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Studying a doubly nonlinear model of slightly compressible Forchheimer flows in rotating porous media

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Abstract: We study the generalized Forchheimer flows of slightly compressible fluids in rotating porous media. In the problem's model, the varying density in the Coriolis force is fully accounted for without any simplifications. It results in a doubly nonlinear parabolic equation for the density. We derive a priori estimates for the solutions in terms of the initial, boundary data and physical parameters, emphasizing on the case of unbounded data. Weighted Poincaré–Sobolev inequalities suitable to the equation's nonlinearity, adapted Moser's iteration, and maximum principle are used and combined to obtain different types of estimates.

Key words: Forchheimer flows, porous media, compressible fluids, rotating fluids, doubly nonlinear equation, Poincaré–Sobolev inequality, Moser iteration, maximum estimates

1. Introduction

We continue the investigation of the Forchheimer flows of slightly compressible fluids in rotating porous media, which was initiated in our previous work [10]. In [10], we simplified the Coriolis force's dependence on the density in the model in order to reduce the complexity of the problem. The resulting partial differential equation (PDE) was of degenerate parabolic type and we were able to understand its key nonlinear structure, and derived various estimates for its solutions. In this paper, we study the full model without any simplifications. As we will see, the PDE becomes a doubly nonlinear parabolic equation. We will analyze this more complicated equation in more general context by realizing its new structure and utilizing other techniques with appropriate adaptations and improvements.

We consider a porous medium, with constant porosity $\tilde{\varnothing} \in (0,1)$ and constant permeability $k > 0$, rotated with a constant angular velocity $\tilde{\Omega}\vec{k}$, where $\tilde{\Omega} \geq 0$ is the constant angular speed, and $\vec{k}$ is a constant unit vector. We study the dynamics of fluid flows in this porous medium.

The equation for the Darcy flows in rotating porous media written in a rotating frame is, see Vadasz [25],

$$\frac{\mu}{k}v + \frac{2\rho\tilde{\Omega}}{\tilde{\varnothing}}\vec{k} \times v + \rho\tilde{\Omega}^2\vec{k} \times (\vec{k} \times x) = -\nabla p + \rho\vec{g}, \tag{1.1}$$

where μ is the dynamic viscosity, v is the velocity, ρ is the fluid density, p is the pressure, x is the position in

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the rotating frame, $\vec{g}$ is the gravitational acceleration, $\tilde{\Omega}^2\vec{k}\times(\vec{k}\times x)$ is centripetal acceleration, and $(2\rho\tilde{\Omega}/\tilde{\varnothing})\vec{k}\times v$ represents the Coriolis effects in the rotating porous medium.

For fluid flows that obey Forchheimer's two-term law, we have

$$\frac{\mu}{k}v + \frac{c_F\rho}{\sqrt{k}}|v|v + \frac{2\rho\tilde{\Omega}}{\tilde{\varnothing}}\vec{k}\times v + \rho\tilde{\Omega}^2\vec{k}\times(\vec{k}\times x) = -\nabla p + \rho\vec{g}, \tag{1.2}$$

where $c_F > 0$ is the Forchheimer constant [26]. Other equations for Forchheimer's three-term and power laws can be obtained similarly.

Equations (1.1) and (1.2) can be written in one general form, namely, the generalized Forchheimer equation in rotating porous media

$$\sum_{i=0}^{N} a_i\rho^{\bar{\alpha}_i}|v|^{\bar{\alpha}_i}v + \frac{2\rho\tilde{\Omega}}{\tilde{\varnothing}}\vec{k}\times v + \rho\tilde{\Omega}^2\vec{k}\times(\vec{k}\times x) = -\nabla p + \rho\vec{g}. \tag{1.3}$$

Regarding the first sum in equation (1.3), the dependence on the density is expressed by the term $\rho^{\bar{\alpha}_i}$ which is obtained by using Muskat's dimension analysis [20].

For the Forchheimer equations and other related models of fluid flows in porous media that differ from the ubiquitous Darcy's law, the interested reader is referred to the books [3, 21, 24]. Regarding their mathematical analysis in the case without rotation, see [4, 5, 11, 13, 19, 22–24] for incompressible fluids, see [2, 6–9, 14–17] for compressible fluids, and references therein. For more information about fluid flows in rotating porous media, see [25] and, also, our previous mathematical study [10].

Hereafter, we fix the integer $N \geq 1$, the powers $\bar{\alpha}_0 = 0 < \bar{\alpha}_1 < \bar{\alpha}_2 < \ldots < \bar{\alpha}_N$, and positive constant coefficients $a_0, a_1, \ldots, a_N$.

Define a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(s) = a_0 + a_1 s^{\bar{\alpha}_1} + \cdots + a_N s^{\bar{\alpha}_N} = \sum_{i=0}^{N} a_i s^{\bar{\alpha}_i} \quad \text{for } s \geq 0. \tag{1.4}$$

In (1.4) and throughout the paper, we conveniently use $0^0 = 1$.

Set $\mathcal{R}(\rho) = 2\rho\tilde{\Omega}/\tilde{\varnothing}$. Multiplying both sides of (1.3) by ρ gives

$$g(|\rho v|)\rho v + \mathcal{R}(\rho)\vec{k}\times(\rho v) = -\rho\nabla p + \rho^2\vec{g} - \rho^2\tilde{\Omega}^2\vec{k}\times(\vec{k}\times x). \tag{1.5}$$

We solve for ρv from (1.5) in terms of the vector on its right-hand side and the $\mathcal{R}(\rho)$. To do that, we define the function $F_z : \mathbb{R}^3 \to \mathbb{R}^3$, for any $z \in \mathbb{R}$, by

$$F_z(v) = g(|v|)v + z\mathbf{J}v \quad \text{for } v \in \mathbb{R}^3, \tag{1.6}$$

where $\mathbf{J}$ is the 3×3 matrix for which $\mathbf{J}x = \vec{k}\times x$ for all $x \in \mathbb{R}^3$.

Equation (1.5) is rewritten as

$$F_{\mathcal{R}(\rho)}(\rho v) = -(\rho\nabla p - \rho^2\vec{g} + \rho^2\tilde{\Omega}^2\mathbf{J}^2 x). \tag{1.7}$$

Thanks to [10, Lemma 1.1], the function F_z is odd and bijective for each $z \in \mathbb{R}$. Then we can invert (1.7) to have

$$\rho v = -F^{-1}_{\mathcal{R}(\rho)}(\rho \nabla p - \rho^2 \vec{g} + \rho^2 \tilde{\Omega}^2 \mathbf{J}^2 x). \tag{1.8}$$

In article [10], $\mathcal{R}(\rho)$ was approximated by a constant $\mathcal{R} = 2\rho_* \tilde{\Omega}/\tilde{\varnothing}$, for some constant density ρ_*. This resulted in a simpler equation than (1.8). On the contrary, we will keep the dependence of $\mathcal{R}(\rho)$ on ρ in the current paper, and treat equation (1.8) in that original form.

