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Multiplication of closed balls in $\mathbb{C}^n$

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Abstract: Motivated by circular complex interval arithmetic, some operations on closed balls in $\mathbb{C}^n$ are considered. Essentially, the properties of possible multiplications for closed balls in $\mathbb{C}^n$, related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$, are studied. In addition, certain equations involving the defined multiplications are solved.

Key words: Closed ball, multiplication, 2-fold vector cross product, Hadamard product of vectors

1. Introduction

Circular complex interval arithmetic, as can be seen in the books [2], due to Alefeld and Herzberger, and [11], by Petković and Petković, deals with closed balls in $\mathbb{C}$. Over the years, research related to interval mathematics, namely [8] and [10], has been produced. In reference [8], Gargantini and Henrici apply circular complex interval arithmetic to the determination of polynomial zeros. Johansson, in [10], exhibits the advantages of ball arithmetic for rigorous algebraic computation with real numbers. More recently, in [6], Beites, Nicolás, and Vitória presented an arithmetic for closed balls in $\mathbb{R}^n$; the particular case $n = 2$ can be identified with $\mathbb{C}$.

In the present work, some operations on closed balls in $\mathbb{C}^n$ are considered. To start with, in section 2, we recall definitions and results related to the complex vector space $\mathbb{C}^n$ endowed with a 2-fold vector cross product when $n \in \{3, 7\}$, closed balls and the Hadamard product of vectors. Nontrivial 2-fold vector cross products exist for 3 and 7-dimensional vector spaces (see [3–5], and cited references), a consequence of the generalized Hurwitz Theorem: over a field of characteristic not 2, a finite-dimensional Hurwitz algebra is isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized octonion algebra.

In section 3, an addition for closed balls in $\mathbb{C}^n$ is examined. In section 4, properties of possible multiplications for these closed balls, related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$, are established. (Anti-)Commutativity, (power-)associativity, the existence of a neutral element and reciprocal of each element, and also its square root(s), are studied. Inclusion monotonicity

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– the basis for diverse applications of interval arithmetic, [2] – and the (sub)distributivity of each multiplication relative to the addition are analysed. Finally, certain equations involving the defined multiplications are solved.

2. Preliminaries

Throughout the work, consider the usual complex vector space $\mathbb{C}^n$. In addition, $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ complex matrices, and we identify $\mathbb{C}^{n\times 1}$ with $\mathbb{C}^n$.

Also throughout the work, we do not use different notations to distinguish scalars from vectors, but what the objects are is clear from the context in which the undifferentiated notation is used.

The complex vector space $\mathbb{C}^n$, together with the standard Hermitian inner product $(\cdot,\cdot)_h : (\mathbb{C}^n)^2 \to \mathbb{C}$, is a complex inner product space. Recall that, for all $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$, $y = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T \in \mathbb{C}^n$,

$$(x,y)_h = \sum_{t=1}^{n} x_t \overline{y_t}$$

and, for all $x, y, z \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{C}$,

$$(\alpha x + \beta y, z)_h = \alpha(x,z)_h + \beta(y,z)_h \text{ (linearity in the first coordinate)}, \tag{2.1}$$

$$(x,y)_h = \overline{(y,x)_h} \text{ (conjugate or Hermitian symmetry)}, \tag{2.2}$$

$$(x,x)_h \in \mathbb{R}_0^+ \text{ and } (x,x)_h = 0 \Leftrightarrow x = 0 \text{ (positive definiteness)}. \tag{2.3}$$

Also, (2.1) and (2.2) imply conjugate or Hermitian linearity in the second coordinate, that is,

$$(x, \alpha y + \beta z)_h = \overline{\alpha}(x,y)_h + \overline{\beta}(x,z)_h. \tag{2.4}$$

The complex vector space $\mathbb{C}^n$, together with the norm $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ induced by $(\cdot,\cdot)_h$, is also a normed linear space. Recall that, for all $x \in \mathbb{C}^n$,

$$\|x\| = \sqrt{(x,x)_h},$$

where $\sqrt{\cdot}$ stands for the real, positive or null root, and, for all $x, y \in \mathbb{C}^n$, $\alpha \in \mathbb{C}$,

$$\|x\| \in \mathbb{R}_0^+ \text{ and } \|x\| = 0 \Leftrightarrow x = 0, \tag{2.5}$$

$$\|\alpha x\| = |\alpha|\|x\|, \tag{2.6}$$

$$\|x+y\| \leq \|x\| + \|y\| \text{ (triangle inequality)}, \tag{2.7}$$

where $|\cdot|$ stands for the modulus of a complex number.

The closed ball A in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r \in \mathbb{R}_0^+$ is defined by

$$A = \langle a; r\rangle = \{x \in \mathbb{C}^n : \|x - a\| \leq r\}.$$

The set of closed balls in $\mathbb{C}^n$ is denoted by $\mathfrak{B}_{\mathbb{C}}$, and by $\mathfrak{B}_{\mathbb{C}}^+$ or $\mathfrak{B}_{\mathbb{C}}^0$ if, respectively, $r \in \mathbb{R}^+$ or $r = 0$.

Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle \in \mathfrak{B}_{\mathbb{C}}$. The closed balls A and B are equal $(A = B)$ if set-theoretic equality holds, that is, $a = b$ and $r_1 = r_2$. A is contained in B $(A \subseteq B)$ if set-theoretic inclusion is valid.

Let $*_{\mathfrak{B}_\mathbb{C}} : \mathfrak{B}_\mathbb{C} \times \mathfrak{B}_\mathbb{C} \to \mathfrak{B}_\mathbb{C}$ be a binary operation. The operation $*_{\mathfrak{B}_\mathbb{C}}$ is inclusion monotonic if, for all $A_m, B_m \in \mathfrak{B}_\mathbb{C}$ such that $A_m \subseteq B_m$, $m \in \{1,2\}$, $A_1 *_{\mathfrak{B}_\mathbb{C}} A_2 \subseteq B_1 *_{\mathfrak{B}_\mathbb{C}} B_2$. The operation $*_{\mathfrak{B}_\mathbb{C}}$ is power-associative if, for all $A \in \mathfrak{B}_\mathbb{C}$ and for all $m, s \in \mathbb{N}$, $A^s *_{\mathfrak{B}_\mathbb{C}} A^m = A^{s+m}$. The operation $*_{\mathfrak{B}_\mathbb{C}}$ is subdistributive with respect to another binary operation $\boxplus_{\mathfrak{B}_\mathbb{C}} : \mathfrak{B}_\mathbb{C} \times \mathfrak{B}_\mathbb{C} \to \mathfrak{B}_\mathbb{C}$ if, for all $A, B, C \in \mathfrak{B}_\mathbb{C}$, $A *_{\mathfrak{B}_\mathbb{C}} (B \boxplus_{\mathfrak{B}_\mathbb{C}} C) \subseteq (A *_{\mathfrak{B}_\mathbb{C}} B) \boxplus_{\mathfrak{B}_\mathbb{C}} (A *_{\mathfrak{B}_\mathbb{C}} C)$.

Let $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{C}^n$. The ∞-norm $\|\cdot\|_\infty$ of x is defined by $\|x\|_\infty = \max_{j\in\{1,\dots,n\}} |x_j| \in \mathbb{R}_0^+$, where $|\cdot|$ stands for the modulus of a complex number.

Let $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T, y = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T \in \mathbb{C}^n$. The Hadamard (componentwise) product $\circ$ of x and y is $x \circ y \in \mathbb{C}^n$ with $k, 1$ entry, $k \in \{1, \dots, n\}$, given by $x_k y_k$.

Endow the complex vector space $\mathbb{C}^n$ with the nondegenerate symmetric bilinear form $(\cdot,\cdot)$ defined by

$$(x, y) = (x, \overline{y})_h.$$

Now consider $n \in \{3, 7\}$ and equip $\mathbb{C}^n$ also with the 2-fold vector cross product $\times : (\mathbb{C}^n)^2 \to \mathbb{C}^n$. Recall that $\times$ is the bilinear map that, for any $x, y \in \mathbb{C}^n$,

$$(x \times y, x) = (x \times y, y) = 0, \tag{2.8}$$

$$(x \times y, x \times y) = \begin{vmatrix} (x,x) & (x,y) \\ (y,x) & (y,y) \end{vmatrix}. \tag{2.9}$$

The trilinear map $(\cdot \times \cdot, \cdot)$ is skew-symmetric due to (2.8), and so $\times$ is anticommutative.

