open set $\mathbb{U}$. Thereby, with the help of the hypothesis of the lemma, of course, under the conditions in (2.5) and (2.8), from (2.12), we then arrive at:

$$\Re e\left\{\frac{1}{1-\beta}\left(\frac{\mathbb{J}_{s,\vartheta}^{\lambda}\{\xi(z)\}}{n-s-\lambda}-\beta\right)\right\} = \Re e\big\{v(z)\big\} > 0\,. \tag{2.16}$$

Thus, under the conditions given by (2.8), the inequality, which is presented by the condition in (2.16), also necessitates the known provision of the theorem, which is the inequality given by (2.7). For this reason, the inevitable proof ends. □

In this last section, for our interested researchers, some specific conclusions regarding our main result will be highlighted or some specific recommendations will be also presented. Especially, when one takes cognizance of the extensive information emphasised in the last part of the first chapter, it can be naturally seen that the main result presented just above also has an excessive number of special consequences. In order to reveal those, it will be sufficient to either choose the parameters in the relevant theorem appropriately or accurately determine the extensive relationships between those operators given in (1.2). Of course, the definition constituted in (2.4) will play a very important role in determining the accentuated results. Now, let us try to give some information associated with those special results.

As the first-special information, for all appropriate values of the mentioned parameters given by the restricted conditions as in (1.8) and also for all n-valent(ly) analytic function possessing the form like the complex function $\xi(z)$ given by (1.1), when taking the value of the parameter λ as $\lambda := 0$ in the definition stated in (2.9), a quite special relationship, which can be easily achieved by the mentioned operators defined by (1.3)-(1.7), can also be determined in the following form:

$$\begin{aligned}
\mathbb{J}_{s,\vartheta}^{0}\{\xi(z)\} &= \frac{z\mathfrak{D}_{\vartheta}^{1+s+0}[\xi(z)]}{\mathfrak{D}_{\vartheta}^{s+0}[\xi(z)]} \\
&= \frac{z\mathbf{D}_z^{1+0}\big[\mathcal{D}_{\vartheta}^{s}[\xi(z)]\big]}{\mathbf{D}_z^{0}\big[\mathcal{D}_{\vartheta}^{s}[\xi(z)]\big]} \\
&= \frac{z\mathbf{D}_z^{1+0}\big[\vartheta\xi^{(s)}(z)+(1-\vartheta)z\xi^{(1+s)}(z)\big]}{\mathbf{D}_z^{0}\big[\vartheta\xi^{(s)}(z)+(1-\vartheta)z\xi^{(s+1)}(z)\big]}
\end{aligned}$$

$$= \frac{z\mathcal{D}_{\vartheta}^{1}\left[\vartheta\xi^{(s)}(z) + (1-\vartheta)z\xi^{(1+s)}(z)\right]}{\mathcal{D}_{\vartheta}^{0}\left[\vartheta\xi^{(s)}(z) + (1-\vartheta)z\xi^{(s+1)}(z)\right]}$$

$$= \frac{z\xi^{(1+s)}(z) + \vartheta z\xi^{(2+s)}(z)}{\vartheta\xi^{(s)}(z) + (1-\vartheta)z\xi^{(s+1)}(z)} \quad \big(z \in \mathbb{U}\big). \tag{2.17}$$

In the same time, of course, if the current parameters s and ϑ in the special result given just above are chosen differently and appropriately, many complex functions with similar fractional types can be formed by the help of the function $\xi(z)$ there. As more special forms of that rational type function, given by (2.13), we also want to present only extra two of them, which are in the rational forms given by

$$s := 0 \;\Rightarrow\; \mathbb{J}_{0,\vartheta}^{0}\{\xi(z)\} := \frac{z\xi'(z) + \vartheta z\xi''(z)}{\vartheta\xi(z) + (1-\vartheta)z\xi'(z)} \tag{2.18}$$

and

$$s := 1 \;\Rightarrow\; \mathbb{J}_{1,\vartheta}^{0}\{\xi(z)\} := \frac{z\xi''(z) + \vartheta z\xi'''(z)}{\vartheta\xi'(z) + (1-\vartheta)z\xi''(z)} \tag{2.19}$$

for some $z \in \mathbb{U}$ and $0 \leq \vartheta \leq 1$.