We recall that the fluid's compressibility for isothermal conditions is

$$\varpi = -\frac{1}{V}\frac{dV}{dp} = \frac{1}{\rho}\frac{d\rho}{dp},$$

where V, here, denotes the fluid's volume. In many cases such as (isothermal) compressible liquids, ϖ is assumed to be a constant [3, 20]. In particular, it is a small positive constant for (isothermal) slightly compressible fluids such as crude oil and water. This condition is commonly used in petroleum and reservoir engineering [1, 12], where the fluid dynamics in porous media have important applications. The current paper is focused on (isothermal) slightly compressible fluids; hence, we study the following equation of state

$$\frac{1}{\rho}\frac{d\rho}{dp} = \varpi, \quad \text{where the constant compressibility } \varpi > 0 \text{ is small.} \tag{1.9}$$

The equation of continuity is

$$\tilde{\varnothing}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0. \tag{1.10}$$

Note, by (1.9), that $\rho \nabla p = \varpi^{-1} \nabla \rho$. Then combining (1.10) with (1.8), we obtain

$$\tilde{\varnothing}\frac{\partial \rho}{\partial t} = \nabla \cdot (F^{-1}_{\mathcal{R}(\rho)}(\varpi^{-1}\nabla \rho - \rho^2 \vec{g} + \rho^2 \tilde{\Omega}^2 \mathbf{J}^2 x)). \tag{1.11}$$

The gravitational field in the rotating frame is $\vec{g}(t) = -\tilde{\mathcal{G}}\tilde{e}_0(t)$, where $\tilde{\mathcal{G}} > 0$ is the gravitational constant, and $\tilde{e}_0 \in C^\infty(\mathbb{R}, \mathbb{R}^3)$ with $|\tilde{e}_0(t)| = 1$ for all $t \in \mathbb{R}$.

We make a simple change of variable $u = \rho/\varpi$, and corresponding scaling of parameters

$$\varnothing = \varpi\tilde{\varnothing} > 0, \quad \mathcal{G} = \varpi^2 \tilde{\mathcal{G}}, \quad \Omega = \varpi \tilde{\Omega}.$$

Note that

$$\mathcal{R}(\rho) = R_* u, \text{ where } R_* = 2\varpi\tilde{\Omega}/\tilde{\varnothing} = 2\varpi\Omega/\varnothing. \tag{1.12}$$

Then we obtain from (1.11) that

$$\varnothing \frac{\partial u}{\partial t} = \nabla \cdot \left(X\left(u, \nabla u + u^2[-\mathcal{G}\tilde{e}_0(t) + \Omega^2 \mathbf{J}^2 x]\right)\right), \tag{1.13}$$

where

$$X(z, y) = F^{-1}_{R_* z}(y) \text{ for } z \in \mathbb{R}, \ y \in \mathbb{R}^3. \tag{1.14}$$

By making another transformation $\tilde{u}(x,t) = u(x, \varnothing t)$ and rewriting equation (1.13) for $\tilde{u}(x,t)$ and then removing the tilde notation, we obtain

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(X\left(u, \nabla u + u^2 \mathcal{Z}(x,t)\right)\right), \tag{1.15}$$

where

$$\mathcal{Z}(x,t) = -\mathcal{G}e_0(t) + \Omega^2 \mathbf{J}^2 x \text{ with } e_0(t) = \tilde{e}_0(\varnothing t). \tag{1.16}$$

We will focus on the Dirichlet boundary condition for $u(x,t)$. Let U be an open, bounded set in $\mathbb{R}^3$ with C^1 boundary $\Gamma = \partial U$. We study the initial boundary value problem (IBVP)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot \left(X\left(u, \nabla u + u^2 \mathcal{Z}(x,t)\right)\right) & \text{in} \quad U \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in} \quad U \\ u(x,t) = \psi(x,t) & \text{in} \quad \Gamma \times (0,\infty), \end{cases} \tag{1.17}$$

where $u_0(x)$ and $\psi(x,t)$ are given.

In a previous article [10], the maximum estimates for the solutions are achieved by the use of the maximum principle. This method requires the initial data to be bounded. In this paper, we aim at treating also unbounded initial data. For that, we will use the Moser iteration. Regarding the newly obtained PDE (1.15), it has extra dependence on u, in addition to $\nabla u + u^2\mathcal{Z}(x,t)$. This dependence turns out to yield new weights, which depend on the solution u itself, in the energy estimates. Therefore, more technical treatments are required. Indeed, we establish suitable weighted Poincaré–Sobolev inequalities to deal with these weights. We are then able to estimate the Lebesgue norms of the solutions, and, by the Moser iteration, their essential supremum. These short-time estimates are combined with the maximum principle to give all time estimates. Moreover, we highlight that our estimates are derived by appropriately handled techniques to provide explicit dependence on physical parameters including the angular speed of the rotation.

The paper is organized as follows. In Section 2, we present crucial properties of the function $X(z,y)$ by recasting the corresponding results in [10] but with explicit dependence on z, see Lemmas 2.1 and 2.2. We also establish some elliptic and parabolic Poincaré-Sobolev inequalities with certain weights. These particular inequalities are then formulated in suitable forms for our treatment of the double nonlinearity in (1.15), see Lemma 2.3, Corollary 2.4 and Lemma 2.5. In Section 3, we study the IBVP (3.3) for $\bar{u}(x,t)$, which, briefly speaking, is a nonnegative solution $u(x,t)$ of (1.17) shifted by the boundary data. We obtain the L^α-estimates, for sufficient large $\alpha \in (0,\infty)$, for $\bar{u}$ in terms of the initial and boundary data, see Theorem 3.2. We also establish in Theorem 3.2 a weighted $L^{2-a}_{x,t}$-estimate for the gradient of $\bar{u}$, with the number $a \in (0,1)$ defined in (2.9) and the weight function depending on the solution u. Section 4 is focused on the L^∞-estimates for $\bar{u}$. By adapting Moser's iteration, we derive, in Theorem 4.5, an upper bound for $\bar{u}$'s L^∞-norm expressed in terms of its L^α-norm for some finite number $\alpha > 0$. The main estimate, for small time $t > 0$, is then obtained in Theorem 4.6 in terms of certain L^α-norms of the initial and boundary data. All estimates' dependence on the physical parameters is expressed via the number χ_*, see (3.4). It is meticulously tracked in each step of the complicated iteration. In Section 5, we establish the maximum principle for classical solutions of (1.15) in Theorem 5.1. Combining this maximum principle with the short-time estimates in Section 4, we obtain the maximum estimates in Theorem 5.2 for nonnegative solutions of the IBVP (1.17) for all time $t > 0$ even when the initial data is unbounded.

2. Preliminaries

2.1. Notation

A vector $x \in \mathbb{R}^n$ is denoted by a n-tuple $(x_1, x_2, \ldots, x_n)$ and considered a column vector, i.e. a $n \times 1$ matrix. Hence, x^{T} is the $1 \times n$ matrix $(x_1\ x_2 \ldots x_n)$.

For two vectors $x, y \in \mathbb{R}^n$, their dot product is $x \cdot y = x^{\mathrm{T}} y = y^{\mathrm{T}} x$, while xy^{T} is the $n \times n$ matrix $(x_i y_j)_{i,j=1,2,\ldots,n}$.

