Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1932 – 1943
© TÜBİTAK
doi:10.55730/1300-0098.3473

Interpolation polynomials associated to linear recurrences

Muhammad Syifa'ul MUFID[1,*], **László SZALAY**[2,3]

[1]Department of Mathematics, Faculty of Science and Analytical Data, Institut Teknologi Sepuluh Nopember, Surabaya, Indonesia

[2]Institute of Informatics and Mathematics, University of Sopron, Sopron, Hungary

[3]Department of Mathematics, J. Selye University, Bratislavská, Slovakia

Received: 28.05.2023 • **Accepted/Published Online:** 06.09.2023 • **Final Version:** 09.11.2023

Abstract: Assume that $(G_n)_{n\in\mathbb{Z}}$ is an arbitrary real linear recurrence of order k. In this paper, we examine the classical question of polynomial interpolation, where the basic points are given by (t, G_t) $(n_0 \le t \le n_1)$. The main result is an explicit formula depends on the explicit formula of G_n and on the finite difference sequence of a specific sequence. It makes it possible to study the interpolation polynomials essentially by the zeros of the characteristic polynomial of (G_n). During the investigations, we developed certain formulae related to the finite differences.

Key words: Linear recurrence, interpolation polynomial, finite difference

1. Introduction

Let k be a positive integer, and assume that the real linear recurrent sequence $(G_n)_{n\in\mathbb{Z}}$ of order k is defined by the recurrence rule

$$G_n = a_1G_{n-1} + \cdots + a_kG_{n-k}, \qquad (n \ge k), \tag{1.1}$$

and by the real initial values $G_0, G_1, \ldots, G_{k-1}$, where all the coefficients are real numbers. For avoiding trivial cases we assume that at least one of the initial values is non-zero, and $a_k \neq 0$. Clearly, (1.1) together with the initial values provides (G_n) for non-negative subscripts. For negative subscripts we consider (1.1) backward:

$$G_{n-k} = \frac{1}{a_k}\left(G_n - a_1G_{n-1} - \cdots - a_{k-1}G_{n-k+1}\right) \qquad (n < k).$$

Probably the most known recursive sequence is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. It is easy to see that its extension for negative subscripts leads to $F_{-n} = (-1)^{n+1}F_n$ $(n \ge 0)$.

This paper studies the properties of the interpolation polynomials $P_m(x)$ induced by the points

$$(t, G_t) \quad \text{for} \quad t = n_0, n_1 = n_0 + 1, \ldots, n_m = n_0 + m \tag{1.2}$$

given in the planar Cartesian system for $n_0 \in \mathbb{Z}$ and $m \in \mathbb{N}^+$.

*Correspondence: syifaul.mufid@matematika.its.ac.id

2020 *AMS Mathematics Subject Classification:* 11B37, 03D20

There exist different ways to find $P_m(x)$, but here, in this paper, we apply Newton's divided differences method. For a wide generalization of Newton's interpolation algorithm see [3], where the author gives a general form of the Newton-like interpolation formula, and a general recurrence relation for divided differences.

Broadly, besides different approaches, Newton's method provides the interpolation polynomial $Q_m(x)$ for the points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_m, y_m)$$

having distinct abscissas. For simplicity, in accordance with our problem, we suppose that $h = x_{t+1} - x_t$ holds for $t = 0, 1, \ldots, m-1$, that is we deal with the so-called equidistant case. Put $X = (x - x_0)/h$. Then Newton's forward divided difference formula gives

$$Q_m(x) = \sum_{t=0}^{m} \binom{X}{t} h^t t! [y_0, y_1, \ldots, y_t], \tag{1.3}$$

where

$$\binom{X}{0} = 1 \text{ and } \binom{X}{t} = \frac{X(X-1)\cdots(X-t+1)}{t!}$$

for $t \geq 1$. Furthermore, the notation $[y_0, y_1, \ldots, y_t]$ stands for the divided differences. This is defined recursively by $[y_i] = y_i$, and by

$$[y_i, \ldots, y_{i+t}] = \frac{[y_{i+1}, \ldots, y_{i+t}] - [y_i, \ldots, y_{i+t-1}]}{x_{i+t} - x_i},$$

where the denominator is obviously $x_{i+t} - x_i = th$.

