Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2023) 47: 910 – 933

doi:10.55730/1300-0098.3403

Global regularity for the 3D axisymmetric incompressible Hall-MHD system with partial dissipation and diffusion

Meilin JIN[1], **Quansen JIU**[1,*], **Huan YU**[2]
[1]School of Mathematical Sciences, Capital Normal University, Beijing, China
[2]School of Applied Science, Beijing Information Science and Technology University, Beijing, China

Received: 30.01.2022 • **Accepted/Published Online:** 10.01.2023 • **Final Version:** 17.03.2023

Abstract: In this paper, we study the Cauchy problem for the 3D incompressible axisymmetric Hall-MHD system with horizontal velocity dissipation and vertical magnetic diffusion. We obtain a unique global smooth solution of which in the cylindrical coordinate system the swirl velocity fields, the radial and the vertical components of the magnetic fields are trivial. This type of solution has been studied for the MHD system in [17], [16] and [15] and for the Hall-MHD system with total dissipation and diffusion in [11]. Some new and fine estimates are obtained in this paper to overcome the difficulties raised from the Hall term and the loss of vertical velocity dissipation and horizontal magnetic diffusion. Finally we can show that the estimates $\int_0^T \|\nabla u(t)\|_{L^\infty} dt$ and $\int_0^T \|\nabla b(t)\|_{L^\infty} dt$ are finite in a priori way and hence obtain the global well-posedness to the system under considered.

Key words: Hall-magnetohydrodynamics system, global regularity, axis-symmetric solutions, horizontal dissipation, vertical magnetic diffusion

1. Introduction

The three-dimensional incompressible Hall-magnetohydrodynamics (Hall-MHD) system reads as

$$\begin{cases} \partial_t u - \mu_x \partial_{xx} u - \mu_y \partial_{yy} u - \mu_z \partial_{zz} u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t b - \gamma_x \partial_{xx} b - \gamma_y \partial_{yy} b - \gamma_z \partial_{zz} b + u \cdot \nabla b + \nabla \times ((\nabla \times b) \times b) = b \cdot \nabla u, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \nabla \cdot u = \nabla \cdot b = 0, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $b = b(x,t)$ denote the velocity and magnetic fields, respectively, and p is a scalar pressure and μ_x, μ_y, μ_z, κ_x, κ_y, κ_z are nonnegative real parameters which denoting the fluid viscosity, resistivity (electrical diffusivity) in all directions, respectively. In the following context, for simplicity, we denote $\pi = p + \frac{1}{2}|b|^2$. Moreover, if b=0, this system becomes the incompressible Navier-Stokes equations. Moreover, the initial data to (1.1) are imposed as

$$(u(x,t), b(x,t))|_{t=0} = (u_0(x), b_0(x)), \ x \in \mathbb{R}^3. \tag{1.2}$$

The Hall-MHD system (1.1) is widely used in current physics, such as those of magnetic reconnection in space plasmas, star formation, neutron stars, and geodynamos (see [7, 8, 13, 14, 22]). In comparison with standard

*Correspondence: jiuqs@cnu.edu.cn
2010 *AMS Mathematics Subject Classification:* 35Q35, 76D05,76D03

MHD system, there appears a Hall term $\nabla\times((\nabla\times b)\times b)$ in Hall-MHD system $(1.1)_2$. The Hall effect plays a significant role in capturing the essential characteristics of the magnetohydrodynamics with strong magnetic reconnection. Lighthill pioneered the systematic study of the Hall-MHD system in [19]. Acheritogaray et al. in [1] formally derived the Hall-MHD system both from a two fluids system and a kinetic model. Then, in [3], Chae et al. showed the global existence of weak solutions and the local well-posedness for initial data $(u_0, b_0) \in H^s(\mathbb{R}^3)\times H^s(\mathbb{R}^3),\ s > \frac{5}{2}$. Chae and Lee [4] obtained blow-up criteria for smooth solutions. In [23], Wan and Zhou weakened the initial condition in [3] to $(u_0, b_0) \in H^s(\mathbb{R}^3)\times H^s(\mathbb{R}^3),\ s > \frac{3}{2}$. Furthermore, Dai in [10] showed that the solution is locally well-posed for initial data $(u_0, b_0) \in H^s(\mathbb{R}^3)\times H^{s+1-\varepsilon}(\mathbb{R}^3)$ for $s > \frac{1}{2}+\varepsilon$ with $\varepsilon > 0$ an arbitrary small number. Danchin and Tan [9] obtained the global well-posedness for small initial conditions u_0, B_0 and $\nabla\times B_0$ in critical spaces $\dot{B}^{\frac{3}{p}-1}_{p,1}$ with $1 \le p < \infty$. For the case $\mu_z = 0$ and $\mu_x = \mu_y = \kappa_x = \kappa_x = \kappa_z = 1$ in $(1.1)_1$, Fei and Xiang in [12] investigated the global well-posedness of the smooth solution for small initial data $(u_0, b_0) \in H^3(\mathbb{R}^3)$. In [6], it is shown that the nonresistive system ($\kappa_x = \kappa_y = \kappa_z = 0$ in the system (1.1)) is not well-posed in any Sobolev space $H^m(\mathbb{R})$ with $m > \frac{7}{2}$ in the sense that either it is locally ill-posed or it is locally well-posed but there exists an axisymmetric solution that loses the initial regularity in finite time.

In this paper, we consider the global regularity for the 3D axis-symmetric Hall-MHD system with $\mu_z = \gamma_x = \gamma_y = 0$ and $\mu_x = \mu_y = \gamma_z = 1$ in (1.1), that is

$$\begin{cases} \partial_t u - \Delta_h u + u\cdot\nabla u + \nabla\pi = b\cdot\nabla b, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t b - \partial_{zz} b + u\cdot\nabla b + \nabla\times((\nabla\times b)\times b) = b\cdot\nabla u, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \nabla\cdot u = \nabla\cdot b = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3. \end{cases} \tag{1.3}$$

A vector field $u(x,t)$ is called axis-symmetric if it can be written as

$$u(x,t) = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z,$$

where

$$e_r = (cos\theta, sin\theta, 0),\ e_\theta = (-sin\theta, cos\theta, 0),\ e_z = (0,0,1),$$

which means that $u(x,t)$ does not depend on the θ coordinate in the cylindrical coordinate systems. Moreover, an axis-symmetric vector field u without swirl means that $u_\theta = 0$. In this paper, we are concerned with the solutions satisfying:

$$u(r,z,t) = u_r(r,z,t)e_r + u_z(r,z,t)e_z,\ b(r,z,t) = b_\theta(r,z,t)e_\theta. \tag{1.4}$$

This type of solution was first studied by Lei in [17] on the MHD system and was further studied in [16] and [15]. And Fan et al. [11] showed the global well-posedness to the Hall-MHD system with total velocity dissipation and magnetic diffusion under assumptions of (1.4). Notice that the gradient operator ∇ and the Laplacian operator Δ in the cylindrical coordinates are

$$\nabla = e_r\partial_r + \frac{1}{r}e_\theta\partial_\theta + e_z\partial_z,\ \Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2.$$

Thus, under assumptions (1.4), direct calculation yields

$$\nabla\times(\nabla\times((\nabla\times b)\times b)) = -\frac{2}{r}(\partial_z b_\theta\cdot\partial_z b_\theta + b_\theta\partial_z^2 b_\theta)e_r + \frac{2}{r}(\partial_r b_\theta\cdot\partial_z b_\theta + b_\theta\cdot\partial_r\partial_z b_\theta)e_z.$$

In the rest of this article, we denote $u_r(r,z,t)$ by u_r for convenience and others are similar. Then, under assumptions (1.4), the system (1.3) can be rewritten as

$$\begin{cases} \partial_t u_r - (\Delta_h - \frac{1}{r^2})u_r + u\cdot\nabla u_r = -\partial_r\pi - \frac{b_\theta^2}{r}, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t u_z - \Delta_h u_z + u\cdot\nabla u_z = -\partial_z\pi, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t b_\theta - \partial_{zz}b_\theta + u\cdot\nabla b_\theta = \frac{u_r b_\theta}{r} + \frac{\partial_z b_\theta^2}{r}, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ (u,b)|_{t=0} = (u_0,b_0), \quad x\in\mathbb{R}^3, \end{cases} \tag{1.5}$$

where $\Delta_h = \partial_r^2 + \frac{1}{r}\partial_r$. It is easy to verify that if the initial data satisfy (1.4), then the smooth solution to (1.5) will keep the form (1.4).

Throughout the paper, we write $\mathbb{R}^3 = \mathbb{R}_h^2\times\mathbb{R}_v^1$, and we agree that $\nabla_h = (\partial_1,\partial_2,0)$. The main result of the paper can be stated as follows.

Theorem 1.1 *Let* $(u_0,b_0)\in H^2(\mathbb{R}^3)\times H^2(\mathbb{R}^3)$ *be axis-symmetric divergence free vector fields satisfying* $u_0^\theta = b_0^r = b_0^z = 0$ *and*

$$\nabla b_0^\theta \in L^\infty(\mathbb{R}^3),\ \frac{b_0^\theta}{r}\in L^\infty(\mathbb{R}^3),\ \frac{b_0^\theta}{r^2}\in L^\infty(\mathbb{R}^3). \tag{1.6}$$

Then the system (1.3) with the initial data (u_0,b_0) *has a unique global classical solution* (u,b) *satisfying (1.4), and for any* $T>0$,

$$(u,b)\in\mathcal{C}([0,T];H^2(\mathbb{R}^3)),\ (\nabla_h\omega,\partial_z k)\in L^\infty([0,T];H^1(\mathbb{R}^3)), \tag{1.7}$$

where $\omega = \nabla\times u$ *and* $k=\nabla\times b$.

New difficulties will be encountered in this paper. The first one is due to the partial dissipation and magnetic diffusion and hence much more complicated estimates are required when we make higher regularity estimates. The second one is due to the Hall term $\nabla\times((\nabla\times b)\times b)$ which is quadratic in the magnetic field and involves the second-order derivatives. As an example, intend to obtain the H^2 estimates of the solution, we invoke the vorticity and current equations to yield

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 + \|\nabla_h\partial_z\omega\|_{L^2}^2 + \|\nabla_h^2\omega\|_{L^2}^2 = \int_{\mathbb{R}^3}(u\cdot\nabla\omega - \frac{u_r\omega}{r})\Delta\omega\,dx + \int_{\mathbb{R}^3}\frac{\partial_z b_\theta^2}{r}e_\theta\cdot\Delta\omega\,dx \tag{1.8}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla k\|_{L^2}^2 + \|\nabla_h\partial_z k\|_{L^2}^2 + \|\partial_{zz}k\|_{L^2}^2 = \int_{\mathbb{R}^3}u\cdot\nabla k\cdot\Delta k\,dx - \int_{\mathbb{R}^3}\alpha\cdot\Delta k\,dx - \int_{\mathbb{R}^3}\beta\cdot\Delta k\,dx, \tag{1.9}$$

where

$$\alpha = [\partial_z u_r\cdot\partial_r b_\theta + \partial_z u_z\cdot\partial_z b_\theta - \frac{1}{r}\partial_z u_r b_\theta + \frac{1}{r}\partial_z^2 b_\theta^2]e_r,$$

$$\beta = [u\cdot\nabla(\frac{b_\theta}{r}) + \frac{1}{r}\partial_r(u_r b_\theta) - \frac{1}{r}\partial_r(ru)\cdot\nabla b_\theta - \frac{1}{r}\partial_r(\partial_z b_\theta^2)]e_z.$$

It will be quite difficult and complicated to close the H^2 estimates since on the left hands of (1.8) and (1.9) there are only partial higher dissipation and magnetic diffusion, while in the end of the expressions of α and β

there appear higher order derivatives of the solution which are from the Hall term. In order to overcome these difficulties, we will prove a new estimate of $\|\frac{b_\theta}{r^2}\|_{L^\infty}$ under the assumptions of Theorem 1.1 (see Lemma 2.6). And we will apply the anisotropic inequalities (see Lemma 2.3) and obtain some delicate estimates to close the H^2 estimates of (u,b). We refer to Lemma 3.2 in Section 4 for more details. In comparison with [11] in which full dissipation and magnetic diffusion are dealt with, and [18] in which horizontal dissipation and full magnetic diffusion are dealt with, we are concerned with the Hall-MHD system with only horizontal velocity dissipation and vertical magnetic diffusion.

This paper is organized as follows. In Sections 2 and 3, we will focus on making H^1 and H^2 estimates respectively. Section 4 is devoted to the proof of the main theorem.

2. H^1 estimates

In this section, we will give a priori H^1-estimates of the solution. Before we make estimates, we first state some useful facts which will be used later.

Lemma 2.1 *([5], [20], [21]) Suppose that u is a smooth and axisymmetric velocity field defined in $\mathbb{R}^3$, satisfying $divu=0$ and $\omega=\nabla\times u=\omega_\theta e_\theta$. Then it holds that*

$$(1)\ \|\nabla\widetilde{u}\|_{L^p}\le C\|\omega_\theta\|_{L^p},\ p\in(1,\infty),\qquad (2)\ \|\frac{u_r}{r}\|_{L^\infty}\le C\|\frac{\omega_\theta}{r}\|_{L^2}^{\frac12}\|\nabla_h(\frac{\omega_\theta}{r})\|_{L^2}^{\frac12},$$

$$(3)\ \|\partial_z(\frac{u_r}{r})\|_{L^p}\le C\|\frac{\omega_\theta}{r}\|_{L^p},\ p\in(1,\infty),\qquad (4)\ \|\frac{u_r}{r}\|_{L^{\frac{3p}{3-p}}}\le C\|\frac{\omega_\theta}{r}\|_{L^p},\ p\in(1,3),$$

where $\widetilde{u}=u_re_r+u_ze_z$.