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be any $n \times n$ matrices of real numbers. Their inner product is

$$\mathbf{A} : \mathbf{B} \stackrel{\text{def}}{=} \operatorname{trace}\left(\mathbf{A}\mathbf{B}^{\mathrm{T}}\right) = \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$

The Euclidean norm of the matrix $\mathbf{A}$ is

$$|\mathbf{A}| = (\mathbf{A} : \mathbf{A})^{1/2} = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

(Note that we do not use $|\mathbf{A}|$ to denote the determinant in this paper.)

When $\mathbf{A}$ is considered a linear operator, another norm is defined by

$$\|\mathbf{A}\|_{\mathrm{op}} = \max\left\{ \frac{|\mathbf{A}x|}{|x|} : x \in \mathbb{R}^n, x \neq 0 \right\} = \max\{|\mathbf{A}x| : x \in \mathbb{R}^n, |x| = 1\}.$$

It is well-known that

$$\|\mathbf{A}\|_{\mathrm{op}} \leq |\mathbf{A}| \leq c_* \|\mathbf{A}\|_{\mathrm{op}}, \tag{2.1}$$

where $c_* = c_*(n)$ is a positive constant independent of $\mathbf{A}$.

Clearly, the matrix $\mathbf{J}$ in (1.6) satisfies

$$|\mathbf{J}x| \leq |\vec{k}||x| = |x| \text{ and } |\mathbf{J}^2 x| \leq |\mathbf{J}x| \leq |x| \text{ for all } x \in \mathbb{R}^3. \tag{2.2}$$

For a function $f = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, its derivative is the $m \times n$ matrix

$$Df = \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n}. \tag{2.3}$$

In particular, when $m = 1$, i.e. $f : \mathbb{R}^n \to \mathbb{R}$, the derivative is

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \ldots & \frac{\partial f}{\partial x_n} \end{pmatrix},$$

while its gradient vector is $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}) = (Df)^{\mathrm{T}}$.

The Hessian matrix is

$$D^2 f = D(\nabla f) = \left(\frac{\partial^2 f}{\partial x_j \partial x_i} \right)_{i,j=1,2,\ldots,n}.$$

We also write $D_x f$ for Df in (2.3) in case the variables need to be indicated explicitly.

2.2. Auxiliary inequalities

The following is a convenient consequence of Young's inequality. If $x_i \geq 0$ and $z_i > 1$ for $i = 1, 2, \ldots, k$ with $k \geq 2$ such that $\sum_{i=1}^{k} 1/z_i = 1$, then

$$\prod_{i=1}^{k} x_i \leq \sum_{i=1}^{k} x_i^{z_i}. \tag{2.4}$$

For the sake of brevity, we call (2.4) Young's inequality in this paper.

For $x, y \geq 0$, one has

$$x^\beta \leq x^\alpha + x^\gamma \text{ for } 0 \leq \alpha \leq \beta \leq \gamma, \tag{2.5}$$

$$(x+y)^p \leq 2^{(p-1)^+}(x^p + y^p) \text{ for } p > 0, \tag{2.6}$$

where $z^+ = \max\{z, 0\}$ for any $z \in \mathbb{R}$. We also frequently use the following alternative form of (2.6)

$$(x+y)^p \leq 2^p(x^p + y^p) \text{ for all } x, y \geq 0, \ p > 0. \tag{2.7}$$

By the triangle inequality and inequality (2.6), we have

$$|x \pm y|^p \geq 2^{-(p-1)^+}|x|^p - |y|^p \quad \text{for all } x, y \in \mathbb{R}^n, \quad p > 0. \tag{2.8}$$

The interpolation inequality for the Lebesgue integrals: if $0 < p < s < q$ and $1/s = \theta/p + (1-\theta)/q$ for $\theta \in (0,1)$, then

$$\left(\int |f|^s d\mu\right)^{\frac{1}{s}} \leq \left(\int |f|^p d\mu\right)^{\frac{\theta}{p}} \left(\int |f|^q d\mu\right)^{\frac{1-\theta}{q}}.$$

2.3. Characteristics of the function $X(z, y)$

Note that $v = \widetilde{X}(z, y) \overset{\text{def}}{=} F_z^{-1}(y)$ is the unique solution of the equation

$$G(z, y, v) \overset{\text{def}}{=} F_z(v) - y = g(|v|)v + z\mathbf{J}v - y = 0 \text{ for } z \in \mathbb{R}, y, v \in \mathbb{R}^3.$$

The partial derivatives of G are

$$\begin{aligned} D_v G(z, y, v) &= DF_z(v) = g'(|v|)\frac{vv^{\mathrm{T}}}{|v|} + g(|v|)\mathbf{I}_3 + z\mathbf{J} \text{ for } v \neq 0, \\ D_v G(z, y, 0) &= DF_z(0) = g(0)\mathbf{I}_3 + z\mathbf{J}, \\ D_z G(z, y, v) &= \mathbf{J}v, \quad D_y G(z, y, v) = -\mathbf{I}_3. \end{aligned}$$

One can verify that $G \in C^1(\mathbb{R}^7)$. Same as in [10, Lemma 2.3], $D_v G$ is invertible on $\mathbb{R}^7$. By the Implicit Function Theorem, the solution $v = \widetilde{X}(z, y)$ belongs to $C^1(\mathbb{R}^4)$. Consequently, the function $X(z, y) = \widetilde{X}(R_* z, y)$ belongs to $C^1(\mathbb{R}^4)$.

Throughout the paper, we denote

$$a = \frac{\bar{\alpha}_N}{1 + \bar{\alpha}_N} \in (0,1), \quad \chi_0 = g(1) = \sum_{i=0}^{N} a_i. \tag{2.9}$$

The properties of the function $X(z,y)$, which is defined by (1.14), are similar to those established in [10, Lemmas 2.1 and 2.4]. Now that X depends on z, we need some explicit dependence on z for the inequalities there. In fact, thanks to (1.12), we can replace $\chi_1 = \chi_0 + \mathcal{R}$ in [10, Lemmas 2.1 and 2.4] with $\chi_0 + R_* z$ and hence rewrite those two lemmas as Lemmas 2.1 and 2.2 below. Denote $\mathbb{R}_+ = [0,\infty)$.

Lemma 2.1 (i) *One has*

$$\frac{c_1(\chi_0+R_*z)^{-1}|y|}{(1+|y|)^a} \le |X(z,y)| \le \frac{c_2(\chi_0+R_*z)^a|y|}{(1+|y|)^a} \text{ for all } z\in\mathbb{R}_+, y\in\mathbb{R}^3, \tag{2.10}$$

where $c_1 = \min\{1,\chi_0\}^a$ *and* $c_2 = 2^a c_1^{-1}\min\{a_0,a_N\}^{-1}$. *Alternatively,*

$$(\chi_0+R_*z)^{-(1-a)}|y|^{1-a} - 1 \le |X(z,y)| \le c_3|y|^{1-a} \text{ for all } z\in\mathbb{R}_+, y\in\mathbb{R}^3, \tag{2.11}$$

where $c_3 = (a_N)^{a-1}$.