The problem we investigate works with $h = 1$, so we have $X = x - x_0$. Consequently, from now on

$$Q_m(x) = \sum_{t=0}^{m} \binom{x - x_0}{t} t! [y_0, y_1, \ldots, y_t]. \tag{1.4}$$

Clearly, the crucial point is to determine the divided differences $[y_0, y_1, \ldots, y_t]$ if $x_i = n_i$ and $y_i = G_{n_i}$ $(i = 0, 1, \ldots, t)$. If once we have $[y_0, y_1, \ldots, y_t] = [G_{n_0}, G_{n_1}, \ldots, G_{n_t}]$, then

$$P_m(x) = \sum_{t=0}^{m} \binom{x - n_0}{t} t! [G_{n_0}, G_{n_1}, \ldots, G_{n_t}] = \sum_{t=0}^{m} \binom{x - n_0}{t} \Delta^t G_{n_0}, \tag{1.5}$$

where $\Delta^t G_{n_0} = t! [G_{n_0}, G_{n_1}, \ldots, G_{n_t}]$ is the corresponding forward finite difference that will be introduced later in Subsection 2.1. This formula is obviously equivalent to the recursive version

$$P_m(x) = P_{m-1}(x) + \binom{x - n_0}{m} \Delta^m G_{n_0}. \tag{1.6}$$

In this work, we analyze the forward finite differences $\Delta^t G_{n_0}$ in detail and reveal several properties. Especially, we can have an explicit formula for $\Delta^t G_{n_0}$, which leads to nice results if the recurrence (G_n) is relatively simple. Although the main result is theoretical, one may apply our approach to given families of linear recurrences to find general features (see, for instance, Example 3).

The paper is organized as follows. In Section Preliminaries, we recall the notion of finite differences and the principal theorem of homogeneous linear recurrences. Theorem 2.1 admits a useful connection between a result of Flajolet and Sedgewick [2] and weighted sums in Pascal's triangle [1] when the weight sequences is recursive. The next section prepares the proof of the main theorems (Theorem 4.1-4.3) of this study. Here Theorem 3.1 yields a new expansion for finite differences of the product of two arbitrary complex sequences. Interestingly, the so-called trinomial coefficients appear in the description. Later we will apply the specific case of the corollary of this statement when one of the sequences is a linear polynomial. Theorem 3.6 examines the finite differences of the sequence $h_n^{(r)} = n^r\omega^n$, later this result will be combined with the fundamental theorem of linear recurrences. Finally, in the last section, Theorem 4.1 provides an explicit formula for $P_m(x)$ (via $P_m(x) - P_{m-1}(x)$) if neither zero of the characteristic polynomial of the linear recurrence is 1. Theorem 4.3 deals with this remaining exceptional case. Then two examples illustrate the results, and finally, Example 3 handles the general case of ternary recurrences with simple zeros of the characteristic polynomial.

2. Preliminaries

2.1. Finite differences

Assume that $(f_n)_{n\in\mathbb{Z}} \in \mathbb{C}^\infty$ is an arbitrary complex sequence. As usual, the forward differences of (f_n) are defined by $\Delta f_n = f_{n+1} - f_n$. The iterated differences are given by $\Delta^K f_n = \Delta(\Delta^{K-1} f_n) = \Delta^{K-1} f_{n+1} - \Delta^{K-1} f_n$ for $K \geq 1$. It is known (see, for example, Section 1 of [2]) that

$$\Delta^K f_n = \sum_{i=0}^{K} \binom{K}{i} (-1)^{K-i} f_{n+i}, \tag{2.1}$$

which is closely related to the so-called Euler transform of (f_n).

If (f_n) is a recursive sequence, then Theorem 3.1 of [1] makes it possible to find the sequence $(\Delta^K f_n)_{K=0}^{\infty}$ appears in (2.1) as a recursive sequence, say $(D_K)_{K=0}^{\infty}$, determined by its recurrence relation (and explicitly under favourable circumstances). Note that the order of (f_n) and (D_n) coincide. Here we record the consequence of Theorem 3.1 [1] for the specific case (2.1). Recall that we deal with the recurrence (G_n) of order k given in (1.1), i.e. $f_n = G_n$ is valid now.

Theorem 2.1 *If $K \geq k$, then we have (with $D_K = \Delta^K f_n$)*

$$D_K = -D_{K-1} - \sum_{j=1}^{k-1} \binom{k-1}{j} (D_{K-j} + D_{K-j-1}) + \sum_{i=1}^{k} a_i \sum_{j=0}^{k-i} \binom{k-i}{j} D_{k-i-j}.$$

Proof Apply Theorem 3.1 [1] with the local parameters $r = 1$, $q = p = 0$, $x = -1$, $y = 1$, and with $\omega = (K-0)/(0+1) = K$. □

Example 2.2 *Let f_n denote the sequence of Tribonacci numbers (i.e. $f_0 = f_1 = 0$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2} + f_{n-3}$; A000073 in [6]). Then Theorem 2.1 provides immediately the formula*

$$D_K = -2D_{K-1} + 2D_{K-3}$$

for $D_K = \Delta^K f_n$ *with initial values* $D_0 = \Delta^0 f_n = f_n$, $D_1 = \Delta^1 f_n = f_{n+1} - f_n$, *and* $D_2 = \Delta^2 f_n = f_{n+2} - 2f_{n+1} + f_n$. *To illustrate the rule above we obtain immediately*

$$\begin{aligned}\Delta^3 f_n = D_3 &= -2(f_{n+2} - 2f_{n+1} + f_n) + 2f_n + (\underbrace{f_{n+3} - f_{n+2} - f_{n+1} - f_n}_{0}) \\ &= f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n.\end{aligned}$$

