Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 898 – 909

doi:10.55730/1300-0098.3402

Duality approach to regularity problems for the Navier-Stokes equations

Gregory SEREGIN*
Oxford University, Oxford, UK, and St. Petersburg Department of Steklov Mathematical Institute (Russian Academy of Sciences), St. Petersburg, Russia

Received: 30.11.2022 • **Accepted/Published Online:** 09.02.2023 • **Final Version:** 17.03.2023

Abstract: In this note, we describe a way to study local regularity of a weak solution to the Navier-Stokes equations, satisfying the simplest scale-invariant restriction, with the help of zooming and duality approach to the corresponding mild bounded ancient solution.

Key words: Navier-Stokes equations, regularity of solutions

1. Introduction

One of the simplest but not yet solved problem of local regularity of weak solutions to the Navier-Stokes equations is as follows. Consider the so-called suitable weak solution $w \in L_\infty(-1,0;L_2(B)) \cap L_2(-1,0;W^1_2(B))$ and $r \in L_{\frac{3}{2}}(Q)$ to the classical Navier-Stokes equations:

$$\partial_t w + w \cdot \nabla w - \Delta w = -\nabla r, \qquad \operatorname{div} w = 0$$

in the unit parabolic space-time ball $Q = B\times]-1,0[\subset \mathbb{R}^3 \times \mathbb{R}$. For a definition of suitable weak solutions, we refer to the paper [1]. Let us assume that function w satisfies the additional restriction

$$|w(x,t)| \leq \frac{c_d}{|x| + \sqrt{-t}}, \quad \forall (x,t) \in Q, \tag{1.1}$$

where $c_d > 0$ is a given constant. The question is whether or not the origin $z = (0,0)$ is a regular point of w, i.e. there exists $\delta > 0$ such that w is essentially bounded in the parabolic ball $Q(\delta) = B(\delta)\times]-\delta^2,0[$. Here, as usual, $B(\delta)$ stands for the ball of radius δ centred at the origin. With a minor modification of what has been done in the paper [4], one can show that if the origin $z = 0$ is a singular point of w then there exists the so-called mild bounded ancient solution $\tilde{u}$ with the following properties:

$$|\tilde{u}| \leq 1$$

in $Q_- = \mathbb{R}^3\times]-\infty,0[$;

$$|\tilde{u}(0)| = 1;$$

there is a pressure field $\tilde{p} \in L_\infty(-\infty,0;BMO(\mathbb{R}^3))$ so that $\tilde{u}$ and $\tilde{p}$ obey the Navier-Stokes equations

$$\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} - \Delta \tilde{u} = -\nabla \tilde{p}, \quad \operatorname{div} \tilde{u} = 0$$

*Correspondence: seregin@maths.ox.ac.uk
2010 *AMS Mathematics Subject Classification:* 35B06, 35B65, 35Q30

in Q_-; and in addition

$$|\tilde{u}(x,t)| \leq \frac{c_d}{|x|+\sqrt{-t}}$$

for all $z=(x,t)\in Q_-$. In our further considerations, we shall work with positive time t, setting $u(x,t)=\tilde{u}(x,-t)$ for $t\geq 0$. Therefore, the velocity field u satisfies:

$$|u|\leq 1 \tag{1.2}$$

in $Q_+=\mathbb{R}^3\times]0,\infty[$;

$$|u(0)|=1; \tag{1.3}$$

there is a pressure field $p\in L_\infty(0,\infty;BMO(\mathbb{R}^3))$ so that u and p obey the backward Navier-Stokes equations

$$-\partial_t u+u\cdot\nabla u-\Delta u=-\nabla p, \quad \operatorname{div} u=0$$

in Q_+; and in addition

$$|u(x,t)|\leq \frac{c_d}{|x|+\sqrt{t}} \tag{1.4}$$

for all $z=(x,t)\in Q_+$.