The 2-fold vector cross product in $\mathbb{C}^n$, $n \in \{3, 7\}$, can be approached from a matrix point of view, [7, 9]. Let $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$. Consider the linear mapping

$$\begin{array}{rccl} a_\times : & \mathbb{C}^n & \to & \mathbb{C}^n \\ & x & \mapsto & a_\times(x) = a \times x. \end{array}$$

For each $a \in \mathbb{C}^n$, there exists a unique matrix $S_a \in \mathbb{C}^{n\times n}$ such that

$$a \times x = S_a x, \tag{2.10}$$

where, for $n = 3$,

$$S_a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2.11}$$

and, for $n = 7$,

$$S_a = \begin{bmatrix} 0 & -a_3 & a_2 & -a_5 & a_4 & -a_7 & a_6 \\ a_3 & 0 & -a_1 & -a_6 & a_7 & a_4 & -a_5 \\ -a_2 & a_1 & 0 & a_7 & a_6 & -a_5 & -a_4 \\ a_5 & a_6 & -a_7 & 0 & -a_1 & -a_2 & a_3 \\ -a_4 & -a_7 & -a_6 & a_1 & 0 & a_3 & a_2 \\ a_7 & -a_4 & a_5 & a_2 & -a_3 & 0 & -a_1 \\ -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & 0 \end{bmatrix}. \tag{2.12}$$

For $n = 7$, these skew-symmetric matrices were studied by Beites, Nicolás and Vitória in [7]. An earlier study for $n = 3$, due to Gross, Trenkler and Troschke, can be found in [9].

3. Addition

Throughout this section, consider the usual complex vector space $\mathbb{C}^n$. Consider also the binary operation $+_{\mathfrak{B}_\mathbb{C}} : \mathfrak{B}_\mathbb{C} \times \mathfrak{B}_\mathbb{C} \to \mathfrak{B}_\mathbb{C}$, hereinafter called addition $+_{\mathfrak{B}_\mathbb{C}}$, defined by

$$A +_{\mathfrak{B}_\mathbb{C}} B = \langle a; r_1 \rangle +_{\mathfrak{B}_\mathbb{C}} \langle b; r_2 \rangle := \langle a + b; r_1 + r_2 \rangle .$$

The subsequent results establish several properties related to $+_{\mathfrak{B}_\mathbb{C}}$.

Theorem 3.1 *The addition $+_{\mathfrak{B}_\mathbb{C}}$ is commutative and associative. Moreover, $\langle 0; 0 \rangle$ is the neutral element relative to $+_{\mathfrak{B}_\mathbb{C}}$.*

Proof Owing to the commutativity and to the associativity of the addition in $\mathbb{C}^n$, as well as to the commutativity and to the associativity of the addition in $\mathbb{R}$, it is straightforward to prove that, for all $A, B, C \in \mathfrak{B}_\mathbb{C}$, $A +_{\mathfrak{B}_\mathbb{C}} B = B +_{\mathfrak{B}_\mathbb{C}} A$ and $(A +_{\mathfrak{B}_\mathbb{C}} B) +_{\mathfrak{B}_\mathbb{C}} C = A +_{\mathfrak{B}_\mathbb{C}} (B +_{\mathfrak{B}_\mathbb{C}} C)$. Taking into account the neutral elements of $\mathbb{C}^n$ and $\mathbb{R}$ relative to the respective additions, it is also direct to prove that $\langle 0; 0 \rangle$ is the neutral element relative to $+_{\mathfrak{B}_\mathbb{C}}$. □

Corollary 3.2 *$(\mathfrak{B}_\mathbb{C}, +_{\mathfrak{B}_\mathbb{C}})$ is a commutative monoid.*

Proof A straightforward consequence of Theorem 3.1. □

Corollary 3.3 *The set of elements of $\mathfrak{B}_\mathbb{C}$ which possess reciprocal relative to the addition $+_{\mathfrak{B}_\mathbb{C}}$ is $\mathfrak{B}_\mathbb{C}^0$. Furthermore, the reciprocal of $\langle a; 0 \rangle \in \mathfrak{B}_\mathbb{C}^0$ relative to $+_{\mathfrak{B}_\mathbb{C}}$ is $\langle -a; 0 \rangle$.*

Proof Let $E = \langle 0; 0 \rangle$. Let $A = \langle a; r_1 \rangle \in \mathfrak{B}_\mathbb{C}$. Suppose that $A' = \langle a'; r_1' \rangle \in \mathfrak{B}_\mathbb{C}$ is the reciprocal of A relative to $+_{\mathfrak{B}_\mathbb{C}}$. From $A +_{\mathfrak{B}_\mathbb{C}} A' = E$, we have $a' = -a$ and $r_1' = -r_1$. □

Lemma 3.4 *Let $A, B \in \mathfrak{B}_\mathbb{C}$. Then $A +_{\mathfrak{B}_\mathbb{C}} B = \{x + y : x \in A \wedge y \in B\}$.*

Proof Let $A = \langle a; r_1 \rangle$, $B = \langle b; r_2 \rangle \in \mathfrak{B}_\mathbb{C}$.

($\subseteq$) Let $u \in A +_{\mathfrak{B}_\mathbb{C}} B = \langle a + b; r_1 + r_2 \rangle$. Then $\|u - (a + b)\| \leq r_1 + r_2$. If $r_1 + r_2 = 0$ then the inclusion holds since $u = a + b$. If $r_1 + r_2 \neq 0$ then the inclusion also holds since $u = v + (u - v)$ with $v = \alpha u + (1 - \alpha)(a + b) - b \in A$, $\alpha = \frac{r_1}{r_1 + r_2}$, and $u - v \in B$. In fact, $\|v - a\| = \alpha \|u - (a + b)\| \leq r_1$ and $\|u - v - b\| = (1 - \alpha)\|u - (a + b)\| \leq r_2$.

($\supseteq$) Let $x \in A$ and $y \in B$. Then $\|x - a\| \leq r_1$, $\|y - b\| \leq r_2$ and $\|x + y - (a + b)\| \leq \|x - a\| + \|y - b\| \leq r_1 + r_2$. Therefore, $x + y \in A +_{\mathfrak{B}_\mathbb{C}} B = \langle a + b; r_1 + r_2 \rangle$. □

Theorem 3.5 *The addition $+_{\mathfrak{B}_\mathbb{C}}$ is inclusion monotonic.*

Proof Let $A_m, B_m \in \mathfrak{B}_\mathbb{C}$ such that $A_m \subseteq B_m$, $m \in \{1, 2\}$. By Lemma 3.4, $A_1 +_{\mathfrak{B}_\mathbb{C}} A_2 = \{x + y : x \in A_1 \wedge y \in A_2\} \subseteq \{x + y : x \in B_1 \wedge y \in B_2\} = B_1 +_{\mathfrak{B}_\mathbb{C}} B_2$. □

4. Multiplications

Throughout this section, unless stated otherwise, consider the usual complex vector space $\mathbb{C}^n$. We start with an auxiliary result for the following subsections, each devoted to a possible multiplication for closed balls in $\mathbb{C}^n$.

Lemma 4.1 *Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle \in \mathfrak{B}_{\mathbb{C}}$. Then $A \subseteq B$ if and only if $\|a-b\| \leq r_2 - r_1$. In particular, if A and B are concentric then $A \subseteq B$ if and only if $r_1 \leq r_2$.*

Proof ($\Rightarrow$) Suppose that $A \subseteq B$. Assume that $\|a-b\| > r_2 - r_1$. Consider the line passing through a and b. This line intersects the border of A at a point x such that $\|x-b\| = \|a-b\| + \|x-a\| > r_2 - r_1 + r_1 = r_2$, which leads to the contradiction $x \notin B$.

($\Leftarrow$) Let $x \in A$. Then $\|x-a\| \leq r_1$. Hence, $x \in B$ since $\|x-b\| \leq \|x-a\| + \|a-b\| \leq r_2$.

The particular result for concentric balls is immediate. □

4.1. Multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$

Consider the binary operation $\circ_{\mathfrak{B}_{\mathbb{C}},r} : \mathfrak{B}_{\mathbb{C}} \times \mathfrak{B}_{\mathbb{C}} \to \mathfrak{B}_{\mathbb{C}}$, hereinafter called multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$, defined by

$$A \circ_{\mathfrak{B}_{\mathbb{C}},r} B = \langle a; r_1\rangle \circ_{\mathfrak{B}_{\mathbb{C}},r} \langle b; r_2\rangle := \langle a \circ b + r_2 a + r_1 b; r_1 r_2\rangle.$$

Even though $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ is not inclusion monotonic, the following properties hold for $\circ_{\mathfrak{B}_{\mathbb{C}},r}$.