It is straightforward to see that

$$\Delta^K f_n = K![f_n, f_{n+1}, \dots, f_{n+K}]$$

(we already foreshadowed this fact in (1.5)). Thus we can obtain $P_m(x)$ in (1.6) with respect to the sequence $(f_n) = (G_n)$ via $(\Delta^K G_n)$. But we intend to give a more detailed description of $(\Delta^K G_n)$ (and then $P_m(x)$) by constructing a new explicit formula for this sequence. In order to do that we collect a short list about the features of finite differences. Three properties are presented for arbitrary complex sequences (f_n) and (g_n). The first two identities are known facts. The third one is made up as a theorem (Theorem 3.1, see later). We will prove it in the next section, and the proof implies an important corollary.

Properties.

1. Linearity: $\Delta(cf_n + dg_n) = c\Delta(f_n) + d\Delta(g_n)$, $(c, d \in \mathbb{C})$.

2. Product rule: $\Delta(f_n g_n) = f_n(\Delta g_n) + (\Delta f_n)g_n + (\Delta f_n)(\Delta g_n)$.

2.2. Linear recurrences

The real companion polynomial of the sequence (G_n) given in (1.1) is the polynomial

$$g(x) = x^k - a_1 x^{k-1} - \cdots - a_k. \tag{2.2}$$

Denote $\alpha_1, \dots, \alpha_s \in \mathbb{C}$ the distinct zeros of the companion polynomial $g(x)$, which now can be written in the form

$$g(x) = (x - \alpha_1)^{m_1} \cdots (x - \alpha_s)^{m_s} \tag{2.3}$$

with $m_1 + m_2 + \cdots + m_s = k$. Note that we assumed $a_k \neq 0$, therefore $\alpha_i \neq 0$ for $1 \leq i \leq s$. The following result plays a key role in the theory of linear recurrence sequences (see e.g. [5]).

Theorem 2.3 *Let* (G_n) *be a sequence satisfying relation (1.1) with* $a_k \neq 0$, *and* $g(x)$ *its companion polynomial with distinct roots* $\alpha_1, \dots, \alpha_s$, *be given in the form (2.3). Then there exist uniquely determined polynomials* $g_i(x) \in \mathbb{C}[x]$ *of degree less than* m_i $(i = 1, \dots, s)$ *such that*

$$G_n = g_1(n)\alpha_1^n + \cdots + g_s(n)\alpha_s^n \qquad (n \in \mathbb{Z}). \tag{2.4}$$

Corollary 2.4 *If the zeros of* $g(x)$ *are simple, then* $s = k$ *holds in the previous theorem, and*

$$G_n = c_1\alpha_1^n + \cdots + c_k\alpha_k^n \qquad (n \in \mathbb{Z}), \tag{2.5}$$

where $c_1, \dots, c_k$ *are suitable complex numbers.*

3. Preparation of the proof of the main theorem

The next theorem can be considered as the third property of the finite differences.

Theorem 3.1 *Assume that K is a non-negative integer. Using the notations above, we have*

$$\Delta^K(f_n g_n) = \sum_{i=0}^{K}\sum_{j=i}^{K}\binom{K}{i, K-j, j-i}(\Delta^{K-i} f_n)(\Delta^j g_n). \tag{3.1}$$

Here the trinomial coefficient $\binom{K}{i,K-j,j-i}$ is located in the Kth layer of the Pascal tetrahedron. It is known that if $k_1+k_2+k_3=K$ for non-negative integers k_1, k_2, k_3, then we have

$$\binom{K}{k_1, k_2, k_3} = \frac{K!}{k_1!\cdot k_2!\cdot k_2!}. \tag{3.2}$$

Proof We use the technique of induction on K. For $K=0$ the statement is obvious. If $K=1$, then we have

$$\binom{1}{0,1,0}(\Delta f_n)g_n + \binom{1}{0,0,1}(\Delta f_n)(\Delta g_n) + \binom{1}{1,0,0} f_n(\Delta g_n),$$

which is the right-hand side of Property 2 in the list above.

Assume that the statement is true for some $K \geq 2$. Then

$$\begin{aligned}\Delta^{K+1}(f_n g_n) &= \Delta\left(\sum_{i=0}^{K}\sum_{j=i}^{K}\binom{K}{i, K-j, j-i}(\Delta^{K-i} f_n)(\Delta^j g_n)\right)\\ &= \sum_{i=0}^{K}\sum_{j=i}^{K}\binom{K}{i, K-j, j-i}\Delta\left((\Delta^{K-i} f_n)(\Delta^j g_n)\right).\end{aligned}$$