(ii) *One has*

$$\frac{c_4(\chi_0+R_*z)^{-2}|y|^2}{(1+|y|)^a} \le X(z,y)\cdot y \le \frac{c_2(\chi_0+R_*z)^a|y|^2}{(1+|y|)^a} \text{ for all } z\in\mathbb{R}_+, y\in\mathbb{R}^3, \tag{2.12}$$

where $c_4 = (\min\{1,a_0,a_N\}/2^{\alpha_N})^{1+a}$. *Alternatively,*

$$c_5(\chi_0+R_*z)^{-2}(|y|^{2-a}-1) \le X(z,y)\cdot y \le c_3|y|^{2-a} \text{ for all } z\in\mathbb{R}_+, y\in\mathbb{R}^3, \tag{2.13}$$

where $c_5 = 2^{-a}c_4$.

Although inequalities (2.10) and (2.12) provide more precise dependence on $|y|$ than (2.11) and (2.13), the latter two are sufficient and more convenient in this paper.

Lemma 2.2 *For all* $z\in\mathbb{R}_+$ *and* $y\in\mathbb{R}^3$, *the matrix* D_yX *of partial derivatives in the variable* y *satisfies*

$$c_6(\chi_0+R_*z)^{-1}(1+|y|)^{-a} \le |D_yX(z,y)| \le c_7(1+\chi_0+R_*z)^a(1+|y|)^{-a}, \tag{2.14}$$

$$\xi^{\mathrm{T}} D_yX(z,y)\xi \ge c_8(\chi_0+R_*z)^{-2}(1+|y|)^{-a}|\xi|^2 \text{ for all } \xi\in\mathbb{R}^3, \tag{2.15}$$

where

$$c_6 = \sqrt{3}(2^{-\alpha_N}\min\{1,a_N\})^a/(\alpha_N+2),\ c_7 = c_*2^{\alpha_N}/\min\{a_0,a_N\},\ c_8 = c_4/(\alpha_N+2)^2$$

with $c_* = c_*(3)$ *given in* (2.1).

To complement the estimate of D_yX in (2.14), we derive in (2.17) and (2.18) below some estimates for D_zX. Taking the partial derivative in z of the equation $G(R_*z,y,X(z,y))=0$, we have

$$0 = R_*\mathbf{J}X(z,y) + DF_{R_*z}(X(z,y))D_zX(z,y) = R_*\mathbf{J}X(z,y) + (D_yX(z,y))^{-1}D_zX(z,y),$$

which implies

$$D_zX(z,y) = -R_*D_yX(z,y)\mathbf{J}X(z,y). \tag{2.16}$$

Combining formula (2.16) with estimates (2.14) and (2.10), respectively, (2.11) yields

$$|D_zX(z,y)| \leq c_2c_7R_*(1+\chi_0+R_*z)^{2a}|y|(1+|y|)^{-2a}, \tag{2.17}$$

respectively,

$$|D_zX(z,y)| \leq c_3c_7R_*(1+\chi_0+R_*z)^{a}|y|^{1-a}(1+|y|)^{-a}. \tag{2.18}$$

2.4. Weighted Poincaré–Sobolev inequalities

In this subsection, we consider the space $\mathbb{R}^n$, with $n \geq 2$, and an open, bounded set $U \subset \mathbb{R}^n$. For a number $p \in [1,n)$, its Sobolev conjugate is $p^* = np/(n-p)$. We establish some specific inequalities of Poincaré–Sobolev type with weight functions.

Lemma 2.3 (Elliptic version) *Suppose p and r_* are positive numbers that satisfy*

$$\frac{n}{n+p} < r_* < 1 \text{ and } \frac{1}{p} \leq r_* < \frac{n}{p}. \tag{2.19}$$

Let r, s, and α be numbers such that

$$r > 0,\ \alpha > 0,\ \alpha \geq s \geq 0,\ \text{and } \alpha > \frac{nr_*(r-p+s)}{r_*(n+p)-n}. \tag{2.20}$$

Denote

$$m = \frac{\alpha - s + p}{p}. \tag{2.21}$$

Let $u(x)$ be a function that vanishes on ∂U with $|u|^m \in W^{1,r_*p}(U)$, and $W(x)$ be a positive function on U. Then one has, for any $\varepsilon > 0$, that

$$\int_U |u|^{\alpha+r}dx \leq \varepsilon \int_U |u|^{\alpha-s}|\nabla u|^p W dx + \varepsilon^{-\frac{\theta}{1-\theta}} (\bar{c}m)^{\frac{\theta p}{1-\theta}} \|u\|_{L^\alpha}^{\alpha+\mu} \|W^{-1}\|_{L^{\frac{r_*}{1-r_*}}}^{\frac{\theta}{1-\theta}}, \tag{2.22}$$

where

$$\theta = \frac{rnr_*}{nr_*(p-s)+\alpha(r_*(n+p)-n)} \in (0,1), \quad \mu = \frac{r+\theta(s-p)}{1-\theta} > -\alpha, \tag{2.23}$$

and positive constant $\bar{c}$, which appears in (2.25) below, depends on U and r_*p, but not on u, W, r, α, s.

Proof We can use the calculations in the proof of Lemma 2.1(ii) up to inequality (2.17) in [9] applied to

$$\bar{p} := r_*p, \quad \bar{\alpha} := r_*\alpha, \text{ and } \bar{s} := r_*s.$$

Then other numbers m in [9, (2.8)] and q in [9, (2.16)] become

$$\bar{m} = \frac{\bar{\alpha} - \bar{s} + \bar{p}}{\bar{p}} = m \text{ and } \bar{q} = \bar{p}^*\bar{m} = (r_*p)^*m = \frac{nr_*(\alpha - s + p)}{n - r_*p}. \tag{2.24}$$

Thanks to the last condition in (2.19) and the first condition in (2.20), one has $1 \le \bar{p} < n$, $\bar{\alpha} \ge \bar{s}$ and $\bar{m} = m \ge 1$. Because $\bar{m} \ge 1$ and u vanishes on ∂U, we have the following Poincaré–Sobolev inequality for $|u|^{\bar{m}}$, which corresponds to inequality [9, (2.14)],

$$\| |u|^{\bar{m}} \|_{L^{\bar{p}^*}} \le \bar{c} \| \nabla (|u|^{\bar{m}}) \|_{L^{\bar{p}}}, \tag{2.25}$$

where $\bar{c} > 0$ depends on U and $\bar{p}$. Elementary calculations, see inequality [9, (2.17)], yield from (2.25) that

$$\begin{aligned} \|u\|_{L^{\bar{q}}} &\le (\bar{c}\bar{m})^{1/\bar{m}} \left(\int_U |u|^{\bar{\alpha}-\bar{s}} |\nabla u|^{\bar{p}} dx \right)^{1/(\bar{\alpha}-\bar{s}+\bar{p})} \\ &= (\bar{c}m)^{\frac{1}{m}} \left(\int_U \left[|u|^{\alpha-s} |\nabla u|^p W(x) \right]^{r_*} \cdot W(x)^{-r_*} dx \right)^{\frac{1}{r_*(\alpha-s+p)}}. \end{aligned}$$

Denote $I = \int_U |u|^{\alpha-s} |\nabla u|^p W dx$ and note that $\alpha - s + p = mp$. Applying Hölder's inequality with powers $1/r_*$ and $1/(1-r_*)$ to the last integral gives

$$\|u\|_{L^{\bar{q}}} \le (\bar{c}m)^{\frac{1}{m}} I^{\frac{1}{mp}} \|W^{-1}\|^{\frac{1}{mp}}_{L^{\frac{r_*}{1-r_*}}}. \tag{2.26}$$

Thanks to the fact $r_* > n/(n+p)$, we have $r_*(n+p) - n > 0$ and $nr_* > n - r_* p$, which, together with the last assumption in (2.20), yield

$$\alpha > \frac{nr_* r + nr_*(s-p)}{r_*(n+p) - n} > \frac{(n - r_* p) r + nr_*(s-p)}{r_*(n+p) - n}.$$

This implies $\alpha + r < \bar{q}$.