In addition, as one of the special consequences of the definition, given by (2.4), and, of course, as a result of different selection of the relevant complex function $\xi(z)$ having the series expansion like the form in (1.1), each one of those fractional type functions with complex variable, determined in (2.13)-(2.15), is a complex function, which has important roles in the theory of complex functions. Moreover, since the point $z = 0$ is a removable singular point, each one of them is an analytic function in the domain $\mathbb{U}$. Especially, in the theory of analytic-geometric function, which is still a very active area of special research, the special roles that are emphasized still remain important for both the multivalent(ly)-analytic functions and the normalized-analytic functions in $\mathbb{U}$ (cf., e.g., [14, 20, 26, 37]).

As the second-special information, by considering the indicated explanations above, it is clear that in the light of similar special information such as the extensive explanations constituted by (2.13)-(2.15), and using the special selections of the parameters used in Theorem 1, a great number of special results can be revealed as various implications of the main result. As some examples, for interested researchers, we particularly recommend focusing on some of the results given in [19, 20]. However, we would like to create only two of the special results indicated above.

Firstly, the following-special implication, which is Proposition 2.3 just below, can be also constituted by considering a combining of the special information designated by (2.13) and using the main result.

Proposition 2.3 *Let $\xi(z)$ be an analytic function that has the series expansion in (1.1). Then, if the statement:*

$$Arg\left[z\frac{d}{dz}\left(\frac{z\xi^{(1+s)}(z) + \vartheta z\xi^{(2+s)}(z)}{\vartheta\xi^{(s)}(z) + (1-\vartheta)z\xi^{(s+1)}(z)}\right)\right] \notin \{0, \pi\}$$

is satisfied, the statement:

$$\Re e\left(\frac{z\xi^{(1+s)}(z) + \vartheta z\xi^{(2+s)}(z)}{\vartheta\xi^{(s)}(z) + (1-\vartheta)z\xi^{(s+1)}(z)}\right) > \beta(n-s)\,,$$

is also satisfied, where

$$s < n \ , \ 0 \leq \vartheta \leq 1 \ , \ n \in \mathbb{N} \ , \ z \in \mathbb{U} \quad \textit{and} \quad 0 \leq \beta < \frac{1}{n-s}.$$

Secondly, as a more special example in relation to our main result, when choosing the related coefficients belonging to the analytic function $\xi(z)$ like the series form in (1.1) with $n := 1$ as in the following forms:

$$\kappa_{n+1} := \frac{1}{n!} \text{ for all } n := 1, 2, 3, \cdots ,$$

the concerned-analytic function $\xi(z)$ is easily arrived at the complex-series-expansion form given by the elementary function with the complex variable z:

$$\begin{aligned}\xi(z) &:= ze^z \\ &= z + \frac{1}{1!}z^2 + \frac{1}{2!}z^3 + \frac{1}{3!}z^4 + \cdots \ \big(z \in \mathbb{U}\big).\end{aligned}$$

Then, in view of the definition (2.4), it is easily seen that

$$\mathbb{J}^0_{0,1}\{\xi(z)\} \equiv \mathbb{J}^0_{0,1}\{ze^z\} = 1 + z \quad \big(z \in \mathbb{U}\big).$$

So, in consideration of those special information just above, the following interesting example can be easily generated.

Example 2.4 *Let* $z \in \big\{w \in \mathbb{U} : \Re e(w) > 0 \ \textit{and} \ \Im m(w) \neq 0\big\} \subset \mathbb{U}$. *Then, the following assertions can be easily received:*

$$Arg\left\{z\frac{d}{dz}\Big(\mathbb{J}^0_{0,1}\{\xi(z)\}\Big)\right\} = Arg\left\{z\frac{d}{dz}\big(z+1\big)\right\} = Arg\{z\}$$

and

$$1 < \Re e\Big(\mathbb{J}^0_{0,1}\{\xi(z)\}\Big) \equiv \Re e\big(z+1\big) = 1 + \Re e\big(z\big) < 2.$$