These equalities rely on the assumption for K, and Property 1, respectively. Applying Property 2 we find that

$$\begin{aligned}\Delta^{K+1}(f_n g_n) = \sum_{i=0}^{K}\sum_{j=i}^{K}\binom{K}{i, K-j, j-i}\times&\\ ((\Delta^{K+1-i} f_n)(\Delta^j g_n) + (\Delta^{K-i} f_n)(\Delta^{j+1} g_n) + (\Delta^{K+1-i} f_n)(\Delta^{j+1} g_n)).&\end{aligned}$$

Now we separate the three terms of the sum after the trinomial coefficients. One can easily see that

$$\sum_{i=0}^{K}\sum_{j=i}^{K}\binom{K}{i, K-j, j-i}(\Delta^{K+1-i} f_n)(\Delta^j g_n) = \sum_{i=0}^{K+1}\sum_{j=i}^{K+1}\binom{K}{i, K-j, j-i}(\Delta^{K+1-i} f_n)(\Delta^j g_n),$$

$$\sum_{i=0}^{K}\sum_{j=i}^{K}\binom{K}{i, K-j, j-i}(\Delta^{K-i} f_n)(\Delta^{j+1} g_n) = \sum_{i=0}^{K+1}\sum_{j=i}^{K+1}\binom{K}{i-1, K+1-j, j-i}(\Delta^{K+1-i} f_n)(\Delta^j g_n),$$

$$\sum_{i=0}^{K}\sum_{j=i}^{K}\binom{K}{i, K-j, j-i}(\Delta^{K+1-j} g_n)(\Delta^{j+1} g_n) = \sum_{i=0}^{K+1}\sum_{j=i}^{K+1}\binom{K}{i, K+1-j, j-i-1}(\Delta^{K+1-i} f_n)(\Delta^j g_n).$$

The addition rule in Pascal tetrahedron admits

$$\binom{K}{i, K-j, j-i} + \binom{K}{i-1, K+1-j, j-i} + \binom{K}{i, K+1-j, j-i-1} = \binom{K+1}{i, K+1-j, j-i},$$

therefore the sums of the three right-hand sides above together provide the desired result. □

Corollary 3.2 *Assume that the sequence* (g_n) *takes polynomial values of a given polynomial* $g(x)$ *(i.e.* $g_n = g(n)$*) of degree* k_g *with complex coefficients. Put* $\kappa = \min\{k_g, K\}$*. Then we have*

$$\Delta^K(f_n g_n) = \sum_{i=0}^{\kappa}\sum_{j=i}^{\kappa}\binom{K}{i, K-j, j-i}(\Delta^{K-i} f_n)(\Delta^j g_n). \tag{3.3}$$

Proof Now the sequence (g_n) is a polynomial sequence of degree k_g, so $\Delta^K g_n = 0$ if $K > k_g$. Thus, from the layer $k_g + 1$ it is sufficient to consider only a particular infinite triangular prism part of the Pascal tetrahedron. The summation of Theorem 3.1 will be modified from K to $\kappa = \min\{k_g, K\}$ at the upper limit. Clearly, we need the whole Pascal tetrahedron as far as layer k_g (see Figure 1 if $k_g = 1$). □

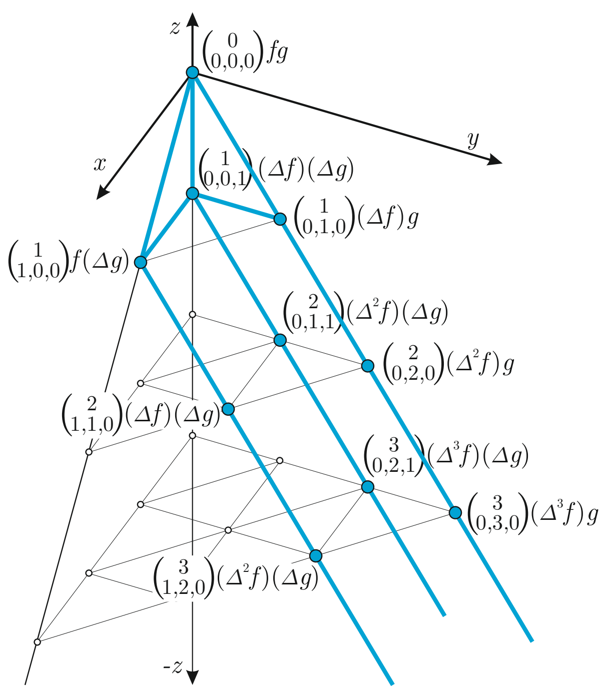

Figure 1. Illustration of Corollary 3.2 with linear $g(x)$.

This corollary will be applied with $k_g = 1$ later. If $k_g = 1$, then $\kappa = 1$, and the double sum in (3.3) contains only three terms. Now we turn our attention to the finite differences of certain well-defined sequences (f_n).