Because $\alpha < \alpha + r < \bar{q}$, we can find a number $\theta_0 \in (0,1)$ such that

$$\frac{1}{\alpha + r} = \frac{\theta_0}{\bar{q}} + \frac{1-\theta_0}{\alpha}.$$

In fact, θ_0 is explicitly given by

$$\theta_0 = \frac{r\bar{q}}{(\alpha + r)(\bar{q} - \alpha)}. \tag{2.27}$$

Applying interpolation inequality and combining it with (2.26), we have

$$\int_U |u|^{\alpha+r} dx \le \left(\|u\|^{\theta_0}_{L^{\bar{q}}} \|u\|^{1-\theta_0}_{L^\alpha} \right)^{\alpha+r} \le (\bar{c}m)^{\frac{\theta_0(\alpha+r)}{m}} I^{\frac{\theta_0(\alpha+r)}{mp}} \|W^{-1}\|^{\frac{\theta_0(\alpha+r)}{mp}}_{L^{\frac{r_*}{1-r_*}}} \|u\|^{(1-\theta_0)(\alpha+r)}_{L^\alpha}. \tag{2.28}$$

Denoting

$$\theta = \frac{\theta_0(\alpha + r)}{mp}, \tag{2.29}$$

we rewrite (2.28) as

$$\int_U |u|^{\alpha+r} dx \le (\varepsilon^\theta I^\theta) \cdot \left(\varepsilon^{-\theta} (\bar{c}m)^{\theta p} \|W^{-1}\|^{\theta}_{L^{\frac{r_*}{1-r_*}}} \|u\|^{(1-\theta_0)(\alpha+r)}_{L^\alpha} \right). \tag{2.30}$$

Using the values of m, $\bar{q}$, and θ_0 in (2.21), (2.24), and (2.27), respectively, we calculate the number θ in (2.29) and find that it is the same as in (2.23).

By, again, the last assumption in (2.20), we have $\theta \in (0,1)$. Then applying Young's inequality (2.4) with powers $1/\theta$ and $1/(1-\theta)$ to the product on the right-hand side of inequality (2.30) gives

$$\int_U |u|^{\alpha+r}dx \le \varepsilon I + \varepsilon^{-\frac{\theta}{1-\theta}}(\bar{c}m)^{\frac{\theta p}{1-\theta}}\|W^{-1}\|_{L^{\frac{r_*}{1-r_*}}}^{\frac{\theta}{1-\theta}}\|u\|_{L^\alpha}^{\frac{(1-\theta_0)(\alpha+r)}{1-\theta}}. \tag{2.31}$$

Recalculating the last power, with the use of identity in (2.29), we have

$$\frac{(1-\theta_0)(\alpha+r)}{1-\theta} = \frac{(\alpha+r)-\theta mp}{1-\theta} = \frac{(\alpha+r)-\theta(\alpha-s+p)}{1-\theta} = \alpha+\mu. \tag{2.32}$$

Then we obtain (2.22) from (2.31). Since $\theta_0, \theta \in (0,1)$ in (2.32), we have $\alpha+\mu>0$, which gives $\mu > -\alpha$ in (2.23). □

Note from (2.19) that $p \ge 1/r_* > 1$. Conversely, if $p>1$, then the set of r_* that satisfies (2.19) is not empty.

Lemma 2.4 *Assume (2.19) and (2.20), and let $\bar{c}, m, \theta, \mu$ be defined as in Lemma 2.3.*

Let $u(x)$ be as in Lemma 2.3, and $\varphi(x)$ be a function on U, and define $v = u+\varphi$. Let $\beta>0$ and $\varepsilon>0$.

(i) *Then one has*

$$\begin{aligned}\int_U |u|^{\alpha+r}dx &\le \varepsilon\int_U |u|^{\alpha-s}|\nabla u|^p(1+|v|)^{-\beta}dx\\ &+2^{\frac{\theta(1+(\beta-1)r_*)}{(1-\theta)r_*}}\varepsilon^{-\frac{\theta}{1-\theta}}(\bar{c}m)^{\frac{\theta p}{1-\theta}}\|u\|_{L^\alpha}^{\alpha+\mu}\Big(\|u\|_{L^{\frac{\beta r_*}{1-r_*}}}^{\frac{\beta\theta}{1-\theta}} + \|1+|\varphi|\|_{L^{\frac{\beta r_*}{1-r_*}}}^{\frac{\beta\theta}{1-\theta}}\Big).\end{aligned} \tag{2.33}$$

(ii) *If, in addition,*

$$\alpha \ge \beta r_*/(1-r_*), \tag{2.34}$$

then one has

$$\begin{aligned}\int_U |u|^{\alpha+r}dx &\le \varepsilon\int_U |u|^{\alpha-s}|\nabla u|^p(1+|v|)^{-\beta}dx\\ &+2^{\frac{\theta(1+(\beta-1)r_*)}{(1-\theta)r_*}}\varepsilon^{-\frac{\theta}{1-\theta}}(\bar{c}m)^{\frac{\theta p}{1-\theta}}\Big(|U|^{\frac{\theta(\alpha(1-r_*)-\beta r_*)}{\alpha r_*(1-\theta)}}\|u\|_{L^\alpha}^{\alpha+\mu+\frac{\beta\theta}{1-\theta}} + \|u\|_{L^\alpha}^{\alpha+\mu}\|1+|\varphi|\|_{L^{\frac{\beta r_*}{1-r_*}}}^{\frac{\beta\theta}{1-\theta}}\Big).\end{aligned} \tag{2.35}$$

Proof (i) Applying inequality (2.22) to $W(x)=(1+|v|)^{-\beta}$, we have

$$\begin{aligned}\int_U |u|^{\alpha+r}dx &\le \varepsilon\int_U |u|^{\alpha-s}|\nabla u|^p(1+|v|)^{-\beta}dx\\ &+\varepsilon^{-\frac{\theta}{1-\theta}}(\bar{c}m)^{\frac{\theta p}{1-\theta}}\|u\|_{L^\alpha}^{\alpha+\mu}\Big(\int_U (1+|v|)^{\frac{\beta r_*}{1-r_*}}dx\Big)^{\frac{(1-r_*)\theta}{r_*(1-\theta)}}.\end{aligned} \tag{2.36}$$