3. Concluding remarks

It is clear that this scientific study consists of two main sections. Moreover, in the first section, necessary basic information and various derivative-integral operators have been first introduced and some elementary results related to multivalent functions have been then presented. In the second section, an auxiliary theorem, the main theorems consisting of our main results, some special forms of those theorems, and some special examples have been also constituted. In addition, as concluding remarks and various recommendations for relevant researchers, in the same time, as it has been underlined in various parts of this scientific note, by choosing the appropriate values of the mentioned parameters used in Theorem 2.2, both quite extensive results (and/or an excessive number of their special consequences) can be revealed and various elementary exemplifications can be recreated. As some special forms of them, only one of the relevant implications and another related example are presented as Proposition 2.3 and Example 2.4 above. Specially, as extra explanatory information, by taking into account the (extra) information given as in the earlier papers [19, 20], it is possible to get a great number of those special results. We also leave those creations knowledgeable with possible implications of this scientific note to the interests of researchers.

References

[1] Aouf MK, Mostafa AO, Bulboacă T. Notes on multivalent Bazilević functions defined by higher order derivatives. Turkish Journal of Mathematics 2021; 45 (2): 624-633. https://doi.org/10.3906/mat-1911-6

[2] Atshan WG, Wanas AK, Murugusundaramoorthy G. Properties and characteristics of certain subclass of multivalent prestarlike functions with negative coefficients. Analele Universitatii din Oradea. Fascicola matematica 2019; 26 (2): 17-24.

[3] Aydoğan SM, Hussain A, Sakar FM. On a nonlinear fractional-order model of Covid-19 under AB-Fractional derivative. Journal of Mathematical Extension 2021; 15 (11): 1-31. https://doi.org/10.30495/JME.SI.2021.1994

[4] Aydoğan SM, Sakar FM. Radius of starlkeness of p-vlently λ-farcational operator. Applied Mathematics and Computations 2019; 357: 374-378. https://doi.org/10.10,16/j.amc.2018.11.067

[5] Badar RS, Noor KI. Generalized q-Srivastava-Attiya operator on multivalent functions. Studia Universitatis Babes-Bolyai Mathematica (to appear).

[6] Chen MP, Irmak H, Srivastava, HM. Some multivalent functions with negative coefficients defined by using a differential operator. Panamerican Mathematical Journal 1996; 6 (2): 55-64.

[7] Chen MP, Irmak H, Srivastava HM. Some families of multivalently analytic functions with negative coefficients. The Journal of Mathematical Analysis and Applications 1997; 214 (2): 674-690.

[8] Choi J. A formula related to fractional calculus. Communications of the Korean Mathematical Society 1997; 12 (2): 457-466.

[9] Deniz E, Orhan E. Certain subclasses of multivalent functions defined by new multiplier transformations. The Arabian Journal for Science and Engineering 2011; 36 (6): 1091-1112.

[10] El-Deeb SM, El-Matary BM. Coefficients bounds of multivalent function connected with q-analogue of Salagean operator. Applied Mathematics and Information Sciences 2020; 14 (6): 1057-1065. https://doi.org/10.18576/amis/140613

[11] Esa Z, Srivastava HM, Kilicman A, Ibrahim RW. A novel subclass of analytic functions specified by a family of fractional derivatives in the complex domain. Filomat 2017; 31 (9): 2837-2849. https://doi.org/10.2298/FIL1709837E

[12] Fernandez A, Fahad HM. On the importance of conjugation relations in fractional calculus. Computational and Applied Mathematics 2022; 41 (6): Paper No. 246. https://doi.org/10.1007/s40314-022-01925-z

[13] Frasin BA, Farzana HA, Stephen BA. Subordination properties of certain subclasses of multivalent functions defined by Srivastava-Wright operator. The Theory and Applications of Mathematics & Computer Science 2016; 6 (2): 110-124.

[14] Goodman AW. Univalent Functions. Vols. I and II. Mariner Publishing Co., Inc., Tampa, FL, 1983.

[15] Hadi SH, Darus M, Park C, Lee JR. Some geometric properties of multivalent functions associated with a new generalized q-Mittag-Leffler function, AIMS Mathematics 2022; 7 (7): 11772-11783.