Lemma 3.3 *Assume that* $f_n = \omega^n$ $(n \in \mathbb{Z})$*, where* $\omega \neq 0$ *is an arbitrary complex number. Then for the integer* $t \geq 0$ *we have*

$$\Delta^t f_n = (\omega - 1)^t \omega^n.$$

Proof The case $\omega = 1$ is trivial with the convention $0^0 = 1$. Assume now that $\omega \neq 1$. Then we prove the lemma by induction on t. For $t = 0$, the statement is obvious. Suppose the statement holds for $t = k$, i.e.

$$\Delta^k f_n = (\omega - 1)^k \omega^n.$$

By changing the index n into $n+1$ on the above expression, we obtain $\Delta^k f_{n+1} = (\omega - 1)^k \omega^{n+1}$. Now for $t = k+1$, we have $\Delta^{k+1} f_n = \Delta^k f_{n+1} - \Delta^k f_n = (\omega - 1)^k \left(\omega^{n+1} - \omega^n\right) = (\omega - 1)^{k+1}\omega^n$. □

Lemma 3.4 *Assume that $h_n = n\omega^n$ $(n \in \mathbb{Z})$, where the non-zero $\omega \in \mathbb{C}$ is arbitrary. Then for $\omega \neq 1$*

$$\Delta^t h_n = ((\omega - 1)n + \omega t)\,(\omega - 1)^{t-1}\omega^n$$

holds for $t \geq 0$.

Proof The statement is trivially true for $t = 0$. Suppose $t \geq 1$. Put $g_n = n$, a linear polynomial sequence, and let $f_n = \omega^n$. Clearly, $\Delta g_n = 1$, and $\Delta^t g_n = 0$ for $t \geq 2$, moreover $\Delta^t f_n$ is given in Lemma 3.3. Now we apply property (3.3) with $\kappa = \min\{1, t\} = 1$. Hence

$$\begin{aligned}
\Delta^t(f_n g_n) &= \sum_{i=0}^{1}\sum_{j=i}^{1}\binom{t}{i, t-j, j-i}(\Delta^{t-i} f_n)(\Delta^j g_n) \\
&= \binom{t}{0, t, 0}(\Delta^t f_n)(g_n) + \binom{t}{0, t-1, 1}(\Delta^t f_n)(\Delta g_n) + \binom{t}{1, t-1, 0}(\Delta^{t-1} f_n)(\Delta g_n) \\
&= (\omega - 1)^t \omega^n n + t(\omega - 1)^t \omega^n + t(\omega - 1)^{t-1}\omega^n \\
&= \begin{cases} ((\omega - 1)n + \omega t)\,(\omega - 1)^{t-1}\omega^n, & \text{if } \omega \neq 1, \\ 0^t n + t0^t + t0^{t-1} & \text{if } \omega = 1. \end{cases}
\end{aligned}$$

The upper case justifies the main result of the theorem. □

Remark 3.5 *If $\omega = 1$, then $h_n = n$, and the second case at the end of the previous proof implies*

$$\Delta^t(h_n) = \begin{cases} n, & \text{if } t = 0, \\ 1, & \text{if } t = 1, \\ 0, & \text{if } t \geq 2. \end{cases}$$

Theorem 3.6 *Let $h_n^{(r)} = n^r \omega^n$ $(r \in \mathbb{N})$ with $\omega \in \mathbb{C} \setminus \{0\}$. Then assuming $\omega \neq 1$*

$$\Delta^t h_n^{(r)} = \Omega_r^{t,\omega}(n)(\omega - 1)^{t-r}\omega^n$$

holds for $t \geq 0$. Here $\Omega_r^{t,\omega}(n)$ is a univariate polynomial in variable n of degree t with parameters r, t and ω that satisfies the equality

$$\Omega_r^{t,\omega}(n) = (n+t)\Omega_{r-1}^{t,\omega}(n) + t\Omega_{r-1}^{t-1,\omega}(n), \qquad r \geq 1.$$

Proof Suppose $\omega \neq 1$. Then Lemma 3.3 shows that if $r = 0$, then the statement is true with the constant polynomial $\Omega_0^{t,\omega}(n) = 1$. Lemma 3.4 justifies the theorem for $r = 1$ with the polynomial $\Omega_1^{t,\omega}(n) = (\omega-1)n+\omega t$.

Now assume that the theorem is true for $r-1$, and look at the case r. We apply Corollary 3.2 with $f_n = h_n^{(r-1)} = n^{r-1}\omega^n$ and with the linear polynomial $g_n = n$. Obviously, $h_n^{(r)} = h_n^{(r-1)} \cdot n$. From Corollary 3.2 we deduce (like in the proof of Theorem 3.4) with $k_g = 1$ that

$$\begin{aligned}
\Delta^t(f_n g_n) &= \Delta^t(n^{r-1}\omega^n)n + t\Delta^t(n^{r-1}\omega^n) + t\Delta^{t-1}(n^{r-1}\omega^n)\\
&= (n+t)\Delta^t(n^{r-1}\omega^n) + t\Delta^{t-1}(n^{r-1}\omega^n)\\
&= (n+t)\Omega_{r-1}^{t,\omega}(n)(\omega-1)^{t-(r-1)}\omega^n + t\Omega_{r-1}^{t-1,\omega}(n)(\omega-1)^{(t-1)-(r-1)}\omega^n\\
&= (\omega-1)^{t-r}\omega^n \underbrace{\left((n+t)\Omega_{r-1}^{t,\omega}(n) + t\Omega_{r-1}^{t-1,\omega}(n)\right)}_{\Omega_r^{t,\omega}(n)}.
\end{aligned}$$