Now, for the last integral on the right hand side of (2.36), using $v = u+\varphi$ and then by applying inequality (2.7) twice, we have

$$\begin{aligned}\Big(\int_U (1+|v|)^{\frac{\beta r_*}{1-r_*}}dx\Big)^{\frac{(1-r_*)\theta}{r_*(1-\theta)}} &\le 2^{\frac{\beta\theta}{1-\theta}}\Big(\int_U |u|^{\frac{\beta r_*}{1-r_*}}dx + \int_U (1+|\varphi|)^{\frac{\beta r_*}{1-r_*}}dx\Big)^{\frac{(1-r_*)\theta}{r_*(1-\theta)}}\\ &\le 2^{\frac{\beta\theta}{1-\theta}}\cdot 2^{\frac{(1-r_*)\theta}{r_*(1-\theta)}}\Big\{\Big(\int_U |u|^{\frac{\beta r_*}{1-r_*}}dx\Big)^{\frac{(1-r_*)\theta}{r_*(1-\theta)}} + \Big(\int_U (1+|\varphi|)^{\frac{\beta r_*}{1-r_*}}dx\Big)^{\frac{(1-r_*)\theta}{r_*(1-\theta)}}\Big\}.\end{aligned}$$

Then we obtain (2.33).

(ii) In case (2.34) is satisfied, by Hölder's inequality

$$\|u\|^{\frac{\beta\theta}{1-\theta}}_{L^{\frac{\beta r_*}{1-r_*}}} \le |U|^{\frac{\theta(\alpha(1-r_*)-\beta r_*)}{\alpha r_*(1-\theta)}}\|u\|^{\frac{\beta\theta}{1-\theta}}_{L^\alpha}.$$

This and (2.33) yield inequality (2.35). □

Next, to carry out Moser's iterations in Section 4 below, we need the following parabolic multiplicative Sobolev inequality.

Lemma 2.5 (Parabolic version) *Assume* (2.19) *and*

$$\alpha > 0,\ \alpha \ge s,\ \alpha > \frac{(s-p)nr_*}{r_*(n+p)-n}. \tag{2.37}$$

Let m and $\bar{c}$ be defined as in Lemma 2.3, and $T>0$. Let $u(x,t)$ be function defined on $U\times(0,T)$ such that $|u(\cdot,t)|^m \in W^{1,r_*p}(U)$ and $u(\cdot,t)$ vanishes on ∂U for all $t\in(0,T)$.

(i) *Suppose $W(x,t)$ is a positive function on $U\times(0,T)$. Then one has*

$$\begin{aligned}\|u\|_{L^{\kappa\alpha}(U\times(0,T))} &\le (\bar{c}m)^{\frac{p}{\kappa\alpha}} \operatorname*{ess\,sup}_{t\in(0,T)} \|W^{-1}(\cdot,t)\|^{\frac{1}{\kappa\alpha}}_{L^{\frac{r_*}{1-r_*}}} \operatorname*{ess\,sup}_{t\in(0,T)}\|u(\cdot,t)\|^{1-\tilde\theta}_{L^\alpha}\\ &\times\Big(\int_0^T\int_U |u|^{\alpha-s}|\nabla u|^p W\,dxdt\Big)^{\frac{1}{\kappa\alpha}},\end{aligned} \tag{2.38}$$

where

$$\kappa = 1 + \frac{r_*(n+p)-n}{nr_*} + \frac{p-s}{\alpha} > 1,\quad \tilde\theta = \frac{1}{1+\frac{\alpha(r_*(n+p)-n)}{nr_*(\alpha-s+p)}} \in (0,1). \tag{2.39}$$

(ii) *Let $\varphi(x,t)$ be a function on $U\times(0,T)$, and define $v=u+\varphi$. Then one has, for any $\beta>0$, that*

$$\begin{aligned}&\|u\|_{L^{\kappa\alpha}(U\times(0,T))} \le 2^{\frac{1}{\kappa\alpha}(\beta+\frac{1-r_*}{r_*})}(\bar{c}m)^{\frac{p}{\kappa\alpha}}\\ &\times\Big(\operatorname*{ess\,sup}_{t\in(0,T)}\|1+|\varphi(\cdot,t)|\|^\beta_{L^{\frac{\beta r_*}{1-r_*}}} + \operatorname*{ess\,sup}_{t\in(0,T)}\|u(\cdot,t)\|^\beta_{L^{\frac{\beta r_*}{1-r_*}}}\Big)^{\frac{1}{\kappa\alpha}} \operatorname*{ess\,sup}_{t\in(0,T)}\|u(\cdot,t)\|^{1-\tilde\theta}_{L^\alpha}\\ &\times\Big(\int_0^T\int_U |u|^{\alpha-s}|\nabla u|^p(1+|v|)^{-\beta}dxdt\Big)^{\frac{1}{\kappa\alpha}}.\end{aligned} \tag{2.40}$$

Proof (i) Let $\bar{q} = (r_* p)^* m$ as in (2.24). Denote

$$I(t) = \int_U |u(x,t)|^{\alpha - s} |\nabla u(x,t)|^p W(x,t) dx.$$

Suppose, at the moment, $\kappa > 1$ and $\tilde{\theta} \in (0,1)$ are two numbers such that

$$\frac{1}{\kappa\alpha} = \frac{\tilde{\theta}}{\bar{q}} + \frac{1-\tilde{\theta}}{\alpha}. \tag{2.41}$$

This particularly implies $\alpha < \kappa\alpha < \bar{q}$. Applying the interpolation inequality gives

$$\|u(\cdot,t)\|_{L^{\kappa\alpha}} \le \|u(\cdot,t)\|_{L^{\bar{q}}}^{\tilde{\theta}} \|u(\cdot,t)\|_{L^\alpha}^{1-\tilde{\theta}}.$$

Applying inequality (2.26) to estimate $\|u(\cdot,t)\|_{L^{\bar{q}}}$ on the right-hand side, and raising both sides of the resulting inequality to power $\kappa\alpha$ yield

$$\int_U |u(x,t)|^{\kappa\alpha} dx \le \left[(\bar{c}m)^{\frac{1}{m}} I(t)^{\frac{1}{mp}} \|W^{-1}(\cdot,t)\|_{L^{\frac{r_*}{1-r_*}}}^{\frac{1}{mp}} \right]^{\tilde{\theta}\kappa\alpha} \|u\|_{L^\alpha}^{(1-\tilde{\theta})\kappa\alpha}. \tag{2.42}$$

We impose the condition

$$mp = \tilde{\theta}\kappa\alpha. \tag{2.43}$$

Then (2.42) becomes

$$\int_U |u(x,t)|^{\kappa\alpha} dx \le (\bar{c}m)^p I(t) \|W^{-1}(\cdot,t)\|_{L^{\frac{r_*}{1-r_*}}} \|u(\cdot,t)\|_{L^\alpha}^{(1-\tilde{\theta})\kappa\alpha}. \tag{2.44}$$

Integrating (2.44) in t from 0 to T, we have

$$\int_0^T \int_U |u(x,t)|^{\kappa\alpha} dx dt \le (\bar{c}m)^p \operatorname*{ess\,sup}_{t\in(0,T)} \|W^{-1}(\cdot,t)\|_{L^{\frac{r_*}{1-r_*}}} \operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot,t)\|_{L^\alpha}^{(1-\tilde{\theta})\kappa\alpha} \int_0^T I(t) dt.$$

Taking power $1/\kappa\alpha$ of both sides of the last inequality gives (2.38).