[16] Hadid SB, Ibrahim RW, Momani S. Multivalent functions and differential operator extended by the Quantum calculus. Fractal and Fractional 2022; 6 (7): 354. https://doi.org/10.3390/fractalfract6070354

[17] Hayman WK. Multivalent functions. Cambridge University Press, London, 2009.

[18] Ibrahim RW. Starlikeness of generalized Srivastava-Owa fractional operators. Proceedings of the Pakistan Academy of Sciences 2013; 50 (1): 85-88.

[19] Irmak H, Kulalı ÖF. An extensive study on various multivalently analytic functions with complex coefficients and some of their argument properties. Journal of the Orissa Mathematical Society 2021; 40 (1-2): 1-16.

[20] Irmak H. A variety of multivalently analytic functions with complex coefficients and some argument properties of their applications. Punjab University Journal of Mathematics 2021; 53 (6): 367-376. https://doi.org/1.52280/pujm.2021.530601

[21] Jenkins JA. Multivalent functions. The Bulletin of the American Mathematical Society 1959; 65 (3): 163-166.

[22] Juma ASR. Study of harmonic multivalent meromorphic functions by using generalized hypergeometric functions. Journal of Nonlinear Analysis and Optimization 2011; 2 (2): 367-372.

[23] Kargar R, Bilavi A, Abdolahi S, Maroufi S. A class of multivalent analytic functions defined by a new linear operator. Journal of Mathematics and Computer Science 2014; 8: 236-334.

[24] Lashin AMY, Aouf MK. Hadamard product of certain multivalent analytic functions with positive real parts. Mathematics 2022; 10: 1506. https://doi.org/10.3390/math10091506

[25] Macphail MS. Some iterated integrals in the fractional calculus. The American Mathematical Society 1938; 44 (10): 707-715.

[26] Nehari N. Conformal Mapping. McGraw-Hill, New York, 1952.

[27] Nishimoto K. Fractional calculus of products of elementary functions. International Journal College Engineering. Nihon University Series B 1987; 28: 21-31.

[28] Nunokawa M. On properties of non-Caratheodory functions. The Proceedings of the Japan Academy Series A 1992; 68 (6): 152-153.

[29] Nunokawa M, Sokól J. On a successive property of strongly starlikeness for multivalent functions. Rendiconti del Circolo Matematico di Palermo Series 2 2020; 69 (3): 939-945. https://doi.org/10.1007/s12215-019-00442-z

[30] Owa S. On the distortion theorems. I. Kyungpook Mathematical Journal 1978; 18: 53-59.

[31] Phung ND, Sakar FM, Etemad S, Rezapour S. A novel fractional structure of a multi-order quantum multi-integro-differentail problem. Advances in Difference Equation 2020; (1): 1-23. https://doi.org/10.1186/s13662-020-03092-z

[32] Robertson MS. On the theory of univalent function. Annals of Mathematics 1936; 37: 374-408.

[33] Ross B. Origins of fractional calculus and some applications. The International Journal of Mathematics and Mathematical Sciences 1992; 1 (1): 21-34.

[34] Saitoh H. On certain class of multivalent functions. Mathematica Japonica 1992; 37 (5): 871-875.

[35] Samko G, Kılbas AA, Marichev OI. Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon, 1993.

[36] Shah SGA, Hussain S, Noor S, Darus M, Ahmad I. Multivalent functions related with an integral operator. International Journal of Mathematics and Mathematical Sciences 2021; Art. ID 5882343, 13 pp. https://doi.org/10.1155/2021/5882343

[37] Srivastava HM, Owa S. Univalent Functions, Fractional Calculus and Their Applications. Halsted Press, John Wiley and Sons, New york, Chieschester, Brisbane, Toronto, 1989.

[38] Srivastava HM. Fractional-order derivatives and integrals: introductory overview and recent developments. Kyungpook Mathematical Journal 2020; 60 (1): 73-116. https://doi.org/10.5666/KMJ.2020.60.1.73

[39] Wietsma HL. Generalized Schur functions as multivalent functions. Complex Analysis and Operator Theory 2021; 15: 19 pp. https://doi.org/10.1007/s11785-020-01071-6