Theorem 4.2 *The multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ is commutative and associative. Moreover, $\langle 0; 1\rangle$ is the neutral element relative to $\circ_{\mathfrak{B}_{\mathbb{C}},r}$.*

Proof As the Hadamard product $\circ$ of vectors is commutative and associative on $\mathbb{C}^n$, so is the multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$. It is straightforward that, for all $\langle a; r_1\rangle \in \mathfrak{B}_{\mathbb{C}}$, $\langle a; r_1\rangle = \langle a; r_1\rangle \circ_{\mathfrak{B}_{\mathbb{C}},r} \langle 0; 1\rangle$. □

Corollary 4.3 *$(\mathfrak{B}_{\mathbb{C}}, \circ_{\mathfrak{B}_{\mathbb{C}},r})$ is a commutative monoid.*

Proof A straightforward consequence of Theorem 4.2. □

Theorem 4.4 *The set of elements of $\mathfrak{B}_{\mathbb{C}}$ which possess reciprocal relative to the multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ is $\mathfrak{R} = \{A = \langle a; r_1\rangle \in \mathfrak{B}_{\mathbb{C}}^+ : a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n \wedge a_l \neq -r_1, l \in \{1, \dots, n\}\}$. Furthermore, the reciprocal of $\langle a; r_1\rangle \in \mathfrak{R}$ relative to $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ is $\langle b; \frac{1}{r_1}\rangle$ with $b = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}^T \in \mathbb{C}^n$ such that $b_l = -\frac{a_l}{r_1(r_1+a_l)}$, $l \in \{1, \dots, n\}$.*

Proof Let A=$\langle a; r_1\rangle \in \mathfrak{B}_{\mathbb{C}}^+$. Let $b = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}^T \in \mathbb{C}^n$ such that $\langle a; r_1\rangle \circ_{\mathfrak{B}_{\mathbb{C}},r} \langle b; 1/r_1\rangle = \langle 0; 1\rangle$. As $a \circ b + \frac{1}{r_1} a + r_1 b = 0$, we get $a_l b_l + \frac{1}{r_1} a_l + r_1 b_l = 0, l \in \{1, \dots, n\}$. □

Let $A \in \mathfrak{B}_{\mathbb{C}}$. We define the powers of A relative to $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ by

$$A^0 = \langle 0; 1\rangle \text{ and } A^k = A^{k-1} \circ_{\mathfrak{B}_{\mathbb{C}},r} A \text{ for } k \in \mathbb{N}.$$

Denote $\begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T$ by $a^{\circ 0}$ and $a^{\circ(k-1)} \circ a$ by $a^{\circ k}$ for $k \in \mathbb{N}$.

Theorem 4.5 *The multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ is power-associative.*

Proof Due to Theorem 4.2, for all $A \in \mathfrak{B}_{\mathbb{C}}$, $A^2 \circ_{\mathfrak{B}_{\mathbb{C}},r} A = A \circ_{\mathfrak{B}_{\mathbb{C}},r} A^2$ and $(A^2 \circ_{\mathfrak{B}_{\mathbb{C}},r} A) \circ_{\mathfrak{B}_{\mathbb{C}},r} A = A^2 \circ_{\mathfrak{B}_{\mathbb{C}},r} A^2$ are valid. The result follows since, invoking [1], this suffices to prove that, for all $A \in \mathfrak{B}_{\mathbb{C}}$ and for all $m, s \in \mathbb{N}$, $A^s \circ_{\mathfrak{B}_{\mathbb{C}},r} A^m = A^{s+m}$. □

Theorem 4.6 *Let $A = \langle a; r_1 \rangle \in \mathfrak{B}_{\mathbb{C}}$. Relative to the multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$, for all $k \in \mathbb{N}$,* $A^k = \langle \sum_{j=1}^{k} \binom{k}{j} r_1^{k-j} a^{\circ j}; r_1^k \rangle$.

Proof We use induction on k. The equality obviously holds for $k = 1$. Suppose that it is true for k. Then we have

$$\begin{aligned} A^{k+1} &= A^k \circ_{\mathfrak{B}_{\mathbb{C}},r} A \\ &= \langle \textstyle\sum_{l=1}^{k} \binom{k}{l} r_1^{k-l} a^{\circ l}; r_1^k \rangle \circ_{\mathfrak{B}_{\mathbb{C}},r} \langle a; r_1 \rangle \\ &= \langle \textstyle\sum_{l=1}^{k} \binom{k}{l} r_1^{k-l} a^{\circ (l+1)} + \sum_{l=1}^{k} \binom{k}{l} r_1^{k+1-l} a^{\circ l} + r_1^k a; r_1^{k+1} \rangle \\ &= \langle a^{\circ (k+1)} + \textstyle\sum_{l=2}^{k} \left[\binom{k}{l-1} + \binom{k}{l} \right] r_1^{k+1-l} a^{\circ l} + (k+1) r_1^k a; r_1^{k+1} \rangle \\ &= \langle \textstyle\sum_{l=1}^{k+1} \binom{k+1}{l} r_1^{k+1-l} a^{\circ l}; r_1^{k+1} \rangle. \end{aligned}$$

□

Theorem 4.7 *Let $A = \langle a; r_1 \rangle \in \mathfrak{B}_{\mathbb{C}}$ with $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$. The square roots of A relative to the multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ are given by $A^{1/2} = \langle b; \sqrt{r_1} \rangle$, with $b = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}^T \in \mathbb{C}^n$ such that $b_l = -\sqrt{r_1} \pm \sqrt{r_1 + a_l}$ for $l \in \{1, \dots, n\}$, where $\sqrt{\cdot}$ stands, accordingly, for the real, positive or null root and for the complex roots.*

Proof Let $B = \langle b; s \rangle \in \mathfrak{B}_{\mathbb{C}}$ such that $A = B^2$. As $\langle a; r_1 \rangle = \langle b^{\circ 2} + 2sb; s^2 \rangle$, we have $s^2 = r_1$ and $b_l^2 + 2sb_l - a_l = 0$ for $l \in \{1, \dots, n\}$. Thus, $b_l = -s \pm \sqrt{s^2 + a_l}$. □

Theorem 4.8 *The multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},r}$ is distributive with respect to the addition $+_{\mathfrak{B}_{\mathbb{C}}}$.*

Proof Owing to the distributivity of $\circ$ with respect to the addition in $\mathbb{C}^n$, to the distributivity of the multiplication with respect to the addition in $\mathbb{R}$, as well as to mixed distributivities, it is straightforward to prove that, for all $A, B, C \in \mathfrak{B}_{\mathbb{C}}$, $A \circ_{\mathfrak{B}_{\mathbb{C}},r} (B +_{\mathfrak{B}_{\mathbb{C}}} C) = (A \circ_{\mathfrak{B}_{\mathbb{C}},r} B) +_{\mathfrak{B}_{\mathbb{C}}} (A \circ_{\mathfrak{B}_{\mathbb{C}},r} C)$. □

Corollary 4.9 $(\mathfrak{B}_{\mathbb{C}}, +_{\mathfrak{B}_{\mathbb{C}}}, \circ_{\mathfrak{B}_{\mathbb{C}},r})$ *is a semiring.*

Proof A straightforward consequence of Theorem 3.1, Theorem 4.2 and Theorem 4.8. □

Theorem 4.10 *Let $A = \langle a; r_1 \rangle \in \mathfrak{B}_{\mathbb{C}}^{+}$ such that $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$ with $a_k \neq -r_1$, $k \in \{1, \dots, n\}$. Let $B = \langle b; r_2 \rangle \in \mathfrak{B}_{\mathbb{C}}$. Then the unique solution of the equation $A \circ_{\mathfrak{B}_{\mathbb{C}},r} X = B$ is given by $X = \langle x; r_3 \rangle \in \mathfrak{B}_{\mathbb{C}}$, where $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{C}^n$, with*

$$x_k = (a_k + r_1)^{-1}(b_k - r_3 a_k), k \in \{1, \dots, n\}, \text{ and } r_3 = r_1^{-1} r_2.$$

Proof From the definition of $\circ_{\mathfrak{B}_{\mathbb{C}},r}$, the equation $A \circ_{\mathfrak{B}_{\mathbb{C}},r} X = B$ assumes the form $\langle a \circ x + r_3 a + r_1 x; r_1 r_3 \rangle = \langle b; r_2 \rangle$, which leads to $(a_k + r_1)x_k = b_k - r_3 a_k$, $k \in \{1, \ldots, n\}$, and $r_1 r_3 = r_2$. □

Theorem 4.11 *Let $B = \langle b; r_2 \rangle$, $C = \langle c; r_1 \rangle \in \mathfrak{B}_{\mathbb{C}}$. Then, the solutions of the equation $X^2 = B \circ_{\mathfrak{B}_{\mathbb{C}},r} X +_{\mathfrak{B}_{\mathbb{C}}} C$ are given by $X = \langle x; r_3 \rangle \in \mathfrak{B}_{\mathbb{C}}$, where $x = [x_1 \ldots x_n]^T \in \mathbb{C}^n$. If $r_1 > 0$, then*

$$x_k = 2^{-1}\left(b_k - \sqrt{r_2^2 + 4r_1} \pm \sqrt{(b_k + r_2)^2 + 4(r_1 + c_k)}\right), \ k \in \{1, \ldots, n\}, \ \text{and } r_3 = 2^{-1}\left(r_2 + \sqrt{r_2^2 + 4r_1}\right).$$