□

Here we list up the first few polynomials $\Omega_r^{t,\omega}(n)$. Put $B = (\omega-1)n + \omega t$.

$$\begin{aligned}
\Omega_0^{t,\omega}(n) &= 1,\\
\Omega_1^{t,\omega}(n) &= (\omega-1)n + \omega t = B,\\
\Omega_2^{t,\omega}(n) &= (\omega-1)^2n^2 + 2(\omega-1)\omega tn + (\omega t-1)\omega t = B^2 - \omega t,\\
\Omega_3^{t,\omega}(n) &= (\omega-1)^3n^3 + 3(\omega-1)^2\omega tn^2 + 3(\omega-1)(\omega t-1)\omega tn + (\omega^2t^2 - 3\omega t + \omega + 1)\omega t\\
&= B^3 - 3\omega tB + \omega t(\omega+1).
\end{aligned}$$

Remark 3.7 *If $\omega = 1$, then $h_n^{(r)} = n^r$, and $\hat{\Omega}_r^t(n) = \Delta^t n^r$ gives a sequence of polynomials with descending degree as r is increasing. In particular, $\deg_n(\Delta^t n^r) = r-t$ if $t \leq r$, and $\Delta^t n^r = 0$ holds when $t > r$. For example, let $r = 3$. Then $\Delta n^3 = 3n^2 + 3n + 1$, $\Delta^2 n^3 = 6n + 6$, and $\Delta^3 n^3 = 6$. One can show the following more general statements:*

$$\begin{aligned}
\hat{\Omega}_r^t(n) &= (n+t)\hat{\Omega}_{r-1}^t(n) + t\hat{\Omega}_{r-1}^{t-1}(n),\\
\hat{\Omega}_r^1(n) &= \sum_{i=1}^{r}\binom{r}{i}n^{r-i},\\
\hat{\Omega}_r^2(n) &= \sum_{i=2}^{r}\binom{r}{i}\left(2^i-2\right)n^{r-i},\\
\hat{\Omega}_r^{r-2}(n) &= \frac{r!}{2}n^2 + \frac{(r-2)r!}{2}n + \frac{(r-2)(3r-5)r!}{24},\\
\hat{\Omega}_r^{r-1}(n) &= r!n + \frac{(r-1)r!}{2},\\
\hat{\Omega}_r^r(n) &= r!.
\end{aligned}$$

4. Main results

Assume that the coefficient polynomials $g_i(x) \in \mathbb{C}[x]$ $(i = 1, \dots, s)$ in (2.4) have the form

$$g_i(x) = a^{(i)}_{m_i-1} x^{m_i-1} + a^{(i)}_{m_i-2} x^{m_i-2} + \cdots + a^{(i)}_0.$$

Consequently,

$$G_n = \sum_{i=1}^{s} p_i(n)\alpha_i^n = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} a^{(i)}_j n^j \alpha_i^n.$$

Hence, if every zeros of the companion polynomial satisfies $\alpha_i \neq 1$, then from Property 1 and Theorem 3.6 we conclude

$$\Delta^t G_n = \Delta^t \left(\sum_{i=1}^{s} \sum_{j=0}^{m_i-1} a^{(i)}_j n^j \alpha_i^n \right) = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} a^{(i)}_j \Delta^t \left(n^j \alpha_i^n \right)$$
$$= \sum_{i=1}^{s} \alpha_i^n \sum_{j=0}^{m_i-1} a^{(i)}_j \Omega^{t,\alpha_i}_j(n)(\alpha_i - 1)^{t-j}.$$

Putting all the things we need together, (1.5) can be given as follows.

Theorem 4.1 *Using the notation above the interpolation polynomial* $P_m(x)$ *crossing the points* (n_t, G_{n_t}) $(t = 0, 1 \dots, m)$ *has the form*

$$P_m(x) = P_{m-1}(x) + \binom{x - n_0}{m} \sum_{i=1}^{s} \alpha_i^{n_0} \sum_{j=0}^{m_i-1} a^{(i)}_j \Omega^{t,\alpha_i}_j(n_0)(\alpha_i - 1)^{t-j}. \tag{4.1}$$

Obviously, expression (4.1) can be converted to the form of (1.5) to obtain a non-recursive explicit, but more complicated formula. This can be also done with (4.2), too.