Lemma 2.2 *([20]) For any smooth scalar function or vector fields u, there exists a constants C such that*

$$\|u\|_{L^\infty}\le C\|\nabla u\|_{L^2}^{\frac12}\|\nabla_h\nabla u\|_{L^2}^{\frac12}. \tag{2.1}$$

For the convenience of readers, the proof is given as follows.

Proof. It suffices to prove 2.1 for the scalar function.

By using the interpolation theorem, we get

$$\|u(x_h,\cdot)\|_{L^\infty(\mathbb{R}^1_v)}\le C\|u(x_h,\cdot)\|_{L^6(\mathbb{R}^1_v)}^{\frac12}\|\Lambda_v^{\frac23}u(x_h,\cdot)\|_{L^2(\mathbb{R}^1_v)}^{\frac12},$$

where $\Lambda:=(-\Delta)^{\frac12}$ is the fractional Laplacian operator, defined through the Fourier transform as

$$\mathcal{F}(\Lambda^\gamma f)(\xi)=|\xi|^\gamma\mathcal{F}(f)(\xi).$$

This together with the Minkowski inequality and the embedding theorem gives

$$\begin{aligned}\|u\|_{L^\infty(\mathbb{R}^3)}\le&C\|\|u\|_{L^\infty(\mathbb{R}^1_v)}\|_{L^\infty(\mathbb{R}^2_h)}\\ \le&C\|\|u\|_{L^6(\mathbb{R}^1_v)}^{\frac12}\|\Lambda_v^{\frac23}u\|_{L^2(\mathbb{R}^1_v)}^{\frac12}\|_{L^\infty(\mathbb{R}^2_h)}\\ \le&C\|\|\Lambda_v^{\frac13}u\|_{L^2(\mathbb{R}^1_v)}^{\frac12}\|\Lambda_v^{\frac23}u\|_{L^2(\mathbb{R}^1_v)}^{\frac12}\|_{L^\infty(\mathbb{R}^2_h)}\\ \le&C\|\|\Lambda_v^{\frac13}u\|_{L^\infty(\mathbb{R}^2_h)}\|_{L^2(\mathbb{R}^1_v)}^{\frac12}\|\|\Lambda_v^{\frac23}u\|_{L^\infty(\mathbb{R}^2_h)}\|_{L^2(\mathbb{R}^1_v)}^{\frac12}.\end{aligned} \tag{2.2}$$

By the same way, we obtain

$$\begin{aligned}\|\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^\infty(\mathbb{R}_h^2)} \le& C\|\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^6(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}\\ \le& C\|\Lambda_h^{\frac{2}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}},\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\|\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^\infty(\mathbb{R}_h^2)} \le& C\|\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^6(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}\\ \le& C\|\Lambda_h^{\frac{1}{3}}\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}.\end{aligned} \tag{2.4}$$

Inserting (2.3) and (2.4) into (2.2), and using the Hölder inequality, we get

$$\begin{aligned}&\|u\|_{L^\infty(\mathbb{R}^3)}\\ \le& C\|\|\Lambda_v^{\frac{1}{3}}u\|_{L^\infty(\mathbb{R}_h^2)}\|_{L^2(\mathbb{R}_v^1)}^{\frac{1}{2}}\|\|\Lambda_v^{\frac{2}{3}}u\|_{L^\infty(\mathbb{R}_h^2)}\|_{L^2(\mathbb{R}_v^1)}^{\frac{1}{2}}\\ \le& C\|\|\Lambda_h^{\frac{2}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}\|_{L^2(\mathbb{R}_v^1)}^{\frac{1}{2}}\|\|\Lambda_h^{\frac{1}{3}}\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}\|_{L^2(\mathbb{R}_v^1)}^{\frac{1}{2}}\\ \le& C\|\Lambda_h^{\frac{2}{3}}\Lambda_v^{\frac{1}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{6}}\|\Lambda_h^{\frac{1}{3}}\Lambda_v^{\frac{2}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{6}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{2}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}}\\ \le& C\|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.\end{aligned}$$

□

The following are some anisotropy inequalities.

Lemma 2.3 *([20]) Suppose that f, g, h are smooth functions in* $\mathbb{R}^3$*. Then it holds that*

$$(1)\int_{\mathbb{R}^3}|fgh|dxdydz \le C\|f\|_{L^{2(p-1)}}^{\frac{p-1}{p}}\|\partial_x f\|_{L^2}^{\frac{1}{p}}\|g\|_{L^2}^{\frac{p-2}{p}}\|\partial_y g\|_{L^2}^{\frac{1}{p}}\|\partial_z g\|_{L^2}^{\frac{1}{p}}\|h\|_{L^2},\ p\in(2,\infty),$$

$$(2)\int_{\mathbb{R}^3}|fgh|dxdydz \le C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_z f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\nabla_h g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\nabla_h h\|_{L^2}^{\frac{1}{2}},$$

$$(3)\int_{\mathbb{R}^3}|fgh|dxdydz \le C\|f\|_{L^6}^{\frac{3}{4}}\|\partial_z f\|_{L^2}^{\frac{1}{4}}\|g\|_{L^2}^{\frac{1}{2}}\|\nabla_h g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}.$$

In the following, we will make use of the vorticity and the current equations to make a priori estimates. It is noted that under assumption (1.4), the vorticity of the velocity is

$$\omega = \nabla\times u = \omega_\theta e_\theta,$$

where

$$\omega_\theta = \partial_z u_r - \partial_r u_z.$$

And the current of the magnetic fields is $k = \nabla\times b = k_r e_r + k_z e_z$ *with* $k_r = -\partial_z b_\theta,\ k_z = \frac{1}{r}\partial_r(rb_\theta)$. One can rewrite the equations of the vorticity and the current as

$$\begin{cases} \partial_t\omega_\theta - (\Delta_h - \frac{1}{r^2})\omega_\theta + u\cdot\nabla\omega_\theta = \frac{u_r}{r}\cdot\omega_\theta - k\cdot\nabla b_\theta + \frac{b_\theta k_r}{r}, \\ \partial_t k_r - \partial_{zz}k_r + u\cdot\nabla k_r = \frac{u_r}{r}\cdot k_r + \partial_z u\cdot\nabla b_\theta - \frac{b_\theta}{r}\partial_z u_r + 2(k_r^2 - b_\theta\partial_z k_r), \\ \partial_t k_z - \partial_{zz}k_z + u\cdot\nabla k_z = u\cdot\nabla(\frac{b_\theta}{r}) + \frac{1}{r}\partial_r(u_r b_\theta) - \frac{1}{r}\partial_r(ru)\cdot\nabla b_\theta \\ \qquad\qquad - \frac{2}{r}\left[-k_r\cdot b_\theta - r\partial_r k_r\cdot b_\theta - rk_r\cdot(\partial_r b_\theta)\right]. \end{cases} \tag{2.5}$$

For the smooth solution to (1.3), it is easy to deduce

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \|\nabla_h u\|_{L^2}^2 + \|\partial_z b\|_{L^2}^2 = 0, \tag{2.6}$$

in which we have used the fact that

$$\int_{\mathbb{R}^3} b\cdot(\nabla\times((\nabla\times b)\times b))dx = \int_{\mathbb{R}^3}(\nabla\times b)\cdot((\nabla\times b)\times b)dx = 0. \tag{2.7}$$

Hence, the usual energy estimate is

$$\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + 2\int_0^T(\|\nabla_h u\|_{L^2}^2 + \|\partial_z b\|_{L^2}^2)dt \le C(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2), \tag{2.8}$$

where C is an absolute constant.

In the following, to make a priori estimates, we always assume that the solution to (1.3) is smooth. To achieve H^1 estimates, we first make a priori estimates of $\frac{b_\theta}{r}$ in $L^\infty([0,T],L^2)$ and $\frac{\omega_\theta}{r}$ in $L^\infty([0,T],L^2)$.

Lemma 2.4 *For any* $p\in(1,\infty]$, *one has*

$$\|\frac{b_\theta}{r}\|_{L^p} \le \|\frac{b_0^\theta}{r}\|_{L^p}, \tag{2.9}$$

where $b_\theta(x,0) = b_0^\theta$.

Proof. By virtue of $(1.5)_3$, the quantity $\Phi = \frac{b_\theta}{r}$ solves

$$\partial_t\Phi - \partial_{zz}\Phi + u\cdot\nabla\Phi = 2\Phi\partial_z\Phi. \tag{2.10}$$

It follows that

$$\|\Phi(t)\|_{L^p} \le \|\Phi_0\|_{L^p},\ p\in(1,\infty).$$

Letting $p\to\infty$ yields

$$\|\Phi(t)\|_{L^\infty} \le \|\Phi_0\|_{L^\infty}.$$

Thus, we have derived

$$\|\Phi(t)\|_{L^p} \le \|\Phi(0)\|_{L^p},\ p\in(1,\infty].$$

□

Lemma 2.5 *Suppose that the assumptions of Theorem 1.1 hold true. Then we have*

$$\|\frac{\omega_\theta}{r}\|_{L^2}^2+\|\frac{b_\theta}{r}\|_{L^2}^2+\int_0^T(\|\nabla_h(\frac{\omega_\theta}{r})\|_{L^2}^2+\|\partial_z(\frac{b_\theta}{r})\|_{L^2}^2)dt\le C(\|u_0\|_{H^2}^2+\|b_0\|_{H^1}^2)exp^{\|\frac{b_0^\theta}{r}\|_{L^\infty}^2T},$$

where $b^\theta(x,0)=b_0^\theta,\ \omega^\theta(x,0)=\omega_0^\theta$ *and* C *is an absolute constant.*

Proof. Let us rewrite the vorticity equation in $(2.5)_1$ in terms of $\Gamma=\frac{\omega_\theta}{r}$ as follows:

$$\partial_t\Gamma+u\cdot\nabla\Gamma=(\Delta_h+\frac{2}{r}\partial_r)\Gamma-\partial_z\Phi^2. \tag{2.11}$$

Taking inner product of (2.10) and (2.11) with $\Phi,\ \Gamma$ respectively, and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\Gamma\|_{L^2}^2+\|\Phi\|_{L^2}^2)+(\|\nabla_h\Gamma\|_{L^2}^2+\|\partial_z\Phi\|_{L^2}^2)=&-\int_{\mathbb{R}^3}\Gamma\partial_z\Phi^2dx\\ \le&\|\Gamma\|_{L^2}\|\Phi\|_{L^\infty}\|\partial_z\Phi\|_{L^2}\\ \le&\frac{1}{2}\|\partial_z\Phi\|_{L^2}^2+\frac{1}{2}\|\Gamma\|_{L^2}^2\|\Phi\|_{L^\infty}^2.\end{aligned}$$

Using $u_0\in H^2$, it holds that

$$|\nabla(\nabla\times u)|^2=|(e_r\partial_r+\frac{1}{r}e_\theta\partial_\theta+e_\theta\partial_z)\omega^\theta e_\theta|^2=|\nabla\omega^\theta|^2+|\Gamma|^2.$$

Hence, we have

$$\|\Gamma_0\|_{L^2}\le\|u_0\|_{H^2}.$$

Furthermore, noting that

$$|\nabla b|^2=|(e_r\partial_r+\frac{1}{r}e_\theta\partial_\theta+e_\theta\partial_z)b^\theta e_\theta|^2=|\nabla b^\theta|^2+|\Phi|^2,$$

and using Sobolev imbedding, one also has

$$\|\Phi_0\|_{L^2}\le\|b_0\|_{H^1}.$$

Finally, using the Gronwall inequality, one can reach that

$$\begin{aligned}\|\Gamma\|_{L^2}^2+\|\Phi\|_{L^2}^2+\int_0^T(\|\nabla_h\Gamma\|_{L^2}^2+\|\partial_z\Phi\|_{L^2}^2)dt\le& C(\|\Gamma_0\|_{L^2}^2+\|\Phi_0\|_{L^2}^2)exp^{\|\frac{b_0^\theta}{r}\|_{L^\infty}^2T}\\ \le& C(\|u_0\|_{H^2}^2+\|b_0\|_{H^1}^2)exp^{\|\frac{b_0^\theta}{r}\|_{L^\infty}^2T},\end{aligned}$$

where C is an absolute constant. □

Lemma 2.6 *For any* $p\in(1,\infty]$, *one has*

$$\|\frac{b_\theta}{r^2}\|_{L^p}\le C\|\frac{b_0^\theta}{r^2}\|_{L^p}, \tag{2.12}$$

where $b_\theta(x,0)=b_0^\theta$, *and* C *depends on* T, $\|u_0\|_{H^2}$ *and* $\|\frac{b_0^\theta}{r}\|_{L^\infty}$.