If $r_1 = 0$, then

$$x_k = 2^{-1}\left(b_k + r_2 \pm \sqrt{(b_k + r_2)^2 + 4c_k}\right), \ k \in \{1, \ldots, n\}, \ \text{and } r_3 = 0,$$

or

$$x_k = 2^{-1}\left(b_k - r_2 \pm \sqrt{(b_k + r_2)^2 + 4c_k}\right), \ k \in \{1, \ldots, n\}, \ \text{and } r_3 = r_2.$$

Proof From the definition of $\circ_{\mathfrak{B}_{\mathbb{C}},r}$, the equation $X^2 = B \circ_{\mathfrak{B}_{\mathbb{C}},r} X +_{\mathfrak{B}_{\mathbb{C}}} C$ takes the form $\langle x \circ x + 2r_3 x; r_3^2 \rangle = \langle b \circ x + r_3 b + r_2 x + c; r_3 r_2 + r_1 \rangle$. So, $r_3^2 - r_2 r_3 - r_1 = 0$ and $r_3 = 2^{-1}\left(r_2 \pm \sqrt{r_2^2 + 4r_1}\right)$. Also, since $r_2 - \sqrt{r_2^2 + 4r_1} \in \mathbb{R}_0^+$ if and only if $r_1 = 0$, we have $r_3 = 0$ or $r_3 = r_2$ if $r_1 = 0$. On the other hand, for each $k \in \{1, \ldots, n\}$, we have $x_k^2 + (2r_3 - b_k - r_2)x_k - (r_3 b_k + c_k) = 0$, which leads to $x_k = 2^{-1}\left(b_k - \sqrt{r_2^2 + 4r_1} \pm \sqrt{(b_k + r_2)^2 + 4(r_1 + c_k)}\right)$ when $r_1 > 0$. If $r_1 = r_3 = 0$ then we obtain $x_k^2 - (b_k + r_2)x_k - c_k = 0$ and $x_k = 2^{-1}\left(b_k + r_2 \pm \sqrt{(b_k + r_2)^2 + 4c_k}\right)$. If $r_1 = 0$ and $r_3 = r_2$ then we get $x_k^2 + (r_2 - b_k)x_k - (r_2 b_k + c_k) = 0$ and $x_k = 2^{-1}\left(b_k - r_2 \pm \sqrt{(b_k + r_2)^2 + 4c_k}\right)$. □

Corollary 4.12 *Let $E = \langle 0; 1 \rangle$. Then the solutions of the equation $X^2 = X +_{\mathfrak{B}_{\mathbb{C}}} E$ are given by the golden balls $X = \langle x; \frac{1+\sqrt{5}}{2} \rangle$, with $x = \left[\begin{array}{ccc} x_1 & \ldots & x_n \end{array}\right]^T \in \mathbb{R}^n$ such that $x_k \in \{-\sqrt{5}, 0\}$ for $k \in \{1, \ldots, n\}$ and where $\sqrt{\cdot}$ stands for the real, positive root.*

Proof By Theorem 4.2, the equation $X^2 = X +_{\mathfrak{B}_{\mathbb{C}}} E$ can be rewritten as $X^2 = E \circ_{\mathfrak{B}_{\mathbb{C}},r} X +_{\mathfrak{B}_{\mathbb{C}}} E$. Then, the result follows from Theorem 4.11. □

4.2. Multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$

Consider the binary operation $\circ_{\mathfrak{B}_{\mathbb{C}},c} : \mathfrak{B}_{\mathbb{C}} \times \mathfrak{B}_{\mathbb{C}} \to \mathfrak{B}_{\mathbb{C}}$, hereinafter called multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$, defined by

$$A \circ_{\mathfrak{B}_{\mathbb{C}},c} B = \langle a; r_1 \rangle \circ_{\mathfrak{B}_{\mathbb{C}},c} \langle b; r_2 \rangle := \langle a \circ b; r_1 \|b\|_\infty + r_2 \|a\|_\infty + r_1 r_2 \rangle.$$

Although $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is not associative, as presented below, $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ possesses diverse properties.

Theorem 4.13 *The multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is commutative. Moreover, $\langle 1; 0 \rangle$ is the neutral element relative to $\circ_{\mathfrak{B}_{\mathbb{C}},c}$.*

Proof As the Hadamard product $\circ$ of vectors is commutative on $\mathbb{C}^n$, it is clear that $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is commutative. Denote $\begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T \in \mathbb{C}^n$ by 1. Let $A = \langle a; r\rangle \in \mathfrak{B}_{\mathbb{C}}$. Then we get $A \circ_{\mathfrak{B}_{\mathbb{C}},c} \langle 1; 0\rangle = \langle a \circ 1; r\rangle = A$. □

Theorem 4.14 *The set of elements of $\mathfrak{B}_{\mathbb{C}}$ which possess reciprocal relative to the multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is $\mathfrak{R} = \{A = \langle a; 0\rangle \in \mathfrak{B}^0_{\mathbb{C}} : a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n \wedge a_l \neq 0, l \in \{1, \dots, n\}\}$. Furthermore, the reciprocal of $\langle a; 0\rangle \in \mathfrak{R}$ relative to $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is $\langle b; 0\rangle$ with $b_l = a_l^{-1}$, $l \in \{1, \dots, n\}$.*

Proof Let $A = \langle a; r\rangle \in \mathfrak{B}_{\mathbb{C}}$. Suppose that $B = \langle b; s\rangle$ is the reciprocal of A relative to $\circ_{\mathfrak{B}_{\mathbb{C}},c}$. Then we have $A \circ_{\mathfrak{B}_{\mathbb{C}},c} B = \langle a; r\rangle \circ_{\mathfrak{B}_{\mathbb{C}},c} \langle b; s\rangle = \langle a \circ b; r\|b\|_\infty + s\|a\|_\infty + rs\rangle = \langle 1; 0\rangle$. Hence, $b_l = a_l^{-1}$, $l \in \{1, \dots, n\}$, whenever $a_l \neq 0$. In addition, $r\|b\|_\infty + s\|a\|_\infty + rs = 0$, which allows to arrive at $r = s = 0$. □

Let $A \in \mathfrak{B}_{\mathbb{C}}$. We define the powers of A relative to $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ by

$$A^0 = \langle 1; 0\rangle \text{ and } A^k = A^{k-1} \circ_{\mathfrak{B}_{\mathbb{C}},c} A \text{ for } k \in \mathbb{N}.$$

Denote $\begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T$ by $a^{\circ 0}$ and $a^{\circ(k-1)} \circ a$ by $a^{\circ k}$ for $k \in \mathbb{N}$.

Theorem 4.15 *The multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is power-associative.*

Proof To prove that, for all $A \in \mathfrak{B}_{\mathbb{C}}$ and for all $m, s \in \mathbb{N}$, $A^s \circ_{\mathfrak{B}_{\mathbb{C}},c} A^m = A^{s+m}$, invoking [1], it suffices to show that $A^2 \circ_{\mathfrak{B}_{\mathbb{C}},c} A = A \circ_{\mathfrak{B}_{\mathbb{C}},c} A^2$ and $(A^2 \circ_{\mathfrak{B}_{\mathbb{C}},c} A) \circ_{\mathfrak{B}_{\mathbb{C}},c} A = A^2 \circ_{\mathfrak{B}_{\mathbb{C}},c} A^2$. By Theorem 4.13, the former equality holds. As for the latter equality, let $A = \langle a; r\rangle \in \mathfrak{B}_{\mathbb{C}}$. On the one hand,

$$\begin{aligned} A^2 \circ_{\mathfrak{B}_{\mathbb{C}},c} A &= \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \circ_{\mathfrak{B}_{\mathbb{C}},c} \langle a; r\rangle \\ &= \langle a^{\circ 3}; r\|a^{\circ 2}\|_\infty + \|a\|_\infty(2r\|a\|_\infty + r^2) + r(2r\|a\|_\infty + r^2)\rangle \\ &= \langle a^{\circ 3}; 3r\|a\|^2_\infty + 3r^2\|a\|_\infty + r^3\rangle \end{aligned}$$

and

$$\begin{aligned} (A^2 \circ_{\mathfrak{B}_{\mathbb{C}},c} A) \circ_{\mathfrak{B}_{\mathbb{C}},c} A &= \langle a^{\circ 3}; 3r\|a\|^2_\infty + 3r^2\|a\|_\infty + r^3\rangle \circ_{\mathfrak{B}_{\mathbb{C}},c} \langle a; r\rangle \\ &= \langle a^{\circ 4}; 4r\|a\|^3_\infty + 6r^2\|a\|^2_\infty + 4r^3\|a\|_\infty + r^4\rangle. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} A^2 \circ_{\mathfrak{B}_{\mathbb{C}},c} A^2 &= \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \circ_{\mathfrak{B}_{\mathbb{C}},c} \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \\ &= \langle a^{\circ 4}; 4r\|a\|^3_\infty + 6r^2\|a\|^2_\infty + 4r^3\|a\|_\infty + r^4\rangle. \end{aligned}$$