To study the Liouville type statement about the above mild bounded ancient solutions, the duality method has been exploited in the paper [3]. In particular, the following Cauchy problem has been considered:

$$\partial_t v-u\cdot\nabla v-\triangle v-\nabla q=-\operatorname{div} F, \qquad \operatorname{div} v=0 \tag{1.5}$$

in $Q_+=\mathbb{R}^3\times]0,\infty[$ and

$$v(x,0)=0, \quad x\in\mathbb{R}^3. \tag{1.6}$$

It has been supposed that a tensor-valued field F is smooth and compactly supported in Q_+. In addition, it has been assumed that F is skew symmetric, and therefore

$$\operatorname{div}\operatorname{div} F=F_{ij,ji}=0. \tag{1.7}$$

Under the above assumptions, the long time behavior of solutions to (1.5), (1.6) has been studied in [3].

As to the drift u, one may assume that u is a bounded divergence free field in Q_+, say $|u|\leq 1$, whose derivatives of any order exist and are bounded in Q_+. It is not so difficult to check that the following identity takes place:

$$\int\limits_{Q_+} u\cdot \operatorname{div} F dxdt=-\lim_{T\to\infty}\int\limits_{\mathbb{R}^3} u(x,T)\cdot v(x,T)dx. \tag{1.8}$$

Therefore, if, for any skew symmetric tensor field F, the solution v to the dual problem (1.5), (1.6) has a certain decay, then u must be identically equal to zero. It, in turns, says that the origin is a regular point of w.

With regards to the long time behavior of v, it has been proved the following.

Theorem 1.1 *Let v be a solution v to (1.5) and (1.6) and let u be divergence free and satisfy (1.4). Then for any $m=0,1...$, two decay estimates are valid:*

$$\|v(\cdot,t)\|_1\leq c(m,c_d,F)\sqrt{t}^{\frac{3}{2}}\frac{1}{\ln^m(t+e)} \tag{1.9}$$

and

$$\|v(\cdot,t)\|_2\leq \frac{c(m,c_d,F)}{\ln^m(t+e)} \tag{1.10}$$

for all $t\geq 0$.

Unfortunately, the above statement does not allow us to conclude that the mild bounded ancient solution u is equal to zero.

In this paper, we would like to examine a certain modification of duality method letting $F=0$ but taking nonzero initial data. To be precise, let us consider the following Cauchy problem

$$\partial_t v-u\cdot\nabla v-\triangle v-\nabla q=0,\qquad \operatorname{div} v=0 \tag{1.11}$$

in $Q_+=\mathbb{R}^3\times]0,\infty[$ and

$$v(x,0)=v_0(x) \tag{1.12}$$

for $x\in\mathbb{R}^3$. Here, v_0 belongs to the space J which is L_2-closure of the set

$$C^\infty_{0,0}(\mathbb{R}^3)=\{v\in C^\infty_0(\mathbb{R}^3):\ \operatorname{div} v=0\}.$$

Formal calculations show that

$$\int\limits_{\mathbb{R}^3}u(\cdot,t)\cdot v(\cdot,t)dx=\int\limits_{\mathbb{R}^3}u(\cdot,0)\cdot v(\cdot,0)dx=\int\limits_{\mathbb{R}^3}u(\cdot,0)\cdot v_0(\cdot)dx \tag{1.13}$$

for all $t\geq 0$. Indeed,

$$\int\limits_{\mathbb{R}^3}u(\cdot,T)\cdot v(\cdot,T)dx-\int\limits_{\mathbb{R}^3}u(\cdot,0)\cdot v_0(\cdot)dx=$$

$$\int\limits_0^T\int\limits_{\mathbb{R}^3}(v\cdot\partial_t+u\cdot\partial_t v)dz+$$

$$\int\limits_0^T\int\limits_{\mathbb{R}^3}(v\cdot(u\cdot\nabla u-\Delta u+\nabla p)+u\cdot(u\cdot\nabla v+\Delta v+\nabla q))dz=0$$