It remains to verify (2.41) and (2.43). We compute κ and $\tilde{\theta}$ explicitly from these two equations. Multiplying (2.41) by $\kappa\alpha$ and then using relation (2.43), we have

$$1 = \frac{mp}{m(r_* p)^*} + \kappa - \frac{mp}{\alpha} = \frac{n - r_* p}{n r_*} + \kappa - \frac{mp}{\alpha}.$$

Solving for κ and using the value of m given in (2.21), we have

$$\kappa = 1 - \frac{n - r_* p}{n r_*} + \frac{mp}{\alpha} = \frac{r_*(n+p) - n}{n r_*} + \frac{\alpha - s + p}{\alpha}. \tag{2.45}$$

Then formula (2.39) of κ follows. Using formula of κ in (2.45), and again, formula (2.21) for m, we calculate $\tilde{\theta}$ from (2.43) by

$$\tilde{\theta} = \frac{mp}{\kappa\alpha} = \frac{\alpha - s + p}{\frac{\alpha(r_*(n+p)-n)}{n r_*} + \alpha - s + p} = \frac{1}{\frac{\alpha(r_*(n+p)-n)}{n r_*(\alpha - s + p)} + 1}.$$

We then obtain the formula of $\tilde{\theta}$ in (2.39). Because $\alpha > 0$, $\alpha \geq s$ and $r_* > n/(n+p)$, we have $\tilde{\theta} \in (0,1)$. By the last condition in (2.37), we have $\kappa > 1$. The proof of (2.38) is complete.

(ii) Applying inequality (2.38) to $W(x,t) = (1+|v|)^{-\beta}$ with $v = u + \varphi$, we have

$$\begin{aligned}\|u\|_{L^{\kappa\alpha}(U\times(0,T))} \leq (\bar{c}m)^{\frac{p}{\kappa\alpha}} \operatorname*{ess\,sup}_{t\in(0,T)} \left(\int_U (1+|v|)^{\frac{\beta r_*}{1-r_*}} dx\right)^{\frac{1-r_*}{r_*\kappa\alpha}} \operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot,t)\|_{L^\alpha}^{1-\tilde{\theta}} \\ \times \left(\int_0^T \int_U |u|^{\alpha-s} |\nabla w|^p (1+|v|)^{-\beta} dx dt\right)^{\frac{1}{\kappa\alpha}}.\end{aligned} \tag{2.46}$$

By triangle inequality and (2.7), we have

$$\begin{aligned}\left(\int_U (1+|v|)^{\frac{\beta r_*}{1-r_*}} dx\right)^{\frac{1-r_*}{r_*\kappa\alpha}} \leq 2^{\frac{\beta}{\kappa\alpha}} \left(\int_U (1+|\varphi|)^{\frac{\beta r_*}{1-r_*}} dx + \int_U |u|^{\frac{\beta r_*}{1-r_*}} dx\right)^{\frac{1-r_*}{r_*\kappa\alpha}} \\ \leq 2^{\frac{\beta}{\kappa\alpha}} 2^{\frac{1-r_*}{r_*\kappa\alpha}} \left\{ \left(\int_U (1+|\varphi|)^{\frac{\beta r_*}{1-r_*}} dx\right)^{\frac{1-r_*}{r_*}} + \left(\int_U |u|^{\frac{\beta r_*}{1-r_*}} dx\right)^{\frac{1-r_*}{r_*}} \right\}^{\frac{1}{\kappa\alpha}}.\end{aligned} \tag{2.47}$$

Combining (2.46) and (2.47) yields (2.40). ◻

3. Estimates for the Lebesgue norms

Let u be a nonnegative solution of problem (1.17) in a domain $U \subset \mathbb{R}^3$. We will derive estimates for the L^α-norms of u for $\alpha > 0$. To do so, it is convenient to shift u by its boundary values and deal with a function vanishing on the boundary.

Let $\Psi(x,t)$ be an extension of the boundary data $\psi(x,t)$ from $\Gamma \times (0,T]$ to $\bar{U} \times [0,T]$.

Define $\bar{u}(x,t) = u(x,t) - \Psi(x,t)$ and $\bar{u}_0(x) = u_0(x) - \Psi(x,0)$. We derive from (1.17) the equations for $\bar{u}$:

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = \nabla \cdot (X(u, \Phi(x,t)) - \Psi_t & \text{on} \quad U \times (0,\infty), \\ \bar{u}(x,0) = \bar{u}_0(x) & \text{on} \quad U, \\ \bar{u}(x,t) = 0 & \text{on} \quad \Gamma \times (0,\infty), \end{cases} \tag{3.1}$$

where $\Psi_t = \partial\Psi/\partial t$ and

$$\Phi(x,t) = \nabla u(x,t) + u^2(x,t)\mathcal{Z}(x,t). \tag{3.2}$$

We will focus on estimating solution $\bar{u}$ of (3.3). Clearly, corresponding estimates for $u = \bar{u} + \Psi$ will easily follow.

By definition (1.16) of $\mathcal{Z}(\cdot,t)$ and (2.2), we have

$$|\mathcal{Z}(x,t)| \leq \mathcal{G} + \Omega^2 |\mathbf{J}^2 x| \leq M_{\mathcal{Z}} \stackrel{\text{def}}{=} \mathcal{G} + \Omega^2 r_U = \mathcal{G} + \frac{\varnothing^2 r_U}{4\varpi^2} R_*^2, \tag{3.3}$$

where $r_U = \max\{|x| : x \in \bar{U}\}$. Set

$$\chi_* = \max\{1, R_*^2, \chi_0^2, M_{\mathcal{Z}}\}. \tag{3.4}$$

We will use χ_* as the main parameter to measure the effect of the rotation. Our estimates in this paper will be expressed in terms of χ_*.

It follows (3.3) and (3.4) that

$$|\mathcal{Z}(x,t)| \le \chi_* \text{ for all } x \in U \text{ and } t > 0. \tag{3.5}$$

We start estimating the Lebesgue norms of the solutions with the following differential inequality.