□

Theorem 4.16 *Let $A = \langle a; r\rangle \in \mathfrak{B}_{\mathbb{C}}$. Relative to the multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$, for all $k \in \mathbb{N}$, $A^k = \langle a^{\circ k}; (\|a\|_\infty + r)^k - \|a\|^k_\infty\rangle$.*

Proof We proceed by induction on k. The equality clearly holds for $k=1$. Suppose that it is also valid for k. Then we have

$$\begin{aligned} A^{k+1} &= A^k \circ_{\mathfrak{B}_{\mathbb{C}},c} A \\ &= \langle a^{\circ k}; \textstyle\sum_{l=1}^{k} \binom{k}{l} r^l \|a\|_\infty^{k-l}\rangle \circ_{\mathfrak{B}_{\mathbb{C}},c} \langle a; r\rangle \\ &= \langle a^{\circ(k+1)}; r\|a^{\circ k}\|_\infty + \textstyle\sum_{l=1}^{k} \binom{k}{l} r^l \|a\|_\infty^{k+1-l} + \sum_{l=1}^{k} \binom{k}{l} r^{l+1} \|a\|_\infty^{k-l}\rangle \\ &= \langle a^{\circ(k+1)}; (k+1) r\|a\|_\infty^k + \textstyle\sum_{l=2}^{k} \left[\binom{k}{l} + \binom{k}{l-1}\right] r^l \|a\|_\infty^{k+1-l} + r^{k+1}\rangle \\ &= \langle a^{\circ(k+1)}; \textstyle\sum_{l=1}^{k+1} \binom{k+1}{l} r^l \|a\|_\infty^{k+1-l}\rangle \\ &= \langle a^{\circ(k+1)}; (\|a\|_\infty + r)^{k+1} - \|a\|_\infty^{k+1}\rangle. \end{aligned}$$

□

Theorem 4.17 *Let $A = \langle a; r\rangle \in \mathfrak{B}_{\mathbb{C}}$ with $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$. The square roots of A relative to the multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ are given by $A^{1/2} = \langle a^{\circ 1/2}; \sqrt{r + \|a\|_\infty} - \sqrt{\|a\|_\infty}\rangle$, with $a^{\circ 1/2} = (\sqrt{a_1}, \dots, \sqrt{a_n})$, where $\sqrt{\cdot}$ stands, accordingly, for the real, positive, or null root and for the complex roots.*

Proof Let $B = \langle b; s\rangle \in \mathfrak{B}_{\mathbb{C}}$ such that $A = B^2$. From $\langle a; r\rangle = \langle b^{\circ 2}; s^2 + 2s\|b\|_\infty\rangle$ we have $b = a^{\circ 1/2}$ and $s^2 + 2\sqrt{\|a\|_\infty} s - r = 0$. □

Theorem 4.18 *The multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is inclusion monotonic.*

Proof Let $A_m = \langle a_m; r_m\rangle$, $B_m = \langle b_m; s_m\rangle \in \mathfrak{B}_{\mathbb{C}}$ such that $A_m \subseteq B_m$, $m \in \{1, 2\}$. We aim to prove that $A_1 \circ_{\mathfrak{B}_{\mathbb{C}},c} A_2 \subseteq B_1 \circ_{\mathfrak{B}_{\mathbb{C}},c} B_2$. By Lemma 4.1, $\|b_m - a_m\| \le s_m - r_m$, $m \in \{1, 2\}$, and it is enough to prove that

$$\|b_1 \circ b_2 - a_1 \circ a_2\| \le s_1\|b_2\|_\infty + s_2\|b_1\|_\infty + s_1 s_2 - r_1\|a_2\|_\infty - r_2\|a_1\|_\infty - r_1 r_2.$$

Observe that

$$\begin{aligned} \|b_1 \circ b_2 - a_1 \circ a_2\| &= \|b_1 \circ b_2 - b_1 \circ a_2 + b_1 \circ a_2 - a_1 \circ a_2\| \\ &\le \|b_1 \circ (b_2 - a_2)\| + \|(b_1 - a_1) \circ a_2\| \\ &\le \|b_1\|_\infty \|b_2 - a_2\| + \|a_2\|_\infty \|b_1 - a_1\| \\ &\le \|b_1\|_\infty (s_2 - r_2) + \|a_2\|_\infty (s_1 - r_1). \end{aligned}$$

In addition, we have

$$\begin{aligned} s_1\|a_2\|_\infty &\le s_1\|b_2\|_\infty + s_1\|a_2 - b_2\|_\infty \le s_1\|b_2\|_\infty + s_1(s_2 - r_2), \\ -r_2\|b_1\|_\infty &\le -r_2\|a_1\|_\infty + r_2\|a_1 - b_1\|_\infty \le -r_2\|a_1\|_\infty + r_2(s_1 - r_1). \end{aligned}$$

The former and the latter inequalities lead to the result. □

Theorem 4.19 *The multiplication $\circ_{\mathfrak{B}_{\mathbb{C}},c}$ is subdistributive with respect to the addition $+_{\mathfrak{B}_{\mathbb{C}}}$.*

Proof Let $A = \langle a; r_1\rangle$, $B = \langle b; r_2\rangle, C = \langle c; r_3\rangle \in \mathfrak{B}_{\mathbb{C}}$. Applying Lemma 4.1,

$$\begin{aligned} A \circ_{\mathfrak{B}_{\mathbb{C}},c} (B +_{\mathfrak{B}_{\mathbb{C}}} C) &= \langle a; r_1\rangle \circ_{\mathfrak{B}_{\mathbb{C}},c} (\langle b; r_2\rangle +_{\mathfrak{B}_{\mathbb{C}}} \langle c; r_3\rangle) \\ &= \langle a \circ (b + c); r_1\|b + c\|_\infty + (r_2 + r_3)\|a\|_\infty + r_1(r_2 + r_3)\rangle \\ &= \langle a \circ b + a \circ c; r_2\|a\|_\infty + r_1 r_2 + r_3\|a\|_\infty + r_1 r_3 + r_1\|b + c\|_\infty\rangle \\ &\subseteq \langle a \circ b + a \circ c; r_1\|b\|_\infty + r_2\|a\|_\infty + r_1 r_2 + r_1\|c\|_\infty + r_3\|a\|_\infty + r_1 r_3\rangle \\ &= (A \circ_{\mathfrak{B}_{\mathbb{C}},c} B) +_{\mathfrak{B}_{\mathbb{C}}} (A \circ_{\mathfrak{B}_{\mathbb{C}},c} C). \end{aligned}$$

□

Theorem 4.20 *Let* $A = \langle a; r_1 \rangle \in \mathfrak{B}_{\mathbb{C}}$ *such that* $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$ *with* $a_k \neq 0$, $k \in \{1, \dots, n\}$. *Let* $B = \langle b; r_2 \rangle \in \mathfrak{B}_{\mathbb{C}}$. *Then the unique solution of the equation* $A \circ_{\mathfrak{B}_{\mathbb{C}},c} X = B$ *is given by* $X = \langle x; r_3 \rangle \in \mathfrak{B}_{\mathbb{C}}$, *where* $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{C}^n$, *with*

$$x_k = a_k^{-1} b_k, k \in \{1, \dots, n\}, \text{ and } r_3 = (\|a\|_\infty + r_1)^{-1}(r_2 - r_1\|x\|_\infty).$$

Proof The rewriting of the stated equation $A \circ_{\mathfrak{B}_{\mathbb{C}},c} X = B$ leads to $\langle b; r_2 \rangle = \langle a \circ x; r_1\|x\|_\infty + r_3\|a\|_\infty + r_1 r_3 \rangle$ From here, we have $a_k x_k = b_k$, $k \in \{1, \dots, n\}$, and $(\|a\|_\infty + r_1) r_3 = r_2 - r_1\|x\|_\infty$. □