Proof. By virtue of $(1.5)_3$, the quantity $\frac{b_\theta}{r^2}$ solves

$$\partial_t(\frac{b_\theta}{r^2}) - \partial_{zz}(\frac{b_\theta}{r^2}) + u\cdot\nabla(\frac{b_\theta}{r^2}) = -\frac{u_r}{r}\cdot\frac{b_\theta}{r^2} + 2\frac{\partial_z b_\theta}{r}\cdot\frac{b_\theta}{r^2}. \tag{2.13}$$

Taking inner product of (2.13) with $|\frac{b_\theta}{r^2}|^{p-2}(\frac{b_\theta}{r^2})$, and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\frac{b_\theta}{r^2}\|_{L^p}^p + \frac{2(p-1)}{p}\int_{\mathbb{R}^3}[\partial_z(\frac{b_\theta}{r^2})^{\frac{p}{2}}]^2\ dx\\
=&-\int_{\mathbb{R}^3}\frac{u_r}{r}\cdot\frac{b_\theta}{r^2}\cdot|\frac{b_\theta}{r^2}|^{p-2}\cdot\frac{b_\theta}{r^2}\ dx + \frac{2}{p}\int_{\mathbb{R}^3}\frac{b_\theta}{r}\cdot\partial_z|\frac{b_\theta}{r^2}|^p\ dx\\
\le&\|\frac{\omega_\theta}{r}\|_{L^2}^{\frac{1}{2}}\|\nabla_h(\frac{\omega_\theta}{r})\|_{L^2}^{\frac{1}{2}}\|\frac{b_\theta}{r^2}\|_{L^p}^p + \frac{2}{p}\|\frac{b_\theta}{r}\|_{L^\infty}(\int_{\mathbb{R}^3}[\partial_z(\frac{b_\theta}{r^2})^{\frac{p}{2}}]^2\ dx)^{\frac{1}{2}}\|\frac{b_\theta}{r^2}\|_{L^p}^{\frac{p}{2}}.
\end{aligned}$$

Using Lemmas 2.4-2.5 and Cauchy inequality, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\frac{b_\theta}{r^2}\|_{L^p}^p + \frac{2(p-1)}{p}\int_{\mathbb{R}^3}[\partial_z(\frac{b_\theta}{r^2})^{\frac{p}{2}}]^2\ dx\\
\le&C\|\frac{b_\theta}{r^2}\|_{L^p}^p + \frac{p-1}{p}\int_{\mathbb{R}^3}[\partial_z(\frac{b_\theta}{r^2})^{\frac{p}{2}}]^2\ dx + \frac{C}{p-1}\|\frac{b_\theta}{r^2}\|_{L^p}^p,
\end{aligned}$$

which implies, together with the Gronwall inequality, that

$$\|\frac{b_\theta}{r^2}\|_{L^p} \le C\|\frac{b_0^\theta}{r^2}\|_{L^p} \tag{2.14}$$

for any $p\in(1,\infty)$, where C is a constant depending on T, $\|u_0\|_{H^2}$ and $\|\frac{b_0^\theta}{r}\|_{L^\infty}$. Letting $p\to\infty$ yields (2.12) and the proof of the lemma is finished. □

Lemma 2.7 *Suppose that the assumptions of Theorem 1.1 hold true. Then we have*

$$\|b_\theta\|_{L^\infty} \le C\|b_0^\theta\|_{L^\infty} exp^{T^{\frac{3}{4}}(\|u_0\|_{H^2}^2+\|b_0\|_{H^1}^2)exp^{T\|\frac{b_0^\theta}{r}\|_{L^\infty}^2}}, \tag{2.15}$$

where $b^\theta(x,0)=b_0^\theta,\ \omega^\theta(x,0)=\omega_0^\theta$ *and* C *is an absolute constant.*

Proof. By Lemma 2.1 and Lemma 2.5, it holds that

$$\begin{aligned}
&\int_0^T\|\frac{u_r}{r}\|_{L^\infty}dt\\
\le&C\sup_{0\le t\le T}\|\frac{\omega_\theta}{r}\|_{L^2}^{\frac{1}{2}}\int_0^T\|\nabla_h(\frac{\omega_\theta}{r})\|_{L^2}^{\frac{1}{2}}dt,\\
\le&CT^{\frac{3}{4}}\sup_{0\le t\le T}\|\frac{\omega_\theta}{r}\|_{L^2}^{\frac{1}{2}}(\int_0^T\|\nabla_h(\frac{\omega_\theta}{r})\|_{L^2}^2dt)^{\frac{1}{4}}\\
\le&CT^{\frac{3}{4}}(\|\frac{\omega_0^\theta}{r}\|_{L^2}^2 + \|\frac{b_0^\theta}{r}\|_{L^2}^2)exp^{T\|\frac{b_0^\theta}{r}\|_{L^\infty}^2},
\end{aligned} \tag{2.16}$$

where C is an absolute constant.

For $p>1$, taking inner product of $(1.5)_3$ with $|b_\theta|^{p-2}b_\theta$ and integrating on $\mathbb{R}^3$, noting that

$$\int_{\mathbb{R}^3}\frac{1}{r}\partial_z b_\theta^2\,|b_\theta|^{p-2}\,b_\theta dx=0,$$

we finally obtain

$$\frac{d}{dt}\|b_\theta\|_{L^p}\leq\|\frac{u_r}{r}\|_{L^\infty}\|b_\theta\|_{L^p}. \tag{2.17}$$

It follows from Gronwall inequality that

$$\|b_\theta\|_{L^p}\leq\|b_0^\theta\|_{L^p}e^{\int_0^T\|\frac{u_r}{r}\|_{L^\infty}dt}. \tag{2.18}$$

Letting $p\to\infty$ derives

$$\begin{aligned}\|b_\theta\|_{L^\infty}\leq&\|b_0^\theta\|_{L^\infty}e^{\int_0^T\|\frac{u_r}{r}\|_{L^\infty}dt}\\ \leq&C\|b_0^\theta\|_{L^\infty}exp^{T^{\frac{3}{4}}(\|\frac{\omega_0^\theta}{r}\|_{L^2}^2+\|\frac{b_0^\theta}{r}\|_{L^2}^2)exp^{T\|\frac{b_0^\theta}{r}\|_{L^\infty}^2}}\\ \leq&C\|b_0^\theta\|_{L^\infty}exp^{T^{\frac{3}{4}}(\|u_0\|_{H^2}^2+\|b_0\|_{H^1}^2)exp^{T\|\frac{b_0^\theta}{r}\|_{L^\infty}^2}},\end{aligned}$$

where C is an absolute constant. □

Lemma 2.8 *Suppose that the assumptions of Theorem 1.1 hold true. Then we have*

$$\|\omega\|_{L^2}^2+\|k\|_{L^2}^2+\int_0^T(\|\nabla_h\omega\|_{L^2}^2+\|\partial_z k\|_{L^2}^2)dt\leq C_1, \tag{2.19}$$

where the constant C_1 depends only on T, $\|u_0\|_{H^2}$, $\|b_0\|_{H^2}$, $\|\frac{b_0^\theta}{r}\|_{L^\infty}$.

Proof. Taking inner product of $(2.5)_1$ with ω_θ and integrating on $\mathbb{R}^3$ lead to

$$\frac{1}{2}\frac{d}{dt}\|\omega_\theta\|_{L^2}^2+\|\nabla_h\omega_\theta\|_{L^2}^2+\|\frac{\omega_\theta}{r}\|_{L^2}^2=\int_{\mathbb{R}^3}\frac{\omega_\theta^2u_r}{r}dx-\int_{\mathbb{R}^3}\frac{\omega_\theta\partial_z b_\theta^2}{r}dx,$$

Using Lemma 2.3 and Lemma 2.6, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\omega_\theta\|_{L^2}^2+\|\nabla_h\omega_\theta\|_{L^2}^2+\|\frac{\omega_\theta}{r}\|_{L^2}^2,\\ \leq&\|\frac{u_r}{r}\|_{L^6}^{\frac{3}{4}}\|\partial_z(\frac{u_r}{r})\|_{L^2}^{\frac{1}{4}}\|\omega_\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\omega_\theta\|_{L^2}^{\frac{1}{2}}\|\omega_\theta\|_{L^2}+\|\frac{b_\theta}{r}\|_{L^\infty}\|\partial_z b_\theta\|_{L^2}\|\omega_\theta\|_{L^2}\\ \leq&\|\frac{\omega_\theta}{r}\|_{L^2}\|\omega_\theta\|_{L^2}^{\frac{3}{2}}\|\nabla_h\omega_\theta\|_{L^2}^{\frac{1}{2}}+\|\partial_z b_\theta\|_{L^2}^2+\|\frac{b_\theta}{r}\|_{L^\infty}^2\|\omega_\theta\|_{L^2}^2\\ \leq&\frac{1}{2}\|\nabla_h\omega_\theta\|_{L^2}^2+C(\|\frac{\omega_\theta}{r}\|_{L^2}^{\frac{4}{3}}+\|\frac{b_\theta}{r}\|_{L^\infty}^2)\|\omega_\theta\|_{L^2}^2+\|\partial_z b_\theta\|_{L^2}^2.\end{aligned}$$

It follows from the Gronwall inequality that

$$
\begin{aligned}
&\|\omega_\theta\|_{L^2}^2 + \int_0^T (\|\nabla_h \omega_\theta\|_{L^2}^2 + \|\frac{\omega_\theta}{r}\|_{L^2}^2) dt \\
\leq & C\|\omega_0^\theta\|_{L^2}^2 e^{\int_0^T (\|\frac{\omega_\theta}{r}\|_{L^2}^{\frac{4}{3}} + \|\frac{b_\theta}{r}\|_{L^\infty}^2) dt} + \int_0^T \|\partial_z b_\theta\|_{L^2}^2 dt \\
\leq & C\|\omega_0^\theta\|_{L^2}^2 e^{T^{\frac{1}{3}} \int_0^T \|\frac{\omega_\theta}{r}\|_{L^2}^2 dt + T\|\frac{b_0^\theta}{r}\|_{L^\infty}^2} + \int_0^T \|\partial_z b_\theta\|_{L^2}^2 dt \\
\leq & C\|\omega_0^\theta\|_{L^2}^2 e^{T^{\frac{1}{3}} (\|\frac{\omega_0^\theta}{r}\|_{L^2}^2 + \|\frac{b_0^\theta}{r}\|_{L^2}^2) e^{T\|\frac{b_0^\theta}{r}\|_{L^\infty}^2} + T\|\frac{b_0^\theta}{r}\|_{L^\infty}^2} + T\|b_0^\theta\|_{L^2}^2.
\end{aligned}
\tag{2.20}
$$

Taking inner product of $(2.5)_2$ and $(2.5)_3$ with k_r, k_z and integrating on $\mathbb{R}^3$ respectively, it is easy to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|k_r\|_{L^2}^2 + \|k_z\|_{L^2}^2) + \|\partial_z k_r\|_{L^2}^2 + \|\partial_z k_z\|_{L^2}^2 \\
= & 2\int_{\mathbb{R}^3} \partial_z u_r \partial_r b_\theta \cdot k_r dx - \int_{\mathbb{R}^3} (2\partial_z u_z + \partial_r u_r) k_r k_r dx + \int_{\mathbb{R}^3} (\partial_r u_z - \partial_z u_r) k_r k_z dx \\
& + \int_{\mathbb{R}^3} u_r \partial_r (\frac{b_\theta}{r}) \cdot k_z dx + \int_{\mathbb{R}^3} (\frac{b_\theta}{r} - \partial_r b_\theta) \partial_r u_r \cdot k_z dx + 2\int_{\mathbb{R}^3} (k_r^2 - b_\theta \partial_z k_r) k_r dx \\
& + 2\int_{\mathbb{R}^3} [k_r \cdot \frac{b_\theta}{r} - \partial_z(\partial_r b_\theta) \cdot b_\theta + \partial_r b_\theta \cdot k_r] k_z dx \\
=: & \sum_{i=1}^{7} I_i.
\end{aligned}
\tag{2.21}
$$

In the following context, we make the estimate on each term on right hand side of (2.21) respectively. Applying Lemma 2.3, one can reach that

$$
\begin{aligned}
I_1 &= 2\int_{\mathbb{R}^3} \partial_z u_r \cdot \partial_r b_\theta \cdot k_r dx \\
&= -2\int_{\mathbb{R}^3} u_r \cdot \partial_z \partial_r b_\theta \cdot k_r dx - 2\int_{\mathbb{R}^3} u_r \cdot \partial_r b_\theta \cdot \partial_z k_r dx \\
&\leq \|u_r\|_{L^\infty} (\|k_r\|_{L^2} \|\partial_z k_z\|_{L^2} + \|k_z\|_{L^2} \|\partial_z k_r\|_{L^2}) \\
&\leq \frac{1}{10}(\|\partial_z k_z\|_{L^2}^2 + \|\partial_z k_r\|_{L^2}^2) + C\|u_r\|_{L^\infty}^2 (\|k_r\|_{L^2}^2 + \|k_z\|_{L^2}^2) \\
&\leq \frac{1}{10}(\|\partial_z k_z\|_{L^2}^2 + \|\partial_z k_r\|_{L^2}^2) + C\|\nabla u_r\|_{L^2} \|\nabla_h \nabla u_r\|_{L^2} (\|k_r\|_{L^2}^2 + \|k_z\|_{L^2}^2) \\
&\leq \frac{1}{10}(\|\partial_z k_z\|_{L^2}^2 + \|\partial_z k_r\|_{L^2}^2) + C(\|\omega_\theta\|_{L^2}^2 + \|\nabla_h \omega_\theta\|_{L^2}^2)(\|k_r\|_{L^2}^2 + \|k_z\|_{L^2}^2).
\end{aligned}
$$