It is also easy to see that equations (1.11) can be replaced with more symmetric ones:

$$\partial_t v-u\cdot\nabla v\mp u\cdot\nabla v-\triangle v-\nabla q=0,\qquad \operatorname{div} v=0 \tag{1.14}$$

in $Q_+=\mathbb{R}^3\times]0,\infty[$ as the following identity is valid:

$$\int\limits_0^T\int\limits_{\mathbb{R}^3} u\cdot(v\cdot\nabla u)dz=0.$$

If we assume that

$$\int\limits_{\mathbb{R}^3} u(\cdot,T)\cdot v(\cdot,T)dx\to 0 \tag{1.15}$$

as $T\to\infty$ for all $v_0\in C^\infty_{0,0}(\mathbb{R}^3)$, then

$$\int\limits_{\mathbb{R}^3} u(\cdot,0)\cdot v_0(\cdot)dx=0$$

for all $v_0\in C^\infty_{0,0}(\mathbb{R}^3)$. The latter, together with (1.2) and (1.4), implies that $u(x,0)=0$ in $\mathbb{R}^3$, which contradicts to (1.3). It would be a proof of the fact that $z=0$ is a regular point w. Therefore, we need to prove a certain time decay of v that would provide (1.15). To this end, let us represent v as a sum of solutions to two Cauchy problems so that

$$v=v^1+v^2; \tag{1.16}$$

$$\partial_t v^1-\Delta v^1=0 \text{ in } Q_+,\quad v^1(\cdot,0)=v_0(\cdot) \text{ in } \mathbb{R}^3; \tag{1.17}$$

$$\partial_t v^2-\Delta v^2+\nabla q=-\mathrm{div}\, v\otimes u,\ \mathrm{div}\, v^2=0 \tag{1.18}$$

in Q_+ with $v^2(\cdot,0)=0$ in $\mathbb{R}^3$.

With regard to v^1, we have the estimates

$$\|v^1(\cdot,t)\|_s\leq\|v_0\|_s \tag{1.19}$$

for all $t\geq 0$ and all $1\leq s\leq\infty$, and thus

$$\int\limits_{\mathbb{R}^3} u(\cdot,T)\cdot v^1(\cdot,T)dx\to 0$$

as $T\to\infty$ for all $v_0\in C^\infty_{0,0}(\mathbb{R}^3)$.

Our aim is to prove results similar to what has been stated in the paper [3]. In particular, we are going to show that Theorem 1.1 remains to be true in the following reduction.

Theorem 1.2 *Let v be a solution to (1.11) and (1.12) and let u be divergence free and satisfy (1.4). Then, for any $m=0,1...$, two decay estimates are valid:*

$$\|v^2(\cdot,t)\|_1\leq c(m,c_d,v_0)\sqrt{t}^{\frac{3}{2}}\frac{1}{\ln^m(t+e)} \tag{1.20}$$

and

$$\|v^2(\cdot,t)\|_2 \leq \frac{c(m,c_d,v_0)}{\ln^m(t+e)} \tag{1.21}$$

for all $t \geq 1$.

Unfortunately, decay bounds in Theorem 1.2 do not provide the above scenario. One needs to improve decay estimates in it.

2. Comments on Proof of Theorem 1.2

Let

$$\mathcal{F} = -v \otimes u.$$

The solution to the problem (1.18), (1.6) has the form, see for instance [2],

$$v^2(x,t) = \int\limits_0^t \int\limits_{\mathbb{R}^3} K(x-y,t-s)\mathcal{F}(y,s)dyds, \tag{2.1}$$

where the potential $K = (K_{ijl})$ defined with the help of the standard heat kernel Γ in the following way

$$\Delta\Phi(x,t) = \Gamma(x,t)$$

and

$$K_{ijl} = \Phi_{,ijl} - \delta_{il}\Phi_{,kkj}.$$

It is easy to check that the following bound is valid:

$$|K(x,t)| \leq \frac{c}{(t+|x|^2)^2}, \tag{2.2}$$

and therefore

$$\int\limits_{\mathbb{R}^3} |K(x,t)|dx \leq \frac{c}{\sqrt{t}}. \tag{2.3}$$

Assuming that

$$p \in]6/5, 2[, \tag{2.4}$$

and repeating the same arguments as in the paper [3], we arrive at a similar estimate