Theorem 4.21 *Let* $B = \langle b; r_2 \rangle$, $C = \langle c; r_1 \rangle \in \mathfrak{B}_{\mathbb{C}}$. *Then, the solutions of the equation* $X^2 = B \circ_{\mathfrak{B}_{\mathbb{C}},c} X +_{\mathfrak{B}_{\mathbb{C}}} C$ *are given by* $X = \langle x; r_3 \rangle \in \mathfrak{B}_{\mathbb{C}}$, *where* $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{C}^n$, *with*

$$x_k = 2^{-1}\left(b_k \pm \sqrt{b_k^2 + 4c_k}\right), \ k \in \{1, \dots, n\},$$

$$r_3 = 2^{-1}\left(r_2 + \|b\|_\infty - 2\|x\|_\infty + \sqrt{(r_2 + \|b\|_\infty - 2\|x\|_\infty)^2 + 4(r_1 + r_2\|x\|_\infty)}\right),$$

where $\sqrt{\cdot}$ *stands for the real, positive root, and*

$$r_3 = 0 \textit{ if } r_1 = 0 \textit{ and } r_2\|x\|_\infty = 0.$$

Proof From the definition of $\circ_{\mathfrak{B}_{\mathbb{C}},c}$, the equation $X^2 = B \circ_{\mathfrak{B}_{\mathbb{C}},c} X +_{\mathfrak{B}_{\mathbb{C}}} C$ takes the form $\langle x \circ x; 2r_3\|x\|_\infty + r_3^2 \rangle = \langle b \circ x + c; r_2\|x\|_\infty + r_3\|b\|_\infty + r_2 r_3 + r_1 \rangle$. So, $r_3^2 + (2\|x\|_\infty - \|b\|_\infty - r_2) r_3 - \|x\|_\infty r_2 - r_1 = 0$ and, thus, we arrive at $r_3 = 2^{-1}\left(\|b\|_\infty + r_2 - 2\|x\|_\infty + \sqrt{(\|b\|_\infty + r_2 - 2\|x\|_\infty)^2 + 4(r_1 + r_2\|x\|_\infty)}\right)$. Notice that $\|b\|_\infty + r_2 - 2\|x\|_\infty - \sqrt{(\|b\|_\infty + r_2 - 2\|x\|_\infty)^2 + 4(r_1 + r_2\|x\|_\infty)} \in \mathbb{R}_0^+$ if and only if $r_1 + r_2\|x\|_\infty = 0$, in which case $r_3 = 0$. On the other hand, for each $k \in \{1, \dots, n\}$ we must have that $x_k^2 = b_k x_k + c_k$, which leads to $x_k = 2^{-1}(b_k \pm \sqrt{b_k^2 + 4c_k})$. □

Corollary 4.22 *Let* $E = \langle 1; 0 \rangle$. *Then the solutions of the equation* $X^2 = X +_{\mathfrak{B}_{\mathbb{C}}} E$ *are given by the balls* $X = \langle x; 0 \rangle$, *with* $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{R}^n$ *such that* $x_k = 2^{-1}(1 \pm \sqrt{5})$ *for* $k \in \{1, \dots, n\}$, *where* $\sqrt{\cdot}$ *stands for the real, positive root.*

Proof By Theorem 4.13, the equation $X^2 = X +_{\mathfrak{B}_{\mathbb{C}}} E$ can be rewritten as $X^2 = E \circ_{\mathfrak{B}_{\mathbb{C}},c} X +_{\mathfrak{B}_{\mathbb{C}}} E$. Then, the result follows from Theorem 4.21. □

4.3. Multiplication $\times_{\mathfrak{B}_{\mathbb{C}},r}$

Throughout this subsection, consider the usual complex vector space $\mathbb{C}^n$ with $n \in \{3, 7\}$. Consider also the binary operation $\times_{\mathfrak{B}_{\mathbb{C}},r} : \mathfrak{B}_{\mathbb{C}} \times \mathfrak{B}_{\mathbb{C}} \to \mathfrak{B}_{\mathbb{C}}$, hereinafter called multiplication $\times_{\mathfrak{B}_{\mathbb{C}},r}$, defined by

$$A \times_{\mathfrak{B}_{\mathbb{C}},r} B = \langle a; r_1 \rangle \times_{\mathfrak{B}_{\mathbb{C}},r} \langle b; r_2 \rangle := \langle a \times b + r_2 a + r_1 b; r_1 r_2 \rangle.$$

Even though commutativity, anticommutativity, associativity, and inclusion monotonicity do not hold, $\times_{\mathfrak{B}_{\mathbb{C}},r}$ satisfies the subsequent properties.

Theorem 4.23 *The neutral element relative to the multiplication* $\times_{\mathfrak{B}_{\mathbb{C}},r}$ *is* $\langle 0;1\rangle$.

Proof Let $A=\langle a;r_1\rangle\in\mathfrak{B}_{\mathbb{C}}$. Then we have $\langle a;r_1\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle 0;1\rangle=\langle a;r_1\rangle=\langle 0;1\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle a;r_1\rangle$. □

Corollary 4.24 *The set of elements of* $\mathfrak{B}_{\mathbb{C}}$ *which possess reciprocal relative to the multiplication* $\times_{\mathfrak{B}_{\mathbb{C}},r}$ *is* $\mathfrak{B}_{\mathbb{C}}^{+}$. *Furthermore, the reciprocal of* $\langle a;r_1\rangle\in\mathfrak{B}_{\mathbb{C}}^{+}$ *relative to* $\times_{\mathfrak{B}_{\mathbb{C}},r}$ *is* $\langle -\frac{1}{r_1^2}a;\frac{1}{r_1}\rangle$.

Proof Let $A=\langle a;r_1\rangle\in\mathfrak{B}_{\mathbb{C}}^{+}$. Then we obtain $\langle a;r_1\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\left\langle -\frac{1}{r_1^2}a;\frac{1}{r_1}\right\rangle=\langle 0;1\rangle=\left\langle -\frac{1}{r_1^2}a;\frac{1}{r_1}\right\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle a;r_1\rangle$. □

Let $A\in\mathfrak{B}_{\mathbb{C}}$. We define the powers of A relative to $\times_{\mathfrak{B}_{\mathbb{C}},r}$ by

$$A^0=\langle 0;1\rangle \text{ and } A^k=A^{k-1}\times_{\mathfrak{B}_{\mathbb{C}},r}A \text{ for } k\in\mathbb{N}.$$

Theorem 4.25 *The multiplication* $\times_{\mathfrak{B}_{\mathbb{C}},r}$ *is power-associative.*

Proof To prove that, for all $A\in\mathfrak{B}_{\mathbb{C}}$ and for all $m,s\in\mathbb{N}$, $A^s\times_{\mathfrak{B}_{\mathbb{C}},r}A^m=A^{s+m}$, invoking [1], it suffices to show that $A^2\times_{\mathfrak{B}_{\mathbb{C}},r}A=A\times_{\mathfrak{B}_{\mathbb{C}},r}A^2$ and $(A^2\times_{\mathfrak{B}_{\mathbb{C}},r}A)\times_{\mathfrak{B}_{\mathbb{C}},r}A=A^2\times_{\mathfrak{B}_{\mathbb{C}},r}A^2$. Let $A=\langle a;r_1\rangle\in\mathfrak{B}_{\mathbb{C}}$. Then we get

$$\begin{aligned}A^2\times_{\mathfrak{B}_{\mathbb{C}},r}A&=\langle 2r_1a;r_1^2\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle a;r_1\rangle\\&=\langle 3r_1^2a;r_1^3\rangle\\&=\langle a;r_1\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle 2r_1a;r_1^2\rangle\\&=A\times_{\mathfrak{B}_{\mathbb{C}},r}A^2,\end{aligned}$$

$$\begin{aligned}(A^2\times_{\mathfrak{B}_{\mathbb{C}},r}A)\times_{\mathfrak{B}_{\mathbb{C}},r}A&=\langle 3r_1^2a;r_1^3\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle a;r_1\rangle\\&=\langle 4r_1^3a;r_1^4\rangle\\&=\langle 2r_1a;r_1^2\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle 2r_1a;r_1^2\rangle\\&=A^2\times_{\mathfrak{B}_{\mathbb{C}},r}A^2.\end{aligned}$$

□

Theorem 4.26 *Let* $A=\langle a;r_1\rangle\in\mathfrak{B}_{\mathbb{C}}$. *Relative to the multiplication* $\times_{\mathfrak{B}_{\mathbb{C}},r}$, *for all* $k\in\mathbb{N}$, $A^k=\langle kr_1^{k-1}a;r_1^k\rangle$.

Proof Let $A=\langle a;r_1\rangle\in\mathfrak{B}_{\mathbb{C}}$. The base case obviously holds. As $A^k=A^{k-1}\times_{\mathfrak{B}_{\mathbb{C}},r}A=\langle (k-1)r_1^{k-2}a;r_1^{k-1}\rangle\times_{\mathfrak{B}_{\mathbb{C}},r}\langle a;r_1\rangle=\langle kr_1^{k-1}a;r_1^k\rangle$, the induction step follows. □

Theorem 4.27 *Let* $A=\langle a;r\rangle\in\mathfrak{B}_{\mathbb{C}}^{+}$. *The square root of* A *relative to the multiplication* $\times_{\mathfrak{B}_{\mathbb{C}},r}$ *is given by* $A^{1/2}=\left\langle \frac{1}{2\sqrt{r}}a;\sqrt{r}\right\rangle$, *where* $\sqrt{\cdot}$ *stands for the real, positive root.*

Proof Let $A = \langle a; r\rangle \in \mathfrak{B}_{\mathbb{C}}^{+}$. Let $B = \langle b; s\rangle \in \mathfrak{B}_{\mathbb{C}}$ such that $B^2 = A$. Thus, $s^2 = r$ and $S_b b + 2sb = a$, which leads to the result by [7, Proposition 4, Property 6] and [9, Property (A)]. □

Theorem 4.28 *The multiplication* $\times_{\mathfrak{B}_{\mathbb{C}},r}$ *is distributive relative to the addition* $+_{\mathfrak{B}_{\mathbb{C}}}$.