Example 4.2 *Let* $G_n = (3n^2 - n + 1)2^n + (-2n + 1)3^n - 15$*, and consider* $P_7(x)$ *if* $n_0 = -1$*. The interpolation polynomial* $P_7(x)$ *(see Figure 2) crosses the points*

$$(-1, -18.5),\ (0, -13),\ (1, 18),\ (2, 2),\ (3, 80),\ (4, 138),\ (5, 100),\ (6, -1442).$$

The coefficient polynomials are $g_1(x) = 3x^2 - x + 1$*,* $g_2(x) = -2x + 1$*,* $g_3(x) = -15$*, and the companion polynomial is*

$$g(x) = (x - 2)^3(x - 3)^2(x + 1) = x^6 - 11x^5 + 45x^4 - 77x^3 + 22x^2 + 84x - 72,$$

the recurrence rule is given by

$$G_n = 11G_{n-1} - 45G_{n-2} + 77G_{n-3} - 22G_{n-4} - 84G_{n-5} + 72G_{n-6}.$$

Further we see that $s = 3$*,* $\alpha_1 = 2$*,* $m_1 = 3$*,* $\alpha_2 = 3$*,* $m_2 = 2$*,* $\alpha_3 = -1$*,* $m_3 = 1$*. Moreover*

$$\Omega^{t,\alpha_i}_0(-1) = 1,\ \Omega^{t,\alpha_1}_1(-1) = 2t - 1,\ \Omega^{t,\alpha_2}_1(-1) = 3t - 2,\ \Omega^{t,\alpha_1}_2(-1) = 4t^2 - 6t + 1.$$

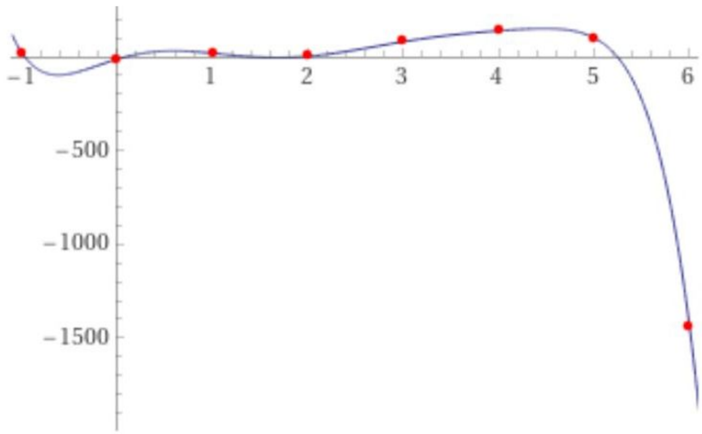

Figure 2. The base points and $P_7(x)$.

Finally, Theorem 4.1 provides

$$\begin{aligned} P_7(x) &= \binom{x+1}{0}\cdot 18.5+\binom{x+1}{1}\cdot(-31.5)+\cdots+\binom{x+1}{7}\cdot\left(-\frac{547}{1120}\right) \\ &= -\frac{547}{1120}x^7+\frac{11443}{1440}x^6-\frac{7741}{180}x^5+\frac{36749}{288}x^4-\frac{24181}{240}x^3-\frac{37547}{360}x^2+\frac{62737}{420}x-13. \end{aligned}$$

If $\alpha_i=1$ holds for certain $i\in\{1,2,\dots\}$, then we handle separately this exceptional case. The next theorem follows from the considerations above.

Theorem 4.3 *Assume that one zero of the characteristic polynomial $g(x)$ is 1, say $\alpha_1=1$. Then*

$$P_m(x)=P_{m-1}(x)+\binom{x-n_0}{m}\sum_{j=0}^{m_1-1}a_j^{(1)}\hat{\Omega}_j^t n_0)+\binom{x-n_0}{m}\sum_{i=2}^{s}\alpha_i^{n_0}\sum_{j=0}^{m_i-1}a_j^{(i)}\Omega_j^{t,\alpha_i}(n_0)(\alpha_i-1)^{t-j}. \qquad (4.2)$$

Example 4.4 *Put $G_n=(n^2-2n+3)$, $n_0=-1$, and consider the interpolation polynomial $P_4(x)$ belonging to the points $(-1,6)$, $(0,3)$, $(1,2)$, $(2,3)$, $(3,6)$. The companion polynomial is $g(x)=(x-1)^3$. Now $s=1$, $\alpha_1=1$, $m_1=3$, furthermore*

$$\hat{\Omega}_j^j(n)=j!\,(j=0,1,2);\ \hat{\Omega}_1^0(n)=n;\ \hat{\Omega}_2^0(n)=n^2;\ \hat{\Omega}_2^1(n)=2n-1.$$

Thus Theorem 4.3 implies

$$P_4(x)=\binom{x+1}{0}\cdot 6+\binom{x+1}{1}\cdot(-3)+\binom{x+1}{2}\cdot 2=x^2-2x+3,$$

and this result was obvious in advance.