For the term I_2, we have

$$
\begin{aligned}
I_2 &= -\int_{\mathbb{R}^3} \partial_z u_z k_r k_r dx + \int_{\mathbb{R}^3} \frac{u_r}{r} k_r k_r dx \\
&\le C\|u_z\|_{L^\infty}\|k_r\|_{L^2}\|\partial_z k_r\|_{L_2} + \|\frac{u_r}{r}\|_{L^\infty}\|k_r\|_{L^2}^2 \\
&\le C\|\nabla u_z\|_{L^2}^{\frac{1}{2}}\|\nabla_h \nabla u_z\|_{L^2}^{\frac{1}{2}}\|k_r\|_{L^2}\|\partial_z k_r\|_{L^2} + \|\frac{u_r}{r}\|_{L^\infty}\|k_r\|_{L^2}^2 \\
&\le \frac{1}{10}\|\partial_z k_r\|_{L^2}^2 + C\|\nabla u_z\|_{L^2}\|\nabla_h \nabla u_z\|_{L^2}\|k_r\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty}\|k_r\|_{L^2}^2 \\
&\le \frac{1}{10}\|\partial_z k_r\|_{L^2}^2 + C(\|\omega_\theta\|_{L^2}^2 + \|\nabla_h \omega_\theta\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty})\|k_r\|_{L^2}^2.
\end{aligned}
$$

For the term I_3, we have

$$
\begin{aligned}
I_3 =& -\int_{\mathbb{R}^3} \partial_r u_z \partial_z b_\theta k_z dx - \int_{\mathbb{R}^3} \partial_z u_r \partial_z k_r k_z dx \\
=& \int_{\mathbb{R}^3} (\partial_r \partial_z u_z) b_\theta k_z dx + \int_{\mathbb{R}^3} \partial_r u_z b_\theta \partial_z k_z dx + \int_{\mathbb{R}^3} u_r \partial_z k_z k_r dx + \int_{\mathbb{R}^3} u_r k_z \partial_z k_r dx \\
\le& \|b_\theta\|_{L^\infty}\|\nabla_h \omega_\theta\|_{L^2}\|k_z\|_{L^2} + \|b_\theta\|_{L^\infty}\|\omega_\theta\|_{L^2}\|\partial_z k_z\|_{L^2} + \|u_r\|_{L^\infty}\|k_r\|_{L^2}\|\partial_z k_z\|_{L^2} \\
& + \|u_r\|_{L^\infty}\|k_z\|_{L^2}\|\partial_z k_r\|_{L^2} \\
\le& \frac{1}{10}(\|\partial_z k_z\|_{L_2}^2 + \|\partial_z k_r\|_{L_2}^2) + \|b_\theta\|_{L^\infty}^2(1 + \|\omega_\theta\|_{L_2}^2) + C(\|\omega_\theta\|_{L_2}^2 + \|\nabla_h \omega_\theta\|_{L_2}^2)(\|k_r\|_{L_2}^2 + \|k_z\|_{L_2}^2).
\end{aligned}
$$

For the term I_4, we have

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} u_r \partial_r (\frac{b_\theta}{r}) \cdot k_z dx \\
&= \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_r b_\theta \cdot k_z dx - \int_{\mathbb{R}^3} \frac{u_r}{r} \frac{b_\theta}{r} \cdot k_z dx \\
&\le \|\frac{u_r}{r}\|_{L^\infty}\|k_z\|_{L^2}^2 + \|\frac{b_\theta}{r}\|_{L^\infty}^2 + \|\omega_\theta\|_{L^2}^2\|k_z\|_{L^2}^2.
\end{aligned}
$$

For the term I_5, we have

$$
\begin{aligned}
I_5 =& \int_{\mathbb{R}^3} (\frac{b_\theta}{r} - \partial_r b_\theta) \partial_r u_r \cdot k_z dx \\
=& -\int_{\mathbb{R}^3} (\frac{b_\theta}{r} - \partial_r b_\theta) \frac{u_r}{r} \cdot k_z dx - \int_{\mathbb{R}^3} (\frac{b_\theta}{r} - \partial_r b_\theta) \partial_z u_z \cdot k_z dx \\
=& -\int_{\mathbb{R}^3} (\frac{b_\theta}{r} - \partial_r b_\theta) \frac{u_r}{r} \cdot k_z dx + \int_{\mathbb{R}^3} \partial_z (\frac{b_\theta}{r} - \partial_r b_\theta) u_z \cdot k_z dx \\
& + \int_{\mathbb{R}^3} (\frac{b_\theta}{r} - \partial_r b_\theta) u_z \cdot \partial_z k_z dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \|\frac{u_r}{r}\|_{L^\infty}\|k_z\|_{L^2}^2 + C(\|u_z\|_{L^\infty}\|k_z\|_{L^2})\|\partial_z k_z\|_{L^2} \\
&\le \frac{1}{10}\|\partial_z k_z\|_{L^2}^2 + C\|\frac{u_r}{r}\|_{L^\infty}\|k_z\|_{L^2}^2 + C\|u_z\|_{L^\infty}^2\|k_z\|_{L^2}^2 \\
&\le \frac{1}{10}\|\partial_z k_z\|_{L^2}^2 + C\|\frac{u_r}{r}\|_{L^\infty}\|k_z\|_{L^2}^2 + C\|\nabla u_z\|_{L^2}\|\nabla_h \nabla u_z\|_{L^2}\|k_z\|_{L^2}^2 \\
&\le \frac{1}{10}\|\partial_z k_z\|_{L^2}^2 + C(\|\frac{u_r}{r}\|_{L^\infty} + \|\omega_\theta\|_{L^2}^2 + \|\nabla_h \nabla \omega_\theta\|_{L^2}^2)\|k_z\|_{L^2}^2.
\end{aligned}
$$

For the term I_6, we have

$$
\begin{aligned}
I_6 =& 2\int_{\mathbb{R}^3}(k_r^2 - b_\theta \partial_z k_r)k_r dx \\
=& 4\int_{\mathbb{R}^3} b_\theta \cdot \partial_z k_r \cdot k_r dx - 2\int_{\mathbb{R}^3} b_\theta \cdot \partial_z k_r \cdot k_r dx \\
=& 2\int_{\mathbb{R}^3} b_\theta \cdot \partial_z k_r \cdot k_r dx \\
\le& C\|b_\theta\|_{L^\infty}\|\partial_z k_r\|_{L^2}\|k_r\|_{L^2} \\
\le& \frac{1}{10}\|\partial_z k_r\|_{L^2}^2 + C\|b_\theta\|_{L^\infty}^2\|k_r\|_{L^2}^2.
\end{aligned}
$$

For the term I_7, we have

$$
\begin{aligned}
I_7 =& 2\int_{\mathbb{R}^3}[k_r \cdot \frac{b_\theta}{r} - \partial_z(\partial_r b_\theta)\cdot b_\theta + \partial_r b_\theta \cdot k_r]k_z dx \\
=& 2\int_{\mathbb{R}^3} \frac{b_\theta}{r}\cdot k_r \cdot k_z + (k_z - \frac{b_\theta}{r})b_\theta \cdot \partial_z k_z dx \\
=& 2\int_{\mathbb{R}^3} \frac{b_\theta}{r}\cdot k_r \cdot k_z - \partial_z(k_z - \frac{b_\theta}{r})\cdot b_\theta \cdot k_z + (k_z - \frac{b_\theta}{r})\cdot k_r \cdot k_z dx \\
\le& C\|\frac{b_\theta}{r}\|_{L^\infty}\|k_r\|_{L^2}\|k_z\|_{L^2} + C\|b_\theta\|_{L^\infty}\|\partial_z k_z\|_{L^2}\|k_z\|_{L^2} \\
\le& \frac{1}{10}\|\partial_z k_z\|_{L^2}^2 + C\|k_r\|_{L^2}^2 + C(\|b_\theta\|_{L^\infty}^2 + \|\frac{b_\theta}{r}\|_{L^\infty}^2)\|k_z\|_{L^2}^2.
\end{aligned}
$$

Substituting the estimates on $I_i(i = 1...,7)$ above into (2.21), we have

$$
\begin{aligned}
&\|k_r\|_{L^2}^2 + \|k_z\|_{L^2}^2 + \int_0^T(\|\partial_z k_r\|_{L^2}^2 + \|\partial_z k_z\|_{L^2}^2)dt \\
&\le \|k_0\|_{L^2}^2 e^{\int_0^T(\|\nabla_h \omega_\theta\|_{L^2}^2 + \|\frac{\omega_\theta}{r}\|_{L^2}^2)dt} + \int_0^T(\|\nabla_h \nabla \omega_\theta\|_{L^2}^2 + \|b_\theta\|_{L^\infty}^2\|\omega_\theta\|_{L^2}^2 + \|\frac{b_\theta}{r}\|_{L^\infty}^2)dt \\
&\le C(T,\ \|\omega_0\|_{L^2},\ \|\frac{\omega_0}{r}\|_{L^2},\ \|b_0^\theta\|_{L^\infty},\ \|\frac{b_0^\theta}{r}\|_{L^2},\ \|\frac{b_0^\theta}{r}\|_{L^\infty},\ \|k_0\|_{L^2}).
\end{aligned}
\tag{2.22}
$$

Since $\omega = \omega_\theta e_\theta$, $k = k_r e_r + k_z e_z$, combining the estimates (2.20) and (2.22), one can derive that

$$
\|\omega\|_{L^2}^2 + \|k\|_{L^2}^2 + \int_0^T(\|\partial_z \omega\|_{L^2}^2 + \|\partial_z k\|_{L^2}^2)dt \le C(T,\ \|u_0\|_{H^2},\ \|b_0\|_{H^2},\ \|\frac{b_0^\theta}{r}\|_{L^\infty}).
$$

□

3. H^2 estimates

This subsection is devoted to getting H^2 estimates of (u,b). To begin with, we first make estimate of $\|\nabla(\frac{b_\theta}{r})\|_{L^\infty([0,T],L^2)}$.

Lemma 3.1 *Suppose that the assumptions of Theorem 1.1 hold true. Then we have*

$$\|\nabla(\frac{b_\theta}{r})\|_{L^2}^2+\int_0^T\|\nabla\partial_z(\frac{b_\theta}{r})\|_{L^2}^2dt\le C, \tag{3.1}$$

where the constant C depends only on T, $\|b_0\|_{H^2}$, $\|\frac{b_0^\theta}{r}\|_{L^\infty}$, $\|u_0\|_{H^2}$.

Proof. Since $\Phi=\frac{b_\theta}{r}$ solves

$$\partial_t\Phi+u\cdot\nabla\Phi=\partial_{zz}\Phi+2\Phi\partial_z\Phi. \tag{3.2}$$

Taking inner product of (3.2) with $-\Delta\Phi$ and integrating on $\mathbb{R}^3$ yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla_h\Phi\|_{L^2}^2+\|\partial_z\Phi\|_{L^2}^2)+\|\nabla_h\partial_z\Phi\|_{L^2}^2+\|\partial_{zz}\Phi\|_{L^2}^2\\&=\int_{\mathbb{R}^3}u\cdot\nabla\Phi\cdot\Delta_h\Phi dx+\int_{\mathbb{R}^3}u\cdot\nabla\Phi\cdot\partial_{zz}\Phi dx-2\int_{\mathbb{R}^3}\Phi\cdot\partial_z\Phi\cdot\Delta\Phi dx\\&=:J_1+J_2+J_3.\end{aligned}$$

Now, we estimate the terms $J_i, i=1,2,3$, respectively. Making use of Lemma 2.1-Lemma 2.3, one can reach that

$$\begin{aligned}J_1=&\int_{-\infty}^{+\infty}\int_0^{+\infty}(u_r\partial_r+u_z\partial_z)\Phi\cdot\partial_r(r\partial_r\Phi)drdz\\=&-\int_{\mathbb{R}^3}\partial_ru_r\partial_r\Phi\partial_r\Phi\ dx-\int_{\mathbb{R}^3}\partial_ru_z\partial_z\Phi\partial_r\Phi\ dx\\=&-\int_{\mathbb{R}^3}u_z\partial_{rz}\Phi\partial_r\Phi\ dx+\int_{\mathbb{R}^3}\frac{u_r}{r}(\partial_r\Phi)^2\ dx-\int_{\mathbb{R}^3}\partial_ru_z\partial_z\Phi\partial_r\Phi\ dx\\\le&C\|u_z\|_{L^\infty}\|\nabla_h\Phi\|_{L^2}\|\nabla_h\partial_z\Phi\|_{L^2}+\|\frac{u_r}{r}\|_{L^\infty}\|\nabla_h\Phi\|_{L^2}^2\\&+\|\partial_ru_z\|_{L^2}^{\frac12}\|\nabla_h\partial_ru_z\|_{L^2}^{\frac12}\|\partial_z\Phi\|_{L^2}^{\frac12}\|\nabla_h\partial_z\Phi\|_{L^2}^{\frac12}\|\partial_r\Phi\|_{L^2}^{\frac12}\|\nabla_r\partial_z\Phi\|_{L^2}^{\frac12}\\\le&\frac14\|\nabla_h\partial_z\Phi\|_{L^2}^2+C(\|\omega_\theta\|_{L^2}^2+\|\nabla_h\omega_\theta\|_{L^2}^2+\|\frac{u_r}{r}\|_{L^\infty})(\|\nabla_h\Phi\|_{L^2}^2+\|\partial_z\Phi\|_{L^2}^2),\\J_2\le&\|u\|_{L^\infty}\|\nabla\Phi\|_{L^2}\|\partial_{zz}\Phi\|_{L^2}\\\le&\frac14\|\partial_{zz}\Phi\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\nabla\Phi\|_{L^2}^2\\\le&\frac14\|\partial_{zz}\Phi\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}\|\nabla\Phi\|_{L^2}^2\\\le&\frac14\|\partial_{zz}\Phi\|_{L^2}^2+C\|\omega_\theta\|_{L^2}\|\nabla_h\omega_\theta\|_{L^2}\|\nabla\Phi\|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
J_3 =& -2\int_{\mathbb{R}^3} \Phi\cdot\partial_z\Phi\cdot\frac{1}{r}\partial_r(r\partial_r\Phi)dx - 2\int_{\mathbb{R}^3}\Phi\cdot\partial_z\Phi\cdot\partial_{zz}\Phi dx\\
=& -2\int_{-\infty}^{+\infty}\int_0^{+\infty}\Phi\cdot\partial_z\Phi\cdot\partial_r(r\partial_r\Phi)drdz - 2\int_{\mathbb{R}^3}\Phi\cdot\partial_z\Phi\cdot\partial_{zz}\Phi dx\\
=&2\int_{\mathbb{R}^3}(\partial_r\Phi\cdot\partial_z\Phi+\Phi\cdot\partial_r\partial_z\Phi)\partial_r\Phi dx - 2\int_{\mathbb{R}^3}\Phi\cdot\partial_z\Phi\cdot\partial_{zz}\Phi dx\\
=&2\int_{\mathbb{R}^3}\partial_r\Phi\cdot\partial_z\Phi\cdot\partial_r\Phi+\Phi\cdot\partial_r\partial_z\Phi\cdot\partial_r\Phi dx - 2\int_{\mathbb{R}^3}\Phi\cdot\partial_z\Phi\cdot\partial_{zz}\Phi dx\\
=& -2\int_{\mathbb{R}^3}\Phi\cdot\partial_r\partial_z\Phi\cdot\partial_r\Phi dx - 2\int_{\mathbb{R}^3}\Phi\cdot\partial_z\Phi\cdot\partial_{zz}\Phi dx\\
\leq& C\|\Phi\|_{L^\infty}\|\nabla_h\partial_z\Phi\|_{L^2}\|\nabla_h\Phi\|_{L^2} + C\|\Phi\|_{L^\infty}\|\partial_z\Phi\|_{L^2}\|\partial_{zz}\Phi\|_{L^2}\\
\leq&\frac{1}{4}(\|\nabla_h\partial_z\Phi\|_{L^2}^2+\|\partial_{zz}\Phi\|_{L^2}^2) + C\|\Phi\|_{L^\infty}^2(\|\nabla_h\Phi\|_{L_2}^2+\|\partial_z\Phi\|_{L^2}^2).
\end{aligned}
$$