Proof Owing to the distributivity of $\times$ with respect to the addition in $\mathbb{C}^n$, to the distributivity of the multiplication with respect to the addition in $\mathbb{R}$, as well as to mixed distributivities, it is straightforward to prove that, for all $A, B, C \in \mathfrak{B}_{\mathbb{C}}$, $A \times_{\mathfrak{B}_{\mathbb{C}},r} (B +_{\mathfrak{B}_{\mathbb{C}}} C) = A \times_{\mathfrak{B}_{\mathbb{C}},r} B +_{\mathfrak{B}_{\mathbb{C}}} A \times_{\mathfrak{B}_{\mathbb{C}},r} C$ and $(B +_{\mathfrak{B}_{\mathbb{C}}} C) \times_{\mathfrak{B}_{\mathbb{C}},r} A = B \times_{\mathfrak{B}_{\mathbb{C}},r} A +_{\mathfrak{B}_{\mathbb{C}}} C \times_{\mathfrak{B}_{\mathbb{C}},r} A$. □

Lemma 4.29 *Let* $a \in \mathbb{C}^n$ *and* $\alpha \in \mathbb{C}$. *The matrix* $S_a + \alpha I_n$ *is invertible if and only if* $\alpha \neq 0$ *and* α *is not a square root of* $-(a, \overline{a})_h$.

Proof From [7, Lemma 9], the result is valid for $n = 7$. For $n = 3$, a straightforward calculation of $\det(S_a + \alpha I_3)$ leads to $\alpha(\alpha^2 + (a, \overline{a})_h)$. In the stated conditions, $\det(S_a + \alpha I_3) = 0$ if and only if $\alpha = 0$ or $\alpha^2 = -(a, \overline{a})_h$. □

Theorem 4.30 *Let* $a \in \mathbb{C}^n$. *Let* $\alpha \in \mathbb{C}\backslash\{0\}$ *such that* α *is not a square root of* $-(a, \overline{a})_h$. *Then* $(S_a + \alpha I_n)^{-1} = -(\alpha^2 + (a, \overline{a})_h)^{-1}(S_a - \alpha I_n - \alpha^{-1} a a^T)$.

Proof By [7, Theorem 10], the result holds for $n = 7$. Now consider $n = 3$. From Lemma 4.29, $S_a + \alpha I_3$ is invertible. Invoking [9, Property (A) and Property (3.1)], we get

$$\begin{aligned}
&(S_a + \alpha I_3)(-(\alpha^2 + (a, \overline{a})_h)^{-1}(S_a - \alpha I_3 - \alpha^{-1} a a^T)) \\
&= -(\alpha^2 + (a, \overline{a})_h)^{-1}(S_a^2 - \alpha S_a - \alpha^{-1} S_a a a^T + \alpha S_a - \alpha^2 I_3 - a a^T) \\
&= -(\alpha^2 + (a, \overline{a})_h)^{-1}(-(a, \overline{a})_h I_3 - \alpha^2 I_3) \\
&= I_3.
\end{aligned}$$

□

Theorem 4.31 *Let* $A = \langle a; r_1\rangle \in \mathfrak{B}_{\mathbb{C}}^{+}$ *such that* r_1 *is not a square root of* $-(a, \overline{a})_h$. *Let* $B = \langle b; r_2\rangle \in \mathfrak{B}_{\mathbb{C}}$. *Then the unique solution of the equation* $A \times_{\mathfrak{B}_{\mathbb{C}},r} X = B$ *is given by* $X = \langle x; r_3\rangle \in \mathfrak{B}_{\mathbb{C}}$, *with*

$$x = -(r_1^2 + (a, \overline{a})_h)^{-1}(S_a - r_1 I_n - r_1^{-1} a a^t)(b - r_3 a) \text{ and } r_3 = r_1^{-1} r_2.$$

Proof By (2.10), the equation $A \times_{\mathfrak{B}_{\mathbb{C}},r} X = B$ assumes the form $\langle b; r_2\rangle = \langle S_a x + r_3 a + r_1 x; r_1 r_3\rangle$, where S_a is given by (2.11)-(2.12). From here, we arrive at $(S_a + r_1 I_n)x = b - r_3 a$ and $r_1 r_3 = r_2$. As $r_1 \in \mathbb{R}\backslash\{0\}$, since $r_1 \in \mathbb{R}^+$, and r_1 is not a square root of $-(a, \overline{a})_h$, by Theorem 4.30, $S_a + r_1 I_n$ is invertible and $(S_a + r_1 I_n)^{-1} = -(r_1^2 + (a, \overline{a})_h)^{-1}(S_a - r_1 I_n - r_1^{-1} a a^t)$. □

Theorem 4.32 *Let* $B = \langle b; r_2\rangle, C = \langle c; r_1\rangle \in \mathfrak{B}_{\mathbb{C}}$. *Then,*

- *if* $r_1 = 0$, $r_2 \in \mathbb{R}^+$ *and* r_2 *is not a square root of* $-(b,\bar{b})_h$, *then the unique solution of the equation* $X^2 = B \times_{\mathfrak{B}_\mathbb{C},r} X +_{\mathfrak{B}_\mathbb{C}} C$ *is given by* $X = \langle x; r_3\rangle \in \mathfrak{B}_\mathbb{C}$, *with*

$$x = (r_2^2 + (b,\bar{b})_h)^{-1}(S_b - r_2 I_n - r_2^{-1} bb^t)c \text{ and } r_3 = 0;$$

- *if* $r_2^2 + 4r_1 \in \mathbb{R}^+$ *and* $\sqrt{r_2^2+4r_1}$ *is not a square root of* $-(b,\bar{b})_h$, *where* $\sqrt{\cdot}$ *stands for the real, positive root, then the unique solution of the equation* $X^2 = B \times_{\mathfrak{B}_\mathbb{C},r} X +_{\mathfrak{B}_\mathbb{C}} C$ *is given by* $X = \langle x; r_3\rangle \in \mathfrak{B}_\mathbb{C}$, *with*

$$x = ((r_2 - 2r_3)^2 + (b,\bar{b})_h)^{-1}(S_b - (r_2-2r_3)I_n - (r_2-2r_3)^{-1} bb^t)(c + r_3 b),$$

$$r_3 = 2^{-1}\left(r_2 + \sqrt{r_2^2+4r_1}\right).$$

Proof From (2.10), the equation $X^2 = B \times_{\mathfrak{B}_\mathbb{C},r} X +_{\mathfrak{B}_\mathbb{C}} C$ may be written as $\langle S_x x + 2r_3 x; r_3^2\rangle = \langle S_b x + r_3 b + r_2 x + c; r_2 r_3 + r_1\rangle$, where S_a is given by (2.11)-(2.12). On the one hand, we have $r_3^2 = r_2 r_3 + r_1$, which leads to $r_3 = 2^{-1}(r_2 \pm \sqrt{r_2^2+4r_1})$, and $r_2 - \sqrt{r_2^2+4r_1} \in \mathbb{R}_0^+$ if and only if $r_1 = 0$. On the other hand, taking into account [7, Proposition 4, Property 6] and [9, Property (A)], we have $(S_b + (r_2-2r_3)I_n)x = -r_3 b - c$. As $r_2 - 2r_3 \in \mathbb{R}\backslash\{0\}$ and $r_2 - 2r_3$ is not a square root of $-(b,\bar{b})_h$ under the stated assumptions, by Theorem 4.30, $S_b + (r_2-2r_3)I_n$ is invertible and $(S_b + (r_2-2r_3)I_n)^{-1} = -((r_2-2r_3)^2 + (b,\bar{b})_h)^{-1}(S_b - (r_2-2r_3)I_n - (r_2-2r_3)^{-1}bb^t)$. □

Corollary 4.33 *Let* $E = \langle 0; 1\rangle$. *Then the unique solution of the equation* $X^2 = X +_{\mathfrak{B}_\mathbb{C}} E$ *is given by the golden ball* $X = \langle 0; \frac{1+\sqrt{5}}{2}\rangle$, *where* $\sqrt{\cdot}$ *stands for the real, positive root.*

Proof By Theorem 4.23, the equation $X^2 = X +_{\mathfrak{B}_\mathbb{C}} E$ can be rewritten as $X^2 = E \times_{\mathfrak{B}_\mathbb{C},r} X +_{\mathfrak{B}_\mathbb{C}} E$. The result then follows from Theorem 4.32. □

4.4. Multiplication $\times_{\mathfrak{B}_\mathbb{C},c}$

Throughout this subsection, consider the usual complex vector space $\mathbb{C}^n$ with $n \in \{3,7\}$. Consider also the binary operation $\times_{\mathfrak{B}_\mathbb{C},c} : \mathfrak{B}_\mathbb{C} \times \mathfrak{B}_\mathbb{C} \to \mathfrak{B}_\mathbb{C}$, hereinafter called multiplication $\times_{\mathfrak{B}_\mathbb{C},c}$, defined by

$$A \times_{\mathfrak{B}_\mathbb{C},c} B = \langle a; r_1\rangle \times_{\mathfrak{B}_\mathbb{C},c} \langle b; r_2\rangle := \langle a \times b; r_2\|a\| + r_1\|b\| + r_1 r_2\rangle.$$

Even though commutativity, anticommutativity, associativity, the existence of a neutral element, and power-associativity do not hold, $\times_{\mathfrak{B}_\mathbb{C},c}$ satisfies the subsequent properties.