$$\|v^2(\cdot,t)\|_p \leq C(p)\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}\Big(\int\limits_{\mathbb{R}^3}|\mathcal{F}(y,s)|^2(\sqrt{s}+|y|)^2dy\Big)^{\frac{1}{2}},$$

where, by (1.4),

$$\int\limits_{\mathbb{R}^3}|\mathcal{F}(y,s)|^2(\sqrt{s}+|y|)^2dy \leq c(c_d\|v(\cdot,s)\|_2)^2, \tag{2.5}$$

and thus

$$\|v^2(\cdot,t)\|_p \leq C(p)\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}c_d\|v(\cdot,s)\|_2ds$$

$$\leq C(p)\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}c_d(\|v^1(\cdot,s)\|_2+\|v^2(\cdot,s)\|_2)ds.$$

Here, we would like to use the following facts about time decay of solutions to the heat equations, see for example [5]:

Lemma 2.1 *Let* $v_0\in L_1(\mathbb{R}^3)$ *and* $M=\int\limits_{\mathbb{R}^3} v_0dx$*. Then*

$$t^{\frac{3}{2}\frac{p-1}{p}}\|v_i^1(\cdot,t)-M_i\Gamma(\cdot,t)\|_p\to 0$$

as $t\to\infty$ *for each* $i=1,2,3$ *and for all* $1\leq p\leq\infty$*.*

From the above lemma, it follows that for any $v_0\in C^\infty_{0,0}(\mathbb{R}^3)$, we have

$$\|v^1(\cdot,t)\|_p\leq c(v_0,p)f^{\frac{3(1-p)}{p}}(t)$$

for any $t\geq 0$ and for any $1\leq p\leq\infty$, where $f(t):=\max\{1,\sqrt{t}\}$.

Therefore,

$$\|v^2(\cdot,t)\|_p\leq C(p)\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}c_d(c(v_0)f^{-\frac{3}{2}}(s)+\|v^2(\cdot,s)\|_2)ds,$$

where we need to evaluate the term

$$I=\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}f^{-\frac{3}{2}}(s)ds.$$

To this end, consider two cases. In the first one, $0\leq t\leq 1$. Then

$$I=\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}ds\leq ct^{\frac{1}{2}-\frac{5p-6}{4p}}=ct^{\frac{3(2-p)}{4p}}\leq c(p).$$

If $t>1$, then

$$I=\int\limits_0^1 \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}ds+\int\limits_1^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}\sqrt{s}^{-\frac{3}{2}}ds=B_1+B_2.$$

Obviously, $B_1\leq c(p)$. As B_2, we have

$$B_2=\int\limits_{\frac{t+1}{2}}^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}\sqrt{s}^{-\frac{3}{2}}ds+\int\limits_1^{\frac{t+1}{2}} \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}\sqrt{s}^{-\frac{3}{2}}ds\leq$$

$$\leq \frac{\sqrt{2}}{\sqrt{t-1}} \int\limits_1^{\frac{t+1}{2}} \sqrt{s}^{-\frac{5p-6}{2p}-\frac{3}{2}} ds + \sqrt{\frac{t-1}{2}}^{-\frac{5p-6}{2p}-\frac{3}{2}} \int\limits_{\frac{t+1}{2}}^{t} \frac{ds}{\sqrt{t-s}}.$$

Assuming further that

$$p \leq 3/2, \tag{2.6}$$

we arrive at:

$$B_2 \leq \frac{\sqrt{2}}{\sqrt{t-1}} \frac{6-4p}{4p} s^{\frac{6-4p}{4p}} \Big|_1^{\frac{t+1}{2}} + \sqrt{\frac{t-1}{2}}^{-\frac{5p-6}{2p}-\frac{3}{2}} 2\sqrt{\frac{t-1}{2}} \leq c(p) f^{\frac{3(1-p)}{p}}(t).$$