Thus, summing up estimates on $J_1 - J_3$ above and making use of Gronwall inequality yield

$$
\begin{aligned}
&\|\nabla_h\Phi\|_{L^2}^2+\|\partial_z\Phi\|_{L^2}^2+\int_0^T(\|\nabla_h\partial_z\Phi\|_{L^2}^2+\|\partial_{zz}\Phi\|_{L^2}^2)dt\\
\leq& C(T,\ \|b_0\|_{H^2},\ \|\frac{b_0^\theta}{r}\|_{L^\infty},\ \|u_0\|_{H^2}).
\end{aligned}
\tag{3.3}
$$

□

Lemma 3.2 *Suppose that the assumptions of Theorem 1.1 hold true. Then we have*

$$
\|\nabla\omega\|_{L^2}^2+\|\nabla k\|_{L^2}^2+\int_0^T(\|\nabla_h\partial_z\omega\|_{L^2}^2+\|\nabla_h^2\omega\|_{L^2}^2)dt+\int_0^T(\|\nabla_h\partial_z k\|_{L^2}^2+\|\partial_{zz}k\|_{L^2}^2)dt\leq C,
$$

where the constant C depends only on $T, \|\frac{b_0}{r}\|_{L^\infty}, \|\frac{b_0}{r^2}\|_{L^\infty}, \|b_0\|_{H^2}, \|u_0\|_{H^2}$.

Proof. Note that $\omega=\omega_\theta e_\theta$ solves the following equation

$$
\partial_t\omega-\Delta_h\omega+u\cdot\nabla\omega=\frac{u_r\omega}{r}-\frac{\partial_z b_\theta^2}{r}e_\theta.
\tag{3.4}
$$

Taking inner product on both sides of (3.4) with $-\Delta\omega$ and integrating on $\mathbb{R}^3$, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla_h\omega\|_{L^2}^2+\|\partial_z\omega\|_{L^2}^2)+(\|\nabla_h\partial_z\omega\|_{L^2}^2+\|\nabla_h^2\omega\|_{L^2}^2)\\
=&\int_{\mathbb{R}^3}(u\cdot\nabla\omega-\frac{u_r\omega}{r})\cdot\Delta\omega dx+\int_{\mathbb{R}^3}\frac{\partial_z b_\theta^2}{r}e_\theta\cdot\Delta\omega dx\\
=&L_1+L_2,
\end{aligned}
$$

$$
\begin{aligned}
L_1=&\int_{\mathbb{R}^3}(u_r\partial_r+u_z\partial_z)\omega\cdot\partial_r(r\partial_r\omega)drdz+\int_{\mathbb{R}^3}(u_r\partial_r+u_z\partial_z)\omega\cdot\partial_{zz}\omega dx-\int_{\mathbb{R}^3}(u_r\partial_r+u_z\partial_z)\omega\cdot\frac{\omega}{r^2}dx\\
&-\int_{\mathbb{R}^3}\frac{u_r\omega}{r}\cdot\partial_r(r\partial_r\omega)dx-\int_{\mathbb{R}^3}\frac{u_r\omega}{r}\cdot\partial_{zz}\omega dx+\int_{\mathbb{R}^3}\frac{u_r\omega}{r}\cdot\frac{\omega}{r^2}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} (\partial_r u_r \partial_r \omega \cdot \partial_r \omega + \partial_r u_z \partial_z \omega \cdot \partial_r \omega) + (\partial_r u_r \partial_r \omega \cdot \partial_z \omega + \partial_z u_z \partial_z \omega \cdot \partial_z \omega) dx - \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z)\omega \cdot \frac{\omega}{r^2} dx \\
&\quad + \int_{\mathbb{R}^3} (\partial_r u_r \frac{\omega}{r} \cdot \partial_r \omega + \frac{u_r}{r} \partial_r \omega \cdot \partial_r \omega) dx + \int_{\mathbb{R}^3} (\frac{\partial_z u_r}{r} \omega \cdot \partial_r \omega + \frac{u_r}{r} \partial_z \omega \cdot \partial_z \omega) dx + \int_{\mathbb{R}^3} \frac{u_r \omega}{r} \cdot \frac{\omega}{r^2} dx \\
&= L_{11} + L_{12} + L_{13} + L_{14} + L_{15}.
\end{aligned}
$$

By Lemma 2.1 and Lemma 2.3, one can reach that

$$
\begin{aligned}
L_{11} \leq & \|\partial_r u_r\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_r u_r\|_{L^2}^{\frac{1}{2}} \|\partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\partial_z \partial_r \omega\|_{L^2}^{\frac{1}{2}} \\
& + \|\partial_r u_z\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_r u_z\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\partial_z \partial_r \omega\|_{L^2}^{\frac{1}{2}} \\
& + \|\partial_z u_r\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z u_r\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \\
& + \|\partial_z u_z\|_{L^2}^{\frac{1}{2}} \|\partial_{zz} u_z\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L_2}^{\frac{1}{2}} \\
\leq & \|\omega_\theta\|_{L^2}^{\frac{1}{2}} \|\nabla_h \omega_\theta\|_{L^2}^{\frac{1}{2}} \|\partial_r \omega\|_{L^2} \|\nabla_h^2 \omega\|_{L^2} + \|\omega_\theta\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega_\theta\|_{L^2}^{\frac{1}{2}} \| \|\partial_z \omega\|_{L^2} \|\nabla_h \partial_z \omega\|_{L^2} \\
& + 2\|\omega_\theta\|_{L^2}^{\frac{1}{2}} \|\nabla_h \omega_\theta\|_{L^2}^{\frac{1}{2}} \| \|\partial_r \omega\|_{L^2}^{\frac{1}{2}} |\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2} \\
\leq & \frac{1}{8} (\|\nabla_h \partial_z \omega\|_{L^2}^2 + \|\nabla_h^2 \omega\|_{L^2}^2) + (\|\omega_\theta\|_{L^2}^2 + \|\nabla_h \omega_\theta\|_{L^2}^2)(\|\partial_r \omega\|_{L^2}^2 + \|\partial_z \omega\|_{L^2}^2), \\
L_{12} = & - \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_r \omega \frac{\omega}{r} dx + \frac{1}{2} \int_{\mathbb{R}^3} \partial_z u_z (\frac{\omega}{r})^2 dx \\
\leq & C \|\frac{u_r}{r}\|_{L^\infty} \|\nabla \omega\|_{L^2}^2 + C \|\partial_z u_z\|_{L^6}^{\frac{3}{4}} \|\partial_{zz} u_z\|_{L^2}^{\frac{1}{4}} \|\frac{\omega}{r}\|_{L^2}^{\frac{1}{2}} \|\nabla_h (\frac{\omega}{r})\|_{L^2}^{\frac{1}{2}} \|\frac{\omega}{r}\|_{L^2} \\
\leq & C \|\frac{u_r}{r}\|_{L^\infty} \|\nabla \omega\|_{L^2}^2 + C \|\omega\|_{L^6}^{\frac{3}{4}} \|\partial_z \omega\|_{L^2}^{\frac{1}{4}} \|\frac{\omega}{r}\|_{L^2}^{\frac{1}{2}} \|\nabla_h (\frac{\omega}{r})\|_{L^2}^{\frac{1}{2}} \|\frac{\omega}{r}\|_{L^2} \\
\leq & C \|\frac{u_r}{r}\|_{L^\infty} \|\nabla \omega\|_{L^2}^2 + C \|\frac{\omega}{r}\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{3}{2}} \|\nabla_h \nabla \omega\|_{L^2}^{\frac{1}{2}} \\
\leq & \frac{1}{8} \|\nabla_h \nabla \omega\|_{L^2}^2 + C (\|\frac{\omega}{r}\|_{L^2}^{\frac{4}{3}} + \|\frac{u_r}{r}\|_{L^\infty}) \|\nabla \omega\|_{L^2}^2, \\
L_{13} \leq & C \|\partial_r u_r\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_r u_r\|_{L^2}^{\frac{1}{2}} \|\partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\frac{\omega}{r}\|_{L^2}^{\frac{1}{2}} \|\frac{\partial_z \omega}{r}\|_{L^2}^{\frac{1}{2}} + C \|\frac{u_r}{r}\|_{L^\infty} \|\partial_r \omega\|_{L^2}^2 \\
\leq & \frac{1}{8} (\|\nabla_h \partial_z \omega\|_{L^2}^2 + \|\nabla_h^2 \omega\|_{L^2}^2) + C (\|\omega_\theta\|_{L^2}^2 + \|\nabla_h \omega_\theta\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty}) \|\nabla_h \omega\|_{L^2}^2, \\
L_{14} \leq & C \|\partial_z (\frac{u_r}{r})\|_{L^2} \|\omega\|_{L^6}^{\frac{3}{4}} \|\partial_z \omega\|_{L^2}^{\frac{1}{4}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} + C \|\frac{u_r}{r}\|_{L^\infty} \|\partial_z \omega\|_{L^2}^2 \\
\leq & C \|\frac{\omega_\theta}{r}\|_{L^2} \|\nabla_h \omega\|_{L^2}^{\frac{3}{4}} \|\partial_z \omega\|_{L^2}^{\frac{3}{4}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} + C \|\frac{u_r}{r}\|_{L^\infty} \|\partial_z \omega\|_{L^2}^2 + C \|\frac{\omega_\theta}{r}\|_{L^2} \|\partial_z \omega\|_{L^2}^{\frac{3}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \\
\leq & \frac{1}{8} (\|\nabla_h \partial_z \omega\|_{L^2}^2 + \|\nabla_h^2 \omega\|_{L^2}^2) + C (\|\omega\|_{L^2}^2 + \|\nabla_h \omega\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty}) \|\nabla_h \omega\|_{L^2}^2, \\
L_{15} \leq & C \|\frac{u_r}{r}\|_{L^\infty} \|\frac{\omega}{r}\|_{L^2}^2 \leq C \|\frac{u_r}{r}\|_{L^\infty} \|\nabla \omega\|_{L^2}^2.
\end{aligned}
$$

To estimate L_2, we rewrite it as follows:

$$
\begin{aligned}
L_2 =&2\int_{\mathbb{R}^3} \frac{b_\theta \partial_z b_\theta}{r} e_\theta \cdot \Delta_h \omega dx + 2\int_{\mathbb{R}^3} \frac{b_\theta \partial_z b_\theta}{r} e_\theta \cdot \partial_{zz}\omega dx \\
=&\int_{\mathbb{R}^3} \frac{b_\theta \partial_z b_\theta}{r} e_\theta \cdot \Delta_h \omega dx - 2\int_{\mathbb{R}^3} \partial_z(\frac{b_\theta}{r})\partial_z b_\theta e_\theta \cdot \partial_z \omega dx - 2\int_{\mathbb{R}^3} \frac{b_\theta}{r} \cdot \partial_{zz} b_\theta e_\theta \cdot \partial_z \omega dx \\
\leq& \|\frac{b_\theta}{r}\|_{L^\infty} \|k_r\|_{L^2} \|\Delta_h \omega\|_{L^2} + C\|\partial_z(\frac{b_\theta}{r})\|_{L^2}^{\frac12} \|\nabla_h \partial_z(\frac{b_\theta}{r})\|_{L^2}^{\frac12} \|k_r\|_{L^2}^{\frac12} \|\partial_z k_r\|_{L^2}^{\frac12} \|\partial_z \omega\|_{L^2}^{\frac12} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac12} \\
&+ \|\frac{b_\theta}{r}\|_{L^\infty} \|\partial_z \omega\|_{L^2} \|\partial_z k_r\|_{L^2} \\
\leq& \frac{1}{10}(\|\nabla_h \partial_z \omega\|_{L^2}^2 + \|\nabla_h^2 \omega\|_{L^2}^2) + (\|\frac{b_\theta}{r}\|_{L^\infty}^2 + \|\partial_z \omega\|_{L^2}^2)\|k_r\|_{L^2}^2 + C(1 + \|\partial_z(\frac{b_\theta}{r})\|_{L^2}^2)\|\partial_z k_r\|_{L^2}^2 \\
&+ \|\frac{b_\theta}{r}\|_{L^\infty}^2 \|\partial_z \omega\|_{L^2}^2 + \|\nabla_h \partial_z \frac{b_\theta}{r}\|_{L^2}^2.
\end{aligned}
$$