Theorem 4.34 *Let* $A = \langle 0; r\rangle \in \mathfrak{B}_\mathbb{C}$. *The square roots of* A *relative to the multiplication* $\times_{\mathfrak{B}_\mathbb{C},c}$ *are given by* $A^{1/2} = \langle b; -\|b\| + \sqrt{\|b\|^2 + r}\rangle$, *with* $b \in \mathbb{C}^n$, *where* $\sqrt{\cdot}$ *stands for the real, positive or null root.*

Proof Let $A = \langle 0; r\rangle \in \mathfrak{B}_\mathbb{C}$. Let $B = \langle b; s\rangle \in \mathfrak{B}_\mathbb{C}$ such that $B^2 = A$. From $s^2 + 2\|b\|s - r = 0$, we have $s = -\|b\| + \sqrt{\|b\|^2 + r} \in \mathbb{R}_0^+$. □

Theorem 4.35 *The multiplication* $\times_{\mathfrak{B}_\mathbb{C},c}$ *is inclusion monotonic.*

Proof Let $A_m = \langle a_m; r_m\rangle$, $B_m = \langle b_m; s_m\rangle \in \mathfrak{B}_\mathbb{C}$ such that $A_m \subseteq B_m$, $m \in \{1,2\}$. We aim to prove that $A_1 \times_{\mathfrak{B}_\mathbb{C},c} A_2 \subseteq B_1 \times_{\mathfrak{B}_\mathbb{C},c} B_2$. From Lemma 4.1, $\|a_m - b_m\| \leq s_m - r_m$, $m \in \{1,2\}$. We also have

$$A_1 \times_{\mathfrak{B}_\mathbb{C},c} A_2 = \langle a_1 \times a_2; r_2\|a_1\| + r_1\|a_2\| + r_1 r_2\rangle$$

and

$$B_1 \times_{\mathfrak{B}_\mathbb{C},c} B_2 = \langle b_1 \times b_2; s_2\|b_1\| + s_1\|b_2\| + s_1 s_2\rangle.$$

As

$$\begin{aligned} &\|a_1 \times a_2 - b_1 \times b_2\| \\ &= \| - b_2 \times (a_1 - b_1) + b_1 \times (a_2 - b_2) + (a_1 - b_1) \times (a_2 - b_2)\| \\ &\leq \|b_2\|\|a_1 - b_1\| + \|b_1\|\|a_2 - b_2\| + \|a_1 - b_1\|\|a_2 - b_2\| \\ &\leq \|b_2\|(s_1 - r_1) + \|b_1\|(s_2 - r_2) + (s_1 - r_1)(s_2 - r_2) \end{aligned}$$

and

$$-\|b_m\| \leq -\|a_m\| + \|a_m - b_m\| \leq -\|a_m\| + s_m - r_m, m \in \{1,2\},$$

we obtain $\|a_1 \times a_2 - b_1 \times b_2\| \leq \beta - \alpha$, where $\beta = s_2\|b_1\| + s_1\|b_2\| + s_1 s_2$ and $\alpha = r_2\|a_1\| + r_1\|a_2\| + r_1 r_2$. Once again by Lemma 4.1, the result follows. □

Theorem 4.36 *The multiplication $\times_{\mathfrak{B}_\mathbb{C},c}$ is subdistributive with respect to the addition $+_{\mathfrak{B}_\mathbb{C}}$.*

Proof Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle, C = \langle c; r_3\rangle \in \mathfrak{B}_\mathbb{C}$. Lemma 4.1 leads to

$$\begin{aligned} A \times_{\mathfrak{B}_\mathbb{C},c} (B +_{\mathfrak{B}_\mathbb{C}} C) &= \langle a; r_1\rangle \times_{\mathfrak{B}_\mathbb{C},c} \langle b + c; r_2 + r_3\rangle \\ &= \langle a \times (b + c); (r_2 + r_3)\|a\| + r_1\|b + c\| + r_1(r_2 + r_3)\rangle \\ &\subseteq \langle a \times b + a \times c; r_2\|a\| + r_1\|b\| + r_1 r_2 + r_3\|a\| + r_1\|c\| + r_1 r_3\rangle \\ &= A \times_{\mathfrak{B}_\mathbb{C},c} B +_{\mathfrak{B}_\mathbb{C}} A \times_{\mathfrak{B}_\mathbb{C},c} C \end{aligned}$$

Thus, left subdistributivity holds. An analogous reasoning leads to the right subdistributivity. □

Theorem 4.37 *Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle \in \mathfrak{B}_\mathbb{C}$ such that $\|a\|$ and r_1 are not simultaneously null, $(a, \overline{a})_h \neq 0$ and $(a, \overline{b})_h = 0$. Then the solutions of the equation $A \times_{\mathfrak{B}_\mathbb{C},c} X = B$ are given by $X = \langle x; r_3\rangle \in \mathfrak{B}_\mathbb{C}$, with*

$$x = -(a, \overline{a})_h^{-1} S_a b + \lambda a, \lambda \in \mathbb{C}, \text{ and } r_3 = (\|a\| + r_1)^{-1}(r_2 - r_1\|x\|).$$

Proof By (2.10), the equation $A \times_{\mathfrak{B}_\mathbb{C},c} X = B$ assumes the form $\langle b; r_2\rangle = \langle S_a x; r_3\|a\| + r_1\|x\| + r_1 r_3\rangle$, where S_a is given by (2.11)-(2.12). Hence, we have $S_a x = b$ and $(\|a\| + r_1)r_3 = r_2 - r_1\|x\|$. The solutions, as $(a, \overline{a})_h \neq 0$ and $(a, \overline{b})_h = 0$, are a consequence of [7, Theorem 14] and [9, Theorem 2]. □

Theorem 4.38 *Let* $B = \langle b; r_2\rangle, C = \langle c; r_1\rangle \in \mathfrak{B}_{\mathbb{C}}$ *such that* $(b, \bar{b})_h \neq 0$ *and* $(b, \bar{c})_h = 0$. *Then the solutions of the equation* $X^2 = B \times_{\mathfrak{B}_{\mathbb{C}},c} X +_{\mathfrak{B}_{\mathbb{C}}} C$ *are given by* $X = \langle x; r_3\rangle \in \mathfrak{B}_{\mathbb{C}}$, *with*

$$x = (b, \bar{b})_h^{-1} S_b c + \lambda b, \lambda \in \mathbb{C},$$

$$r_3 = 2^{-1}\left(r_2 + \|b\| - 2\|x\| + \sqrt{(r_2 + \|b\| - 2\|x\|)^2 + 4(r_1 + r_2\|x\|)}\right),$$

where $\sqrt{\cdot}$ *stands for the real, positive or null root, and*

$$r_3 = 0 \text{ if } r_1 = 0 \text{ and } r_2\|x\| = 0.$$

Proof From (2.10), the equation $X^2 = B \times_{\mathfrak{B}_{\mathbb{C}},c} X +_{\mathfrak{B}_{\mathbb{C}}} C$ may be written in the form $\langle S_x x; 2r_3\|x\| + r_3^2\rangle = \langle S_b x + c; r_2\|x\| + r_3\|b\| + r_2 r_3 + r_1\rangle$, where S_b is given by (2.11)-(2.12). Observe that, applying [7, Proposition 4, Property 6] and [9, Property (A)], we have $S_b x = -c$, whose solutions follow from [7, Theorem 14] and [9, Theorem 2]. In addition, $r_3^2 - (r_2 + \|b\| - 2\|x\|)r_3 - r_1 - r_2\|x\| = 0$, that is, $r_3 = 2^{-1}\left(r_2 + \|b\| - 2\|x\| \pm \sqrt{(r_2 + \|b\| - 2\|x\|)^2 + 4r_1 + 4r_2\|x\|}\right)$.

Observe that $2^{-1}\left(r_2 + \|b\| - 2\|x\| - \sqrt{(r_2 + \|b\| - 2\|x\|)^2 + 4r_1 + 4r_2\|x\|}\right) \in \mathbb{R}_0^+$ if and only if $4r_1 + 4r_2\|x\| = 0$. □

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