Therefore, letting

$$A_p(t) := \int\limits_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{-\frac{5p-6}{2p}} c_d \|v^2(\cdot,s)\|_2 ds, \tag{2.7}$$

we can rewrite the previous estimate

$$\|v^2(\cdot,t)\|_p \leq C(p)(c(c_d,v_0,p) f^{\frac{3(1-p)}{p}}(t) + A_p(t)). \tag{2.8}$$

Now, one can repeat the above arguments for $p=1$ and find

$$\|v^2(\cdot,t)\|_1 \leq \int\limits_0^t \frac{c}{\sqrt{t-s}} \int\limits_{\mathbb{R}^3} |\mathcal{F}(y,s)| dy ds.$$

Since

$$|\mathcal{F}(y,s)| \leq \frac{c_d |v(y,s)|}{\sqrt{s}+|y|},$$

the latter estimate can be transformed as follows:

$$\|v^2(\cdot,t)\|_1 \leq c \int\limits_0^t \frac{ds}{\sqrt{t-s}} \int\limits_{\mathbb{R}^3} \frac{c_d |v(y,s)|}{\sqrt{s}+|y|} dy$$

$$\leq c \int\limits_0^t \frac{ds}{\sqrt{t-s}} \Big(\int\limits_{\mathbb{R}^3} \Big(\frac{1}{\sqrt{s}+|y|} \Big)^{\frac{6+5\varepsilon}{1+5\varepsilon}} dy \Big)^{\frac{1+5\varepsilon}{6+5\varepsilon}} \Big(\int\limits_{\mathbb{R}^3} (c_d |v(y,s)|)^{\frac{6+5\varepsilon}{5}} dy \Big)^{\frac{5}{6+5\varepsilon}}$$

for some positive $0 < \varepsilon < 3/10$. Hence,

$$\|v^2(\cdot,t)\|_1 \leq C_1(\varepsilon) \int\limits_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{3\frac{1+5\varepsilon}{6+5\varepsilon}-1} \Big(\int\limits_{\mathbb{R}^3} (c_d |v(y,s)|)^{\frac{6+5\varepsilon}{5}} dy \Big)^{\frac{5}{6+5\varepsilon}}$$

with

$$C_1(\varepsilon) := \Big(\int\limits_{\mathbb{R}^3} \Big(\frac{1}{1+|z|} \Big)^{\frac{6+5\varepsilon}{1+5\varepsilon}} dz \Big)^{\frac{1+5\varepsilon}{6+5\varepsilon}}.$$

Simplifying slightly the previous bound, we have

$$\|v^2(\cdot,t)\|_1\leq C_2(\varepsilon,c_d)\int\limits_0^t\frac{ds}{\sqrt{t-s}}\sqrt{s}^{\frac{-3+10\varepsilon}{6+5\varepsilon}}(\|v^1(\cdot,s)\|_{\frac{6+5\varepsilon}{5}}+\|v^2(\cdot,s)\|_{\frac{6+5\varepsilon}{5}})ds.$$

To estimate terms with v^1 and v^2, we are going to use Lemma 2.1 and (2.8) with $p=6/5+\varepsilon$, respectively:

$$\|v^2(\cdot,t)\|_1\leq C_2(\varepsilon,c_d)\int\limits_0^t\frac{ds}{\sqrt{t-s}}\sqrt{s}^{\frac{-3+10\varepsilon}{6+5\varepsilon}}(c(c_d,v_0,\varepsilon)(\sqrt{s}^{-\frac{3(1+5\varepsilon)}{6+5\varepsilon}}+f^{-\frac{3(1+5\varepsilon)}{6+5\varepsilon}}(s))$$

$$+A_{\frac{6}{5}+\varepsilon}(s)))ds\leq\leq C_2(\varepsilon,c_d)\int\limits_0^t\frac{ds}{\sqrt{t-s}}\sqrt{s}^{\frac{-3+10\varepsilon}{6+5\varepsilon}}(c(c_d,v_0,\varepsilon)\sqrt{s}^{-\frac{3(1+5\varepsilon)}{6+5\varepsilon}}+A_{\frac{6}{5}+\varepsilon}(s)))ds$$

$$\leq C_3(\varepsilon,c_d,v_0)+C_4(\varepsilon,c_d)\int\limits_0^t\frac{ds}{\sqrt{t-s}}\sqrt{s}^{\frac{-3+10\varepsilon}{6+5\varepsilon}}A_{\frac{6}{5}+\varepsilon}(s)ds.$$