Adding up the estimates L_1 and L_2 yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla \omega\|_{L^2}^2 + (\|\nabla_h \partial_z \omega\|_{L^2}^2 + \|\nabla_h \omega\|_{L^2}^2) \\
\leq& C(\|\omega_\theta\|_{L_2}^2 + \|\nabla_h \omega_\theta\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty} + \|\frac{\omega}{r}\|_{L^2}^{\frac43} + \|\frac{b_\theta}{r}\|_{L^\infty}^2 + \|k\|_{L^2}^2)\|\nabla \omega\|_{L^2}^2 \\
&+ \|\frac{b_\theta}{r}\|_{L^\infty}^2 \|k\|_{L^2}^2 + \|\nabla_h \partial_z(\frac{b_\theta}{r})\|_{L^2}^2 + C(\|\partial_z(\frac{b_\theta}{r})\|_{L^2}^2 + 1)\|\partial_z k_r\|_{L^2}^2.
\end{aligned}
$$

Combining with Gronwall inequality, it is clear that

$$
\begin{aligned}
&\|\nabla \omega\|_{L^2}^2 + \int_0^T (\|\nabla_h \partial_z \omega\|_{L^2}^2 + \|\nabla_h^2 \omega\|_{L^2}^2) dt \\
\leq& C(\|\omega_0\|_{L^2},\ \|\nabla \omega_0\|_{L^2},\ \|\frac{\omega_0}{r}\|_{L^2},\ \|b_0\|_{L^\infty},\ \|\frac{b_0}{r}\|_{L^\infty},\ \|\nabla(\frac{b_0}{r})\|_{L^2},\ \|k_0\|_{L^2}).
\end{aligned}
$$

Similarly, by $(2.5)_2$ and $(2.5)_3$, $k = k_r e_r + k_z e_z$ solves

$$
\partial_t k - \partial_{zz} k + u \cdot \nabla k = \alpha + \beta, \tag{3.5}
$$

where

$$
\alpha = \left[\partial_z u_r \cdot \partial_r b_\theta + \partial_z u_z \cdot \partial_z b_\theta - \frac{1}{r}\partial_z(u_r b_\theta) + \frac{1}{r}\partial_z^2 b_\theta^2\right] e_r,
$$

and

$$
\beta = \left[u \cdot \nabla(\frac{b_\theta}{r}) + \frac{1}{r}\partial_r(u_r b_\theta) - \frac{1}{r}\partial_r(ru) \cdot \nabla b_\theta - \frac{1}{r}\partial_r(\partial_z b_\theta^2)\right] e_z.
$$

Taking inner product of (3.5) with $-\Delta k$ and integrating on $\mathbb{R}^3$, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla k\|_{L^2}^2 + \|\nabla_h \partial_z k\|_{L^2}^2 + \|\partial_{zz} k\|_{L^2}^2 \\
=&\int_{\mathbb{R}^3} u \cdot \nabla k \cdot \Delta k dx - \int_{\mathbb{R}^3} \alpha \cdot \Delta k dx - \int_{\mathbb{R}^3} \beta \cdot \Delta k dx \\
=&M_1 + M_2 + M_3.
\end{aligned}
$$

For the first term on the right hand side, we can decompose it as

$$
\begin{aligned}
M_1 =&\int_{\mathbb{R}}\int_0^\infty (u_r\partial_r + u_z\partial_z)k\cdot\partial_r(r\partial_r k)drdz + \int_{\mathbb{R}^3} u\cdot\nabla k\cdot\partial_{zz}kdx - \int_{\mathbb{R}^3}(u_r\partial_r+u_z\partial_z)k_r\cdot\frac{k_r}{r^2}dx\\
=&\int_{\mathbb{R}^3}\partial_z u_z\partial_r k\cdot\partial_r kdx + \int_{\mathbb{R}^3}\frac{u_r}{r}\partial_r k\cdot\partial_r kdx - \int_{\mathbb{R}^3}\partial_r u_z\partial_z kdx\cdot\partial_r kdx\\
&+\int_{\mathbb{R}^3} u\cdot\nabla k\cdot\partial_{zz}kdx - \int_{\mathbb{R}^3}(u_r\partial_r+u_z\partial_z)k_r\cdot\frac{k_r}{r^2}dx\\
=&-\int_{\mathbb{R}^3}\partial_r u_z\partial_z k\cdot\partial_r kdx + \int_{\mathbb{R}^3}\frac{u_r}{r}(\partial_r k\cdot\partial_r k - \partial_r k_r\cdot\frac{k_r}{r})dx\\
&-\int_{\mathbb{R}^3} u_z(\frac{\partial_z k_r}{r}\cdot\frac{k_r}{r} + 2\partial_r k\cdot\partial_{rz}k)dx + \int_{\mathbb{R}^3} u\cdot\nabla k\cdot\partial_{zz}kdx\\
\leq&\|\partial_r u_z\|_{L^2}^{\frac12}\|\nabla_h\partial_r u_z\|_{L^2}^{\frac12}\|\partial_z k\|_{L^2}^{\frac12}\|\nabla_h\partial_z k\|_{L^2}^{\frac12}\|\partial_z k\|_{L^2}^{\frac12}\|\partial_r\partial_z k\|_{L^2}^{\frac12} + C\|\frac{u_r}{r}\|_{L^\infty}\|\nabla k_r\|_{L^2}^2\\
&+\|u_z\|_{L^\infty}\|\frac{\partial_z k_r}{r}\|_{L^2}\|\frac{k_r}{r}\|_{L^2} + \|u\|_{L^\infty}\|\nabla k\|_{L^2}(\|\partial_{zz}k\|_{L^2} + \|\partial_{zr}k\|_{L^2})\\
\leq&\frac14(\|\nabla_h\partial_z k\|_{L^2}^2 + \|\partial_{zz}k\|_{L^2}^2) + C(\|\omega_\theta\|_{L^2}^2 + \|\nabla_h\omega_\theta\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty})\|\nabla k\|_{L^2}^2,\\
M_2 =&-\int_{\mathbb{R}^3}[\partial_z u_r\partial_r b_\theta + \partial_z u_z\partial_z b_\theta - \frac1r\partial_z(u_r b_\theta)]\cdot[\frac1r\partial_r(r\partial_r k_r) - \frac{k_r}{r^2} + \partial_{zz}k_r]dx\\
&-\int_{\mathbb{R}^3}\frac1r\partial_z^2 b_\theta^2\cdot[\frac1r\partial_r(r\partial_r k_r) - \frac{k_r}{r^2} + \partial_{zz}k_r]dx\\
=&M_{21} + M_{22},\\
M_{21} =&-\int_{-\infty}^\infty\int_0^\infty[\partial_z u_r(\partial_r b_\theta - \frac{b_\theta}{r}) - \partial_z u_z k_r]\partial_r(r\partial_r k_r)drdz\\
&-\int_{-\infty}^\infty\int_0^\infty u_r k_r\partial_{rr}k_r drdz + \int_{\mathbb{R}^3}\partial_z u_r(\partial_r b_\theta - \frac{b_\theta}{r})\frac{k_r}{r^2}dx\\
&-\int_{\mathbb{R}^3}(\partial_z u_z - \frac{u_r}{r})k_r\cdot\frac{k_r}{r^2}dx - \int_{\mathbb{R}^3}\partial_z u_r(\partial_r b_\theta - \frac{b_\theta}{r})\partial_{zz}k_z dx + \int_{\mathbb{R}^3}(2\partial_z u_z + \partial_r u_r)k_z\cdot\partial_{zz}k_z dx\\
=&\int_{\mathbb{R}^3}\partial_r[\partial_z u_r(\partial_r b_\theta - \frac{b_\theta}{r}) - \partial_z u_z k_r]\partial_r k_r dx + \int_{\mathbb{R}^3}[\partial_z u_r(\partial_r b_\theta - \frac{b_\theta}{r}) - (2\partial_z u_z + \partial_r u_r)k_z]\partial_{zz}k_r dx\\
&+\int_{\mathbb{R}^3} -\frac{u_r}{r}\cdot\partial_r k_r\cdot\partial_r k_r + 2u_z\cdot\frac{\partial_z k_r}{r}\cdot\frac{k_r}{r} + \frac{u_r}{r}\cdot\frac{k_r}{r}\cdot\frac{k_r}{r}dx\\
\leq&\|\partial_r k_r\|_{L^2}^{\frac12}\|\partial_r\partial_z k_r\|_{L^2}^{\frac12}\|k\|_{L^2}^{\frac12}\|\nabla_h k\|_{L^2}^{\frac12}\|\partial_z\omega_\theta\|_{L^2}^{\frac12}\|\nabla_h\partial_z\omega_\theta\|_{L^2}^{\frac12} + \|\omega_\theta\|_{L^6}^{\frac34}\|\partial_z\omega_\theta\|_{L^2}^{\frac14}\|k\|_{L^2}^{\frac12}\|\nabla_h k\|_{L^2}^{\frac12}\|\partial_{zz}k_r\|_{L^2}\\
&+\|\frac{u_r}{r}\|_{L^\infty}\|\nabla k\|_{L^2}^2 + \|u_z\|_{L^\infty}\|\frac{\partial_z k_r}{r}\|_{L^2}\|\nabla k\|_{L^2}\\
\leq&\frac14(\|\nabla_h\partial_z k\|_{L^2}^2 + \|\partial_{zz}k\|_{L^2}^2) + (1 + \|k\|_{L^2}^2 + \|\nabla\omega_\theta\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty} + \|u_z\|_{L^\infty}^2)\|\nabla k\|_{L^2}^2\\
&+(1 + \|\nabla\omega_\theta\|_{L^2}^2)\|k\|_{L^2} + C\|\nabla_h\partial_z\omega_\theta\|_{L^2}^2,\\
M_{22} =&\int_{\mathbb{R}^3}\frac1r\partial_z^2 b_\theta^2(-\frac1r\partial_r(r\partial_r k_r) + \frac{k_r}{r^2} - \partial_{zz}k_r)dx
\end{aligned}
$$