Consider now the case when all the zeros of the companion polynomial $g(x)$ in (2.2) are simple (belonging to the sequence (G_n) given by (1.1) with arbitrary initial values, and they are different from 0 and 1). Now in formula (4.1) we have $s=k$, $m_i=1$. We will simply write c_i instead of $a_0^{(i)}$. We even have $\Omega_0^{t,\alpha_i}(n_0)=1$. This specific case can be reformulated in

Corollary 4.5 *With the condition above we have*

$$\Delta^t G_n = \sum_{i=1}^{k} c_i(\alpha_i - 1)^t \alpha_i^n, \qquad \text{in particular} \qquad \Delta^t G_0 = \sum_{i=1}^{k} c_i(\alpha_i - 1)^t, \tag{4.3}$$

and then

$$P_m(x) = P_{m-1}(x) + \binom{x-n_0}{m} \sum_{i=1}^{k} c_i(\alpha_i - 1)^t \alpha_i^{n_0}. \tag{4.4}$$

The expression $\sum_{i=1}^{k} c_i(\alpha_i - 1)^t \alpha_i^{n_0}$ in (4.4) is exactly an explicit formula of a linear recursive sequence of order k, when the zeros of the characteristic polynomial are $\alpha_i - 1$ $(i = 1, 2, \ldots, k)$, and the initial values imply the constant multipliers $c_i\alpha_i^{n_0}$. For instance, in case of $k = 3$, Theorem 2.1 provides the following recursive rule for the sequence.

Example 4.6 *Let $G_n = a_1G_{n-1} + a_2G_{n-2} + a_3G_{n-3}$ such that the three zeros α_1, α_2 and α_3 of the characteristic polynomial $g(x) = x^3 - a_1x^2 - a_2x - a_3$ are distinct.*

Firstly, Theorem 2.1 (independently of the simple zeros) returns with

$$D_t = (a_1 - 3)D_{t-1} + (2a_1 + a_2 - 3)D_{t-2} + (a_1 + a_2 + a_3 - 1)D_{t-3},$$

subsequently, we obtain

$$P_m(x) = P_{m-1}x + \binom{x-n}{m} D_m.$$

On the other hand, $a_1 = \alpha_1 + \alpha_2 + \alpha_3$, $a_2 = -(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)$, $a_3 = \alpha_1\alpha_2\alpha_3$. A straightforward manipulation shows that

$$\begin{aligned}
a_1^\star &= (\alpha_1 - 1) + (\alpha_2 - 1) + (\alpha_3 - 1) = a_1 - 3,\\
a_2^\star &= -(\alpha_1 - 1)(\alpha_2 - 1) - (\alpha_1 - 1)(\alpha_3 - 1) - (\alpha_2 - 1)(\alpha_3 - 1) = 2a_1 + a_2 - 3,\\
a_3^\star &= (\alpha_1 - 1)(\alpha_2 - 1)(\alpha_3 - 1) = a_1 + a_2 + a_3 - 1,
\end{aligned}$$

i.e. the characteristic polynomial of the recursive sequence $(D_t) = (\Delta^t G_n)$ has zeros $\alpha_1 - 1$, $\alpha_2 - 1$, and $\alpha_3 - 1$, each is simple.

Note that in the specific case when $a_1 = a_2 = a_3 = 1$ we find $a_1^\star = -2$, $a_2^\star = 0$, and $a_3^\star = 2$, which are the coefficients in $D_t = -2D_{t-1} + 2D_{t-3}$, see Example 2.2 after (2.1) with the Tribonacci sequence.)

5. Conclusion

This work addresses the polynomials which interpolated from linear recurrences. While such polynomials can be generated using the interpolation method (e.g., Newton's divided differences), we show that they can be obtained by using a new approach concentrating on the inner structure of the explicit formula of the recurrence.

Acknowledgements

For L. Szalay the research was supported in part by National Research, Development and Innovation Office Grant 2019-2.1.11-TÉT-2020-00165, by Hungarian National Foundation for Scientific Research Grant No. 128088, and No. 130909, and by the Slovak Scientific Grant Agency VEGA 1/0776/21.

References

[1] Ahmia M, Szalay L. On the weighted sum associated to rays in generalized Pascal triangle. Indian Journal of Discrete Mathematics; 1 (2016): 8-17.

[2] Flajolet P, Sedgewick R. Mellin transforms and asymptotics: Finite differences and Rice's integrals. Theoretical Computer Science; 144 (1995): 101-124.

[3] Mühlbach G. The general recurrence relation for divided differences and the general Newton-interpolation-algorithm with application to trigonometric interpolation. Numerical Mathematics; 32 (1979): 393-408.

[4] Mufid MS, Asfihani T, Hanafi L. On the Lagrange interpolation of Fibonacci sequence. International Journal of Computing Science and Applied Mathematics; 2 (2016): 38-40.

[5] Shorey TN, Tijdeman R. Exponential Diophantine Equations. Cambridge, UK: Cambridge University Press, 1986.

[6] OEIS Foundation Inc. (2023). Entry A000073 in The On-Line Encyclopedia of Integer Sequences.