On the other hand,

$$A(p)(t)\leq c_d\|v\|_{2,\infty}\int\limits_0^t\frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}ds\leq c_d\|v_0\|_2C_2(p)\sqrt{t}^{\frac{3}{2}\frac{2-p}{p}}.$$

Therefore, we have

$$\|v^2(\cdot,t)\|_1\leq C_4(\varepsilon,c_d,v_0)(1+\|v_0\|_2\sqrt{t}^{\frac{3}{2}})\leq c(\varepsilon,c_d,v_0)f^{\frac{3}{2}}(t).$$

3. Improvement for L_2-norm

Following [3], we have the energy inequality

$$\partial_t y(t)+\|\nabla v(\cdot,t)\|_2^2\leq 0 \tag{3.1}$$

with $y(t)=\|v(\cdot,t)\|_2^2$.

The Fourier transform and Plancherel identity give us

$$\partial_t y(t)\leq-\int\limits_{\mathbb{R}^3}|\xi|^2|\widehat{v}(\xi,t)|^2d\xi=-\int\limits_{|\xi|>g(t)}|\xi|^2|\widehat{v}(\xi,t)|^2d\xi-\int\limits_{|\xi|\leq g(t)}|\xi|^2|\widehat{v}(\xi,t)|^2d\xi,$$

where $g(t)$ is a given function which will be specified later on. The latter implies

$$y'(t)+g^2(t)y(t)\leq\int\limits_{|\xi|\leq g(t)}(g^2(t)-|\xi|^2)|\widehat{v}(\xi,t)|^2d\xi.$$

Taking the Fourier transform of the Navier-Stokes equation, we find

$$\partial_t\widehat{v}+|\xi|^2\widehat{v}=-\widehat{H},$$

where $H=-\mathrm{div}\,(v\otimes u+\mathbb{I}q)$. Clearly,

$$\widehat{v}(\xi,t)=-\int_0^t \exp\{-|\xi|^2(t-s)\}\widehat{H}(\xi,s)ds+\widehat{v}_0(\xi)\exp\{-|\xi|^2t\}$$

and

$$|\widehat{H}(\xi,s)|\leq|\xi|\||v(\cdot,s)||u(\cdot,s)|\|_1.$$

Denoting

$$a(s)=\||v(\cdot,s)||u(\cdot,s)|\|_1,$$

we find

$$|\widehat{v}(\xi,t)|\leq c\int_0^t \exp\{-|\xi|^2(t-s)\}|\xi|a(s)ds+|\widehat{v}_0(\xi)|\exp\{-|\xi|^2t\}.$$

Applying the Hölder inequality, we get

$$y'(t)+g^2(t)y(t)\leq$$

$$\leq c\int\limits_{|\xi|\leq g(t)}(g^2(t)-|\xi|^2)\Big(\int_0^t \exp\{-|\xi|^2(t-s)\}|\xi|a(s)ds+|\widehat{v}_0(\xi)|\exp\{-|\xi|^2t\}\Big)^2d\xi\leq$$

$$\leq c\int\limits_{|\xi|\leq g(t)}(g^2(t)-|\xi|^2)\Big[\int_0^t a^2(s)ds\int_0^t \exp\{-2|\xi|^2(t-s_1)\}|\xi|^2ds_1+$$

$$+|\widehat{v}_0(\xi)|^2\exp\{-2|\xi|^2t\}\Big]d\xi\leq I_1+I_2.$$

For the first term, we have

$$I_1\leq c\int_0^t a^2(s)ds\int_0^t\int\limits_{|\xi|\leq g(t)}(g^2(t)-|\xi|^2)\exp\{-|\xi|^2(t-s_1)\}|\xi|^2ds_1d\xi.$$