$$
\begin{aligned}
&=-\int_{\mathbb{R}^3}\frac{b_\theta}{r}\frac{k_r}{r}\partial_{rz}k_r dx+3\int_{\mathbb{R}^3}\frac{b_\theta}{r}\partial_r k_r\partial_{rz}k_r dx+\int_{\mathbb{R}^3}\frac{b_\theta}{r}\partial_r k_r\partial_{zz}k_z dx+\int_{\mathbb{R}^3}\frac{b_\theta}{r}\partial_z k_z\partial_{rz}k_r dx\\
&\quad+\int_{\mathbb{R}^3}\frac{k_r}{r}k_z\partial_{rz}k_r dx+\int_{\mathbb{R}^3}\frac{b_\theta}{r^2}\partial_r k_r\partial_z k_r dx+2\int_{\mathbb{R}^3}\frac{b_\theta}{r^2}\frac{k_r}{r}\partial_z k_r dx+10\int_{\mathbb{R}^3}\frac{b_\theta}{r}\partial_z k_r\partial_{zz}k_r dx\\
&\le\frac{1}{8}(\|\nabla_h\partial_z k\|_{L^2}^2+\|\partial_{zz}k\|_{L^2}^2)+C(\|k\|_{L^2}^2+\|\frac{b_\theta}{r}\|_{L^\infty}^2+\|\frac{b_\theta}{r^2}\|_{L^\infty}^2)\|\nabla k\|_{L^2}^2,\\
M_3&=-\int_{\mathbb{R}^3}[u\cdot\nabla(\frac{b_\theta}{r})+\frac{1}{r}\partial_r(u_r b_\theta)-\frac{1}{r}\partial_r(ru)\cdot\nabla b_\theta](\partial_{rr}+\frac{1}{r}\partial_r+\partial_{zz})k_z dx\\
&\quad-\int_{\mathbb{R}^3}\frac{1}{r}\partial_r(\partial_z b_\theta^2)(\partial_{rr}+\frac{1}{r}\partial_r+\partial_{zz})k_z dx\\
&=M_{31}+M_{32},\\
M_{31}&=-\int_{\mathbb{R}^3}[u\cdot\nabla(\frac{b_\theta}{r})+\frac{1}{r}\partial_r(u_r b_\theta)-\frac{1}{r}\partial_r(ru)\cdot\nabla b_\theta](\partial_{rr}+\frac{1}{r}\partial_r+\partial_{zz})k_z dx\\
&=-\int_{\mathbb{R}^3}[u_r\cdot\partial_r(\frac{b_\theta}{r})+r\partial_r u_r\partial_r(\frac{b_\theta}{r})](\partial_{rr}+\frac{1}{r}\partial_r)k_z dx-\int_{\mathbb{R}^3}[u_r\cdot\partial_r(\frac{b_\theta}{r})+r\partial_r u_r\cdot\partial_r(\frac{b_\theta}{r})]\partial_{zz}k_z dx\\
&\quad-\int_{\mathbb{R}^3}\partial_r u_z\cdot k_r(\partial_{rr}+\frac{1}{r}\partial_r+\partial_{zz})k_z dx\\
&=N_1+N_2+N_3,\\
N_1&=-\int_{\mathbb{R}^3}[r\partial_{rr}u_r\cdot\partial_r(\frac{b_\theta}{r})+r\partial_r u_r\cdot\partial_{rr}(\frac{b_\theta}{r})-u_r\cdot\partial_{rr}(\frac{b_\theta}{r})]\partial_r k_z dx\\
&\quad-\int_{\mathbb{R}^3}\partial_{rr}u_r(\partial_r b_\theta-\frac{b_\theta}{r})\partial_r k_z dx+2\int_{\mathbb{R}^3}\frac{u_r}{r}[\partial_{rr}b_\theta+2\frac{b_\theta}{r^2}-2\frac{\partial_r b_\theta}{r}]\partial_r k_z dx\\
&\quad+\int_{\mathbb{R}^3}\partial_z u_z[\partial_{rr}b_\theta+2\frac{b_\theta}{r^2}-2\frac{\partial_r b_\theta}{r}]\partial_r k_z dx\\
&\le\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\nabla_h^2\omega\|_{L^2}^{\frac{1}{2}}\|k\|_{L^2}^{\frac{1}{2}}\|\nabla k\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_z k\|_{L^2}^{\frac{1}{2}}+\|\frac{u_r}{r}\|_{L^\infty}\|\nabla k\|_{L^2}^2\\
&\quad+\|u\|_{L^\infty}\|\nabla k\|_{L^2}\|\nabla_h\partial_z k\|_{L^2}\\
&\le\frac{1}{4}\|\nabla_h\partial_z k\|_{L^2}^2+C(\|\omega_\theta\|_{L^2}^2+\|\nabla\omega_\theta\|_{L^2}^2+\|\frac{u_r}{r}\|_{L^\infty})\|\nabla k\|_{L^2}^2+C\|\nabla_h^2\omega\|_{L^2}^2,\\
N_2&=-\int_{\mathbb{R}^3}u_r\cdot\partial_r(\frac{b_\theta}{r})\partial_{zz}k_z dx+\int_{\mathbb{R}^3}\partial_r u_r(\partial_r b_\theta-\frac{b_\theta}{r})\partial_{zz}k_z dx\\
&\le(\|u\|_{L^\infty}\|\nabla k\|_{L^2}+\|\partial_r u_r\|_{L^2}^{\frac{3}{4}}\|\partial_z\partial_r u_r\|_{L^2}^{\frac{1}{4}}\|k_z\|_{L^2}^{\frac{1}{2}}\|\nabla_h k_z\|_{L^2}^{\frac{1}{2}})\|\partial_{zz}k\|_{L^2}\\
&\le\frac{1}{4}\|\nabla_h\partial_z k\|_{L^2}^2+C(\|\omega_\theta\|_{L^2}^2+\|\nabla\omega_\theta\|_{L^2}^2)\|\nabla k\|_{L^2}^2+\|k\|_{L^2}^2\|\nabla\omega_\theta\|_{L^2}^2,\\
N_3&=-\int_{\mathbb{R}^3}\partial_r u_z\cdot k_r\partial_r(r\partial_r k_z)drdz-\int_{\mathbb{R}^3}\partial_r u_z\cdot k_r\cdot\partial_{zz}k_z dx\\
&=\int_{\mathbb{R}^3}\partial_{rr}u_z\cdot k_r\cdot\partial_r k_z dx+\int_{\mathbb{R}^3}\partial_r u_z\cdot\partial_r k_r\cdot\partial_r k_z dx-\int_{\mathbb{R}^3}\partial_r u_z\cdot k_r\cdot\partial_{zz}k_z dx\\
&\le\|\partial_{rr}u_z\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_{rr}u_z\|_{L^2}^{\frac{1}{2}}\|k_r\|_{L^2}^{\frac{1}{2}}\|\nabla_h k_r\|_{L^2}^{\frac{1}{2}}\|\partial_r k_z\|_{L^2}^{\frac{1}{2}}\|\partial_r\partial_z k_z\|_{L^2}\\
&\quad+\|\partial_r u_z\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_r u_z\|_{L^2}^{\frac{1}{2}}\|\nabla k\|_{L^2}^2+\|\partial_r u_z\|_{L^6}^{\frac{3}{4}}\|\partial_r\partial_z u_z\|_{L^2}^{\frac{1}{4}}\|k_r\|_{L^2}^{\frac{1}{2}}\|\nabla_h k_r\|_{L^2}^{\frac{1}{2}}\|\partial_{zz}k_r\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{4}\|\nabla_h\partial_z k\|_{L^2}^2 + C(1+\|\omega_\theta\|_{L^2}^2+\|\nabla\omega_\theta\|_{L^2}^2)\|\nabla k\|_{L^2}^2 + \|k\|_{L^2}^2\|\nabla\omega_\theta\|_{L^2}^2 + \|\nabla_h^2\omega_\theta\|_{L^2}^2,\\
M_{32} =& -2\int_{\mathbb{R}^3}\frac{1}{r}(\partial_r b_\theta\partial_z b_\theta + b_\theta\partial_{rz}b_\theta)\frac{1}{r}\partial_r(r\partial_r k_z)dx - 2\int_{\mathbb{R}^3}\frac{1}{r}(\partial_r b_\theta\partial_z b_\theta - b_\theta\partial_{rz}b_\theta)\partial_{zz}k_z dx\\
=& -2\int_{\mathbb{R}^3}\frac{1}{r^2}(\partial_r b_\theta\partial_z b_\theta + b_\theta\partial_{rz}b_\theta)\frac{1}{r}\partial_r k_z dx + 2\int_{\mathbb{R}^3}\frac{1}{r}\partial_r(\partial_r b_\theta\partial_z b_\theta + b_\theta\partial_{rz}b_\theta)\partial_r k_z dx\\
&-2\int_{\mathbb{R}^3}\frac{1}{r}(\partial_r b_\theta\partial_z b_\theta - b_\theta\partial_{rz}b_\theta)\partial_{zz}k_z dx\\
=&P_1+P_2+P_3,\\
P_1 =& -\int_{\mathbb{R}^3}(\frac{2}{r^2}\partial_r b_\theta\partial_z b_\theta + \frac{2}{r^2}b_\theta\partial_r\partial_z b_\theta)\partial_r k_z dx\\
=& -\int_{\mathbb{R}^3}\frac{2}{r^2}\partial_r b_\theta\partial_z b_\theta\partial_r k_z dx - \int_{\mathbb{R}^3}\frac{2}{r^2}b_\theta\partial_r\partial_z b_\theta\partial_r k_z dx\\
=& \int_{\mathbb{R}^3}\frac{2b_\theta}{r^2}(\partial_z\partial_r b_\theta\partial_r k_z + \partial_r b_\theta\partial_z\partial_r k_z)dx - \int_{\mathbb{R}^3}\frac{2}{r^2}b_\theta\partial_r\partial_z b_\theta\partial_r k_z dx\\
=& \int_{\mathbb{R}^3}\frac{2b_\theta}{r^2}\partial_r b_\theta\cdot\partial_z\partial_r k_z dx\\
=& \int_{\mathbb{R}^3}\frac{2b_\theta}{r^2}(k_z - \frac{b_\theta}{r})\cdot\partial_z\partial_r k_z dx\\
=& \int_{\mathbb{R}^3}\frac{2b_\theta}{r^2}k_z\cdot\partial_z\partial_r k_z dx - 4\int_{\mathbb{R}^3}\frac{b_\theta}{r^2}\frac{k_r}{r}\cdot\partial_r k_z dx\\
\le& \frac{1}{10}\|\nabla_h\partial_z k\|_{L^2}^2 + C\|\frac{b_\theta}{r^2}\|_{L^\infty}^2\|k\|_{L^2}^2 + C\|\frac{b_\theta}{r^2}\|_{L^\infty}\|\nabla k\|_{L^2}^2,\\
P_2 =& 2\int_{\mathbb{R}^3}\frac{1}{r}\partial_{rr}(b_\theta\partial_z b_\theta)\partial_r k_z dx\\
=& 2\int_{\mathbb{R}^3}(k_z - \frac{b_\theta}{r})[\partial_r(\frac{b_\theta}{r}) - \frac{b_\theta}{r^2}]\partial_{rz}k_z dx\\
=& 2\int_{\mathbb{R}^3}k_z\cdot\partial_r(\frac{b_\theta}{r})\partial_{rz}k_z dx + 2\int_{\mathbb{R}^3}\frac{b_\theta}{r}\cdot\frac{b_\theta}{r^2}\cdot\partial_{rz}k_z dx - 2\int_{\mathbb{R}^3}(\frac{b_\theta}{r}\cdot\partial_r(\frac{b_\theta}{r}) + k_z\cdot\frac{b_\theta}{r^2})\partial_{rz}k_z dx\\
=& 2\int_{\mathbb{R}^3}k_z\cdot\partial_r(\frac{b_\theta}{r})\partial_{rz}k_z dx + 4\int_{\mathbb{R}^3}\frac{b_\theta}{r^2}\cdot\frac{k_r}{r}\cdot\partial_r k_z dx + -2\int_{\mathbb{R}^3}(\frac{b_\theta}{r}\cdot\partial_r(\frac{b_\theta}{r}) + k_z\cdot\frac{b_\theta}{r^2})\partial_{rz}k_z dx\\
\le& C\|\partial_r(\frac{b_\theta}{r})\|_{L^2}^{\frac12}\|\partial_z\partial_r(\frac{b_\theta}{r})\|_{L^2}^{\frac12}\|k_z\|_{L^2}^{\frac12}\|\nabla k_z\|_{L^2}\|\partial_z\partial_r k_z\|_{L^2} + +C\|\frac{b_\theta}{r^2}\|_{L^\infty}\|\nabla k\|_{L^2}^2\\
&+\frac{1}{10}\|\nabla_h\partial_z k\|_{L^2}^2 + C\|\frac{b_\theta}{r}\|_{L^\infty}^2\|k\|_{L^2}^2 + \|\frac{b_\theta}{r}\|_{L^\infty}^2\|\nabla(\frac{b_\theta}{r})\|_{L^2}^2,\\
P_3 =& \int_{\mathbb{R}^3}\frac{1}{r}\partial_r(\partial_z b_\theta^2)\cdot\partial_{zz}k_z dx\\
=& \int_{\mathbb{R}^3}\frac{2}{r}(\partial_r b_\theta\partial_z b_\theta + b_\theta\partial_r\partial_z b_\theta)\partial_{zz}k_z dx
\end{aligned}
$$

$$
\begin{aligned}
&= -2\int_{\mathbb{R}^3} \partial_r(\frac{b_\theta}{r}) \cdot k_r \cdot \partial_{zz} k_z dx + 2\int_{\mathbb{R}^3} \frac{b_\theta}{r} \cdot \frac{k_r}{r} \cdot \partial_{zz} k_z dx - 2\int_{\mathbb{R}^3} \frac{b_\theta}{r} \partial_r k_r \cdot \partial_{zz} k_z dx \\
&\leq C\|\partial_r(\frac{b_\theta}{r})\|_{L^2}^{\frac{1}{2}} \|\partial_z \partial_r(\frac{b_\theta}{r})\|_{L^2}^{\frac{1}{2}} \|k_r\|_{L^2}^{\frac{1}{2}} \|\nabla k\|_{L^2} \|\partial_{zz} k\|_{L^2} + C\|\frac{b_\theta}{r}\|_{L^\infty} \|\frac{k_r}{r}\|_{L^2} \|\partial_{zz} k\|_{L^2} \\
&\leq \frac{1}{10}\|\partial_{zz} k\|_{L^2}^2 + C(\|\nabla(\frac{b_\theta}{r})\|_{L^2}^2 + \|\partial_z \nabla(\frac{b_\theta}{r})\|_{L^2}^2 + \|k\|_{L^2}^2 + \|\frac{b_\theta}{r}\|_{L^\infty}^2)\|\nabla k\|_{L^2}^2.
\end{aligned}
$$

By Gronwall inequality, we have

$$
\begin{aligned}
&\|\nabla k\|_{L^2}^2 + \int_0^T (\|\nabla_h \partial_z k\|_{L^2}^2 + \|\partial_{zz} k\|_{L^2}^2) dt \\
\leq& C(T, \|\frac{b_0}{r}\|_{L^\infty}, \|\frac{b_0}{r^2}\|_{L^\infty}, \|b_0\|_{H^2}, \|u_0\|_{H^2}).
\end{aligned}
\tag{3.6}
$$

The proof of the lemma is finished. □

4. Proof of Theorem 1.1

4.1. $L_t^1 L_x^\infty$ estimates.

Lemma 4.1 *Suppose that the assumptions of Theorem 1.1 hold true. Then we have*

$$
\int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty}) dt \leq C, \tag{4.1}
$$

where the constant C depends only on T, $\|\frac{b_0}{r}\|_{L^\infty(\mathbb{R}^3)}$, $\|\omega_0\|_{H^1(\mathbb{R}^3)}$ and $\|k_0\|_{H^1(\mathbb{R}^3)}$.