It can be estimated in the same way as in [3]:

$$I_1\leq cg^6(t)\sqrt{t}\int_0^t a^2(s)ds.$$

As to the second term, we proceed as follows:

$$I_2\leq c\|v_0\|_1^2\int\limits_{|\xi|\leq g(t)}(g^2(t)-|\xi|^2)\exp\{-|\xi|^2t\}d\xi\leq$$

$$\leq c\|v_0\|_1^2\int\limits_0^{g(t)}(g^2(t)-r^2)\exp\{-r^2t\}r^2dr\leq c\|v_0\|_1^2g^5(t).$$

Therefore, we find

$$K(t):=I_1+I_2\leq cg^6(t)\sqrt{t}\int_0^t a^2(s)ds+c(v_0)g^5(t),$$

and thus solution to our inequality has the form:

$$y(t) \leq c \int_0^t \exp\Big\{ - \int_s^t g^2(\tau)d\tau \Big\} K(s)ds + y(0)\exp\{-\int_0^t g^2(\tau)d\tau\}. \tag{3.2}$$

4. Proof of Theorem 1.2

As in the paper [3], we use the induction in m. The basis of induction has been already established in Section 2. Let us assume that our statement is true for m and show that it is true for $m+1$.

We can present the right hand side of (3.2) as a sum so that

$$y(t) \leq y_1(t) + y_2(t).$$

Then we select our function $g(t) = h'(t)/h(t)$ with $h(t) = \ln^k(t+e)$ and $k > 2m+2$, for example, $k = 2m+3$. Next, we observe that

$$a(t) \leq \||v^1(\cdot,t)||u(\cdot,t)|\|_1 + \||v^2(\cdot,t)||u(\cdot,t)|\|_1 \leq \|v_0\|_1 + \||v^2(\cdot,t)||u(\cdot,t)|\|_1$$

and, for $t > 1$, by induction,

$$\int_0^t a^2(s)ds \leq 2\int_0^t \|v_0\|_1^2 + 2\int_0^t \||v^2(\cdot,s)||u(\cdot,s)|\|_1^2 ds \leq$$

$$\leq c(v_0)t + 2\int_0^1 \|v^2(\cdot,s)\|_1^2 ds + 2\int_1^t \frac{c_d^2\||v^2(\cdot,s)|\|_1^2}{s} ds \leq$$

$$\leq c(v_0,c_d)t + c(v_0,c_d,m)\int_1^t \sqrt{s}\ln^{-2m}(s+e)ds.$$

The function $y_1(t)$ is estimated in a similar same way as it has been done in [3]. Indeed, we are going to use the following simple statements.

Lemma 4.1 *Let l be a real number and $\gamma > -1$.*

(i) There exists a positive constant $c(\gamma,l)$ such that

$$\int_1^t s^\gamma \ln^{-l}(s+e)ds \leq c(\gamma,l)t^{\gamma+1}\ln^{-l}(t+e)$$

for all $t \geq 1$;

(ii) There exists a positive constant $c(\gamma,l)$ such that

$$\int_1^t \frac{1}{\sqrt{t-s}} s^\gamma \ln^{-l}(s+e)ds \leq c(\gamma,l)t^{\gamma+1/2}\ln^{-l}(t+e), \quad \forall t \geq 1.$$

Therefore, by Lemma 4.1, we have

$$\int_0^t a^2(s)ds \leq c(v_0)t + c(v_0,c_d,m)t^{\frac{3}{2}}\ln^{-2m}(t+e) \leq c(v_0,c_d,m)t^{\frac{3}{2}}\ln^{-2m}(t+e)$$

for all $t \geq 1$.

Now, we can estimate $K(t)$ for $t \geq 1$. Indeed,

$$K(t) \leq c(v_0,c_d,m)g^6(t)t^2\ln^{2m}(t+e) + c(v_0)g^5(t) \leq c(v_0,c_d,m)g^6(t)t^2\ln^{2m}(t+e)$$

for all $t \geq 1$.