Proof. We first prove

$$
\int_0^T \|\nabla u\|_{L^\infty} dt \leq C.
$$

Since u solves

$$
\partial_t u - \triangle_h u + u \cdot \nabla u = -\nabla \pi + b \cdot \nabla b,
$$

and

$$
u \cdot \nabla u = (\nabla \times u) \times u + \frac{1}{2}\nabla |u|^2,
$$

it follows that

$$
\partial_t(\nabla \times u) - \Delta_h(\nabla \times u) = \nabla \times ((\nabla \times u) \times u) - \partial_z(\frac{b_\theta^2}{r}) e_\theta. \tag{4.2}
$$

Thanks to (2.8) and Lemma 3.2, one has

$$
\nabla \times u \in L^\infty([0,T], L^6), \quad u \in L^\infty([0,T], L^2).
$$

Since $divu = 0$, thus by imbedding inequality, one can get that

$$
\|u\|_{L^\infty([0,T],L^\infty)} \leq \|u\|_{L^\infty([0,T],L^2)} + \|\nabla \times u\|_{L^\infty([0,T],L^6)},
$$

which along with the Hölder inequality implies that

$$\|(\nabla\times u)\times u\|_{L^1([0,T],L^6)}\le \|u\|_{L^\infty([0,T],L^\infty)}\|\nabla\times u\|_{L^\infty([0,T],L^6)}T. \tag{4.3}$$

Moreover, Lemma 2.6 and Lemma 2.7 give that

$$\|\frac{b_\theta^2}{r}\|_{L^1([0,T],L^6)}\le \|b_\theta\|_{L^\infty([0,T],L^\infty)}\|\frac{b_\theta}{r}\|_{L^\infty([0,T],L^6)}, \tag{4.4}$$

which implies that $\frac{b_\theta^2}{r}e_\theta\in L^1([0,T],L^6)$.

Therefore, making use of estimates (4.3), (4.4) and regular estimates of the velocity in the horizontal direction for (4.2), see Lemma 3.4 in [20], one has

$$\|\nabla_h\nabla\times u\|_{L^1([0,T],L^6)}\le C, \tag{4.5}$$

which along with the Sobolev embedding implies that

$$\|\nabla_h u\|_{L^1([0,T],L^\infty)}\le C. \tag{4.6}$$

Then (4.6) ensures that

$$\|\partial_r u_r\|_{L^1([0,T],L^\infty)}+\|\partial_r u_z\|_{L^1([0,T],L^\infty)}\le \|\nabla_h u\|_{L^1([0,T],L^\infty)}\le C. \tag{4.7}$$

Moreover, it follows from Lemma 2.2 and Lemma 3.2 that

$$\begin{aligned}\|\partial_z u_r\|_{L^1([0,T],L^\infty)}&\le C\int_0^T\|\partial_z\nabla u\|_{L^2}^{\frac12}\|\nabla_h\partial_z\nabla u\|_{L^2}^{\frac12}dt\\&\le CT^{\frac12}\|\partial_z\omega\|_{L^\infty([0,T],L^2)}^{\frac12}\|\nabla_h\partial_z\omega\|_{L^2([0,T],L^2)}^{\frac12}\\&\le C.\end{aligned} \tag{4.8}$$

Due to (2.16), (4.7), (4.8), one can reach that

$$\begin{aligned}\|\nabla u\|_{L^1([0,T],L^\infty)}&\le C\|\partial_r u_r\|_{L^1([0,T],L^\infty)}+\|\frac{u_r}{r}\|_{L^1([0,T],L^\infty)}\\&\quad+\|\partial_r u_z\|_{L^1([0,T],L^\infty)}+\|\partial_z u_r\|_{L^1([0,T],L^\infty)}+\|\partial_z u_z\|_{L^1([0,T],L^\infty)}\\&\le 2\|\frac{u_r}{r}\|_{L^1([0,T],L^\infty)}+2\|\partial_r u_r\|_{L^1([0,T],L^\infty)}+\|\partial_r u_z\|_{L^1([0,T],L^\infty)}+\|\partial_z u_r\|_{L^1([0,T],L^\infty)}\\&\le C.\end{aligned} \tag{4.9}$$

Now we prove

$$\int_0^T\|\nabla b\|_{L^\infty}dt\le C.$$

By $(1.5)_3$, one has

$$\begin{aligned}\partial_t\nabla b_\theta-\partial_{zz}\nabla b_\theta+u\cdot\nabla\nabla b_\theta&=-\nabla u\cdot\nabla b_\theta+\frac{u_r}{r}\cdot\nabla b_\theta+(\nabla u_r-\frac{u_r}{r})\frac{b_\theta}{r}\\&\quad+2(\partial_z\nabla b_\theta-\frac{\partial_z b_\theta}{r})\frac{b_\theta}{r}+2\frac{\partial_z b_\theta\cdot\nabla b_\theta}{r}.\end{aligned} \tag{4.10}$$

Then for any $p > 1$, taking inner product with $|\nabla b_\theta|^{p-2}\nabla b_\theta$ and integrating on $\mathbb{R}^3$, one obtains

$$\frac{d}{dt}\|\nabla b_\theta\|_{L^p} \le C(\|\nabla u\|_{L^\infty} + \|\frac{b_\theta}{r}\|_{L^\infty})\|\nabla b_\theta\|_{L^p}.$$

By Gronwall ineuqlity, there holds that

$$\|\nabla b_\theta\|_{L^p} \le \|\nabla b_0\|_{L^\infty} e^{C\int_0^T(\|\nabla u\|_{L^\infty}dt + \|\frac{b_\theta}{r}\|_{L^\infty})dt}.$$

Together with estimates (4.9) and Lemma 2.6, it can be reached that

$$\|\nabla b_\theta\|_{L^p} \le \|\nabla b_0\|_{L^\infty} e^{C\int_0^T\|\nabla u\|_{L^\infty}dt + CT\|\frac{b_0^\theta}{r}\|_{L^\infty}}.$$

The proof of the Lemma will be finished after letting $p \to \infty$. □

4.2. Proof of Theorem 1.1.

Theorem 1.1 can be shown by the classical Friedrichs method (see [2] for more details). For $n \ge 1$, let J_n be the spectral cut-off defined by

$$\widehat{J_n f(\xi)} = 1_{[0,n]}(|\xi|)\widehat{f}(\xi),\ \xi \in \mathbb{R}^3,$$

and $\mathcal{P}$ is the projection operator. Recall that $J_n^2 = J_n$, $\mathcal{P}^2 = \mathcal{P}$ and $J_n\mathcal{P} = \mathcal{P}J_n$.

We consider the follow approximate system:

$$\begin{cases} \partial_t u_n - \Delta_h u_n + \mathcal{P}J_n div(u_n \cdot \nabla u_n) = \mathcal{P}J_n(b_n \cdot \nabla b_n), \\ \partial_t b_n - \partial_{zz} b_n + \mathcal{P}J_n(u_n \cdot \nabla b_n) + \mathcal{P}J_n(\nabla \times ((\nabla \times b_n) \times b_n)) = \mathcal{P}J_n(b_n \cdot \nabla u_n), \\ \nabla \cdot u_n = \nabla \cdot b_n = 0, \\ (u_n, b_n)|_{t=0} = J_n(u_0, b_0). \end{cases} \tag{4.11}$$

Then for any $T > 0$, the system (4.11) has a unique solution $(u_n, b_n)_{n\in\mathbf{N}}$ satisfying

$$u_n \in \mathcal{C}([0,T); H^2(\mathbb{R}^3)), \quad b_n \in \mathcal{C}([0,T); H^2(\mathbb{R}^3)). \tag{4.12}$$

Furthermore, by using a priori estimates obtained in (2.7), Lemma 3.2 and Lemma 4.1 and applying standard compactness argument, we obtain that the approximate solutions $(u_n, b_n)_{n\in\mathbf{N}}$ converge to $(u, b) \in \mathcal{C}([0,\infty); H^2(\mathbb{R}^3))$ which is a solution of system (1.3). Moreover, the uniqueness of the solution can be shown by standard energy method. We omit the details here.

Acknowledgment

Q. Jiu was partially supported by the National Natural Science Foundation of China (NNSFC) (No. 11931010, No. 12061003). H. Yu was partially supported by the National Natural Science Foundation of China (NNSFC) (No. 11901040) and Beijing Municipal Education Commission (KM202011232020).

References

[1] Acheritogaray M, Degond P, Frouvelle A, Liu JG. Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system. Kinetic and Related Models 2011; 4 (4): 901-918. https://doi.org/10.3934/krm.2011.4.901

[2] Bahouri H, Chemin J-Y, Danchin R. Fourier analysis and nonlinear partial differential equations. Grundlehren der mathematischen Wissenschaften. Springer, Heidelberg, 2011.

[3] Chae D, Degond P. Liu JG. Well-posedness for Hall-magnetohydrodynamics, Annales de l'Institut Henri Poincare (C) Non Linear Analysis 2014; 31: 555-565. https://doi.org/10.1016/j.anihpc.2013.04.006

[4] Chae D, Lee J. On the blow-up criterion and small data global existence for the Hall- magnetohydrodynamics. Journal of Differential Equations 2014; 256 (11): 3835-3858. https://doi.org/10.1016/j.jde.2014.03.003

[5] Chae D, Lee J. On the regularity of the axisymmetric solutions of the Navier-Stokes equations. Mathematische Zeitschrift 2002; 239 : 645-671. https://doi.org/10.1007/s002090100317

[6] Chae D, Weng S. Singularity formation for the incompressible Hall-MHD equations without resistivity. Annales de l'Institut Henri Poincaré C. Analyse Non Linéaire 2016; 4 (33): 1009-1022. https://doi.org/10.1016/J.ANIHPC.2015.03.002

[7] Chemin J-Y, Desjardins B, Gallagher I, Grenier E. Mathematical Geophysics: An Introduction to Rotating fluids and to the Navier-Stokes Equations. Oxford: Oxford Unversity Press, 2006.

[8] Chemin J-Y, Desjardins B, Gallagher I, Grenier E. Fluids with anisotropic viscosity. ESAIM. Mathematical Modelling and Numerical Analysis 2000; 34 (2): 315-335. https://doi.org/10.1051/m2an:2000143

[9] Danchin R, Tan J. On the well-posedness of the Hall-magnetohydrodynamics system in critical spaces. Communications in Partial Differential Equations 2021; 46 (1): 31-65. https://doi.org/10.1080/03605302.2020.1822392

[10] Dai M. Local well-posedness for the Hall-MHD system in optimal Sobolev spaces. Journal of Differential Equations 2021; 289: 159-181. https://doi.org/10.1016/j.jde.2021.04.019.

[11] Fan J, Huang S, Nakamura G. Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodyna -mic equations. Applied Mathematics Letters 2013; 26 (9): 963-967. https://doi.org/10.1016/j.aml.2013.04.008.

[12] Fei MG, Xiang ZY. On the blow-up criterion and small data global existence for the Hall- magnetohydrodynamics with horizontal dissipation. Journal of Mathematical Physics 2015; 56 (5): 051504, 13 pp. https://doi.org/10.1063/1.4921653

[13] Forbes TG. Magnetic reconnection in solar flares,Geophysical Astrophysical Fluid Dynamics 1991; 62: 15-36. https://doi.org/10.1080/03091929108229123

[14] Homann H, Grauer R. Bifurcation analysis of magnetic reconnection in Hall-MHD-systems. Physica D. Nonlinear Phenomena 2005; 208 (1-2): 59-72. https://doi.org/10.1016/j.physd.2005.06.003

[15] Jiu QS, Liu JT. Global regularity for the 3D axisymmetric MHD Equations with horizontal dissipation and vertical magnetic diffusion. Discrete and Continuous Dynamical Systems. Series A 2015; 35 (1): 301-322. https://doi.org/10.3934/dcds.2015.35.301

[16] Jiu QS, Yu H, Zheng XX. Global well-posedness for axisymmetric MHD system with only vertical viscosity. Journal of Differential Equations 2017; 263 (5): 2954-2990. https://doi.org/10.1016/j.jde.2017.04.021

[17] Lei Z. On axially symmetric incompressible magnetohydrodynamics in three dimensions. Journal of Differential Equations 2015; 259 (7): 3202-3215. https://doi.org/10.1016/j.jde.2015.04.017

[18] Li Z, Cui M. Global Well-Posedness for the 3D Axisymmetric HallMHD System with Horizontal Dissipation, Journal of Nonlinear Mathematical Physics 2022; 29: 794-817. https://doi.org/10.1007/s44198-022-00063-8

[19] Lighthill MJ. Studies on magneto-hydrodynamic waves and other anisotropic wave motions. Philosophical Transactions of the Royal Society of London. Series A 1960; 252 (1014): 397-430. https://doi.org/10.1098/rsta.1960.0010

[20] Miao CX, Zheng XX. On the global well-posedness for the Boussinesq system with horizontal dissipation. Communications in Mathematical Physics 2013; 321 (1): 33-67. https://doi.org/10.1007/s00220-013-1721-2

[21] Miao CX, Zheng XX. Global well-posedness for axisymmetric Boussinesq system with horizontal viscosity. Journal de Mathématiques Pures et Appliquées. Neuvième Série 2014; 101 (6): 842-872. https://doi.org/10.1016/j.matpur.2013.10.007

[22] Simakov AN, Chacón L. Quantitative, Analytical model for magnetic reconnection in Hall magnetohydrodynamics. Physical Review Letters 2008; 101 (10): 105003. https://doi.org/10.1103/PhysRevLett.101.105003

[23] Wan RH, Zhou Y. Low regularity well-posedness for the 3D generalized Hall-MHD system. Acta Applicandae Mathematicae 2017; 147: 95-111. https://doi.org/10.1007/s10440-016-0070-5