For $0 < t \leq 1$, we have

$$\int_0^t a^2(s)ds \leq c(v_0)$$

, and thus

$$K(t) \leq c(v_0,c_d,m)g^5(t).$$

Now, we can find estimates y_1 and y_2. Let us start with y_2:

$$y_2(t) \leq y_0\int_0^t \frac{h'(s)}{h(s)}ds = y(0)\frac{h(0)}{h(t)} \leq c(v_0,m)\min\Big\{\frac{1}{\ln^{2m+2}(1+e)}, \frac{1}{\ln^{2m+2}(t+e)}\Big\}.$$

Now, we shall treat y_1. For $0 < t \leq 1$, it can be done easily. So that, we get $y_1(t) \leq c(v_0,m)$ for this time interval.

What happens if $t \geq 1$? By the choice of the function g, we have

$$y_1(t) \leq \frac{1}{h(t)}\int_0^t h(s)K(s)ds \leq \frac{1}{h(t)}\Big[\int_0^1 h(s)K(s)ds + \int_1^t h(s)K(s)ds\Big]$$

$$\leq \frac{c(v_0,c_d,m)}{h(t)}\Big[1 + \int_0^t h(s)g^6(s)s^2\ln^{-2m}(s+e)ds\Big].$$

The second term has been evaluated in the paper [3]. Therefore, finally, we find

$$y_1(t) \leq c(v_0,c_d,m)\ln^{-2m-2}(t+2)$$

for all $t \geq 1$. So, induction for L_2-norm is proved.

Now, we need to prove our statement for L_1-norm. To this end, we need to consider $A_{\frac{6}{5}+\varepsilon}(t)$. For $0 < t \leq 1$, the estimate is simple: $A_{\frac{6}{5}+\varepsilon}(t) \leq c(v_0,c_d,m)$.

In the case $t \geq 1$, we can use Lemma 4.1. Indeed,

$$A_{\frac{6}{5}+\varepsilon}(t) \leq c(v_0,c_d,m)(1 + \int_1^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{25\varepsilon}{2(6+5\varepsilon)}}\frac{1}{\ln^{m+1}(s+e)} \leq c(v_0,c_d,m)t^{\frac{12-15\varepsilon}{4(6+5\varepsilon)}}\frac{1}{\ln^{m+1}(t+e)}.$$

Then, for $t \geq 1$, by Lemma 4.1,

$$\|v^2(\cdot,t)\|_1 \leq c(v_0,c_d,m)\Big(1+\int\limits_0^t \frac{ds}{\sqrt{t-s}} s^{\frac{-3+10\varepsilon}{2(6+5\varepsilon)}} A_{\frac{6}{5}+\varepsilon}(s)\Big)$$

$$\leq c(v_0,c_d,m)\Big(1+\int\limits_1^t \frac{ds}{\sqrt{t-s}} s^{\frac{1}{4}} \frac{1}{\ln^{m+1}(t+e)}\Big) leqc(v_0,c_d,m)\frac{t^{\frac{3}{2}}}{\ln^{m+1}(t+e)}$$

for $t \geq 1$. Theorem 1.2 is proven.

Acknowledgment

The work is supported by the grant RFBR 20-01-00397.

References

[1] Caffarelli L, Kohn RV, Nirenberg L. Partial regularity of suitable weak solutions of the Navier-Stokes equations, Communications on Pure and Applied Mathematics Vol. XXXV (1982), pp. 771–831.

[2] Koch G, Nadirashvili N, Seregin G, Šverák V. Liouville theorems for the Navier-Stokes equations and applications. Acta Mathematica 203 (2009), no. 1, 83–105.

[3] Schonbek ME, Seregin G. Time decay for solutions to the Stokes equations with drift. Communications in Contemporary Mathematics 20 (2018), no. 3, 1750046, 19 pp.

[4] Seregin G, Šverák V. On Type I singularities of the local axi-symmetric solutions of the Navier-Stokes equations, Communications in PDE's, 34(2009), pp. 171-201.

[5] Vazquez JL. Asymptotic behaviour methods for the Heat Equation. Convergence to the Gaussian. arXiv:1706.10034v3 [math.AP] 22 November 2018.