Lemma 3.1 *Assume*

$$\frac{3}{5-a} < r_* < 1, \quad r_* \ge \frac{1}{2-a}, \tag{3.6}$$

and

$$\alpha \ge \frac{2r_*}{1-r_*}, \quad \alpha > \frac{3r_*(4-3a)}{r_*(5-a)-3}. \tag{3.7}$$

Then one has

$$\begin{aligned}&\frac{d}{dt}\int_U |\bar u|^\alpha dx + \frac{c_4}{8}\chi_*^{-1}\int_U (1+u)^{-2}|\nabla \bar u|^{2-a}|\bar u|^{\alpha-2}dx\\ &\le C_0\chi_*^{\bar\mu}\left\{\left(\int_U |\bar u|^\alpha dx\right)^{1+\gamma} + \left(\int_U |\bar u|^\alpha dx\right)^{1+\gamma'} \|1+|\Psi|\|^{\frac{2\theta}{1-\theta}}_{L^{\frac{2r_*}{1-r_*}}}\right\} + C_0 M(t),\end{aligned} \tag{3.8}$$

where

$$\gamma = \frac{\mu}{\alpha} + \frac{2\theta}{\alpha(1-\theta)} > 0, \quad \gamma' = \frac{\mu}{\alpha} > 0, \quad \bar\mu = \frac{3-2a+\theta}{1-\theta} > 0 \tag{3.9}$$

with

$$\theta = \frac{12(1-a)r_*}{\alpha((5-a)r_*-3)-3ar_*} \in (0,1), \quad \mu = \frac{4(1-a)+a\theta}{1-\theta} > 0, \tag{3.10}$$

positive constants C_0 is defined in (3.32) below, which depends on α, and

$$\begin{aligned}M(t) &= \chi_*^{3-2a}\int_U (1+|\Psi(x,t)|)^{\alpha+4(1-a)}dx + \chi_*^{1-a}\int_U (1+|\nabla\Psi(x,t)|)^{\frac{\alpha+4(1-a)}{2}}dx\\ &+ \chi_*^{-\frac{(3-2a)(\alpha-1)}{5-4a}}\int_U |\Psi_t(x,t)|^{\frac{\alpha+4(1-a)}{5-4a}}dx.\end{aligned} \tag{3.11}$$

Proof We proceed in two steps. In the calculations below, the constants c_i's are independent of α and χ_*, while C_j's are independent of χ_*, but dependent on α.

Step 1. We derive a weaker version of (3.8) first, namely, inequality (3.28) below.

Since $r_* \ge 1/(2-a) > 1/2$, one has

$$\alpha \ge \frac{2r_*}{1-r_*} > \frac{2(1/2)}{1-1/2} = 2. \tag{3.12}$$

Then we can use the following identities

$$\frac{\partial}{\partial t}(|\bar u|^\alpha) = \alpha|\bar u|^{\alpha-2}\bar u\frac{\partial \bar u}{\partial t}, \quad \nabla(|\bar u|^{\alpha-2}\bar u) = (\alpha-1)|\bar u|^{\alpha-2}\nabla\bar u.$$

Multiplying the PDE in (3.3) by $|\bar{u}|^{\alpha-2}\bar{u}$, integrating over domain U, and using integration by parts, we have

$$\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha dx = -(\alpha-1)\int_U X(u,\Phi(x,t))\cdot(|\bar{u}|^{\alpha-2}\nabla\bar{u})dx - \int_U \Psi_t|\bar{u}|^{\alpha-2}\bar{u}dx.$$

On the right-hand side, we write $\nabla\bar{u} = \nabla u - \nabla\Psi$ and use relation (3.2) for ∇u to have

$$\nabla\bar{u}(x,t) = \Phi(x,t) - u^2(x,t)\mathcal{Z}(x,t) - \nabla\Psi = \Phi(x,t) - (\bar{u}+\Psi)^2\mathcal{Z}(x,t) - \nabla\Psi.$$

We obtain

$$\begin{aligned}\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha dx &= -(\alpha-1)\int_U X(u,\Phi(x,t))\cdot\Phi|\bar{u}|^{\alpha-2}dx \\ &\quad + (\alpha-1)\int_U X(u,\Phi(x,t))\cdot\mathcal{Z}|\bar{u}|^{\alpha-2}(\bar{u}+\Psi)^2dx \\ &\quad + (\alpha-1)\int_U X(u,\Phi(x,t))\cdot\nabla\Psi|\bar{u}|^{\alpha-2}dx - \int_U \Psi_t|\bar{u}|^{\alpha-2}\bar{u}dx.\end{aligned} \tag{3.13}$$

For the first integral on the right-hand side of (3.13), we use the first inequality in (2.13) to estimate

$$\begin{aligned}X(u,\Phi)\cdot\Phi &\ge c_5(\chi_0+R_*u)^{-2}(|\Phi|^{2-a}-1) \\ &\ge c_5\max\{\chi_0,R_*\}^{-2}(1+u)^{-2}|\Phi|^{2-a} - c_5\chi_0^{-2} \\ &\ge c_5\chi_*^{-1}(1+u)^{-2}|\Phi|^{2-a} - c_5\chi_0^{-2}.\end{aligned}$$

For the second integral on the right-hand side of (3.13), we use Cauchy-Schwarz inequality, the second inequality in (2.11) and (3.5) to have

$$|X(u,\Phi)\cdot\mathcal{Z}|(\bar{u}+\Psi)^2 \le |X(u,\Phi)||\mathcal{Z}|\cdot 2(\bar{u}^2+\Psi^2) \le 2c_3|\Phi|^{1-a}\chi_*(\bar{u}^2+\Psi^2).$$

By the triangle inequality, (3.5) and then (2.6),

$$|\Phi|^{1-a} \le (|\nabla u| + u^2\chi_*)^{1-a} \le |\nabla u|^{1-a} + |u|^{2(1-a)}\chi_*^{1-a}.$$

Therefore,

$$|X(u,\Phi)\cdot\mathcal{Z}|(\bar{u}+\Psi)^2 \le 2c_3\chi_*(|\nabla u|^{1-a} + \chi_*^{1-a}|u|^{2(1-a)})(\bar{u}^2+\Psi^2).$$

Similarly, for the third integral on the right-hand side of (3.13),

$$|X(u,\Phi)\cdot\nabla\Psi| \le |X(u,\Phi)||\nabla\Psi| \le c_3|\Phi|^{1-a}|\nabla\Psi| \le c_3(|\nabla u|^{1-a} + \chi_*^{1-a}|u|^{2(1-a)})|\nabla\Psi|.$$

Combining the above estimates of the terms in (3.13) gives

$$\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha dx \le (\alpha-1)(-I_0+I_1+I_2+I_3+I_4)+I_5, \tag{3.14}$$

where

$$I_0 = c_5\chi_*^{-1}\int_U (1+u)^{-2}|\Phi|^{2-a}|\bar u|^{\alpha-2}dx, \qquad I_1 = c_9\int_U |\bar u|^{\alpha-2}dx,$$

$$I_2 = 2c_3\int_U (\chi_*|\nabla u|^{1-a} + \chi_*^{2-a}|u|^{2(1-a)})|\bar u|^{\alpha}dx,$$

$$I_3 = 2c_3\int_U (\chi_*|\nabla u|^{1-a} + \chi_*^{2-a}|u|^{2(1-a)})|\bar u|^{\alpha-2}\Psi^2dx,$$

$$I_4 = c_3\int_U (|\nabla u|^{1-a} + \chi_*^{1-a}|u|^{2(1-a)})|\nabla\Psi|\,|\bar u|^{\alpha-2}dx, \qquad I_5 = \int_U |\bar u|^{\alpha-1}|\Psi_t|dx$$

with $c_9 = c_5\chi_0^{-2}$.

Estimation of I_0. Applying inequality (2.8) and estimate (3.5), we have

$$\begin{aligned}|\Phi|^{2-a} &= |\nabla u + u^2\mathcal{Z}|^{2-a} \ge 2^{a-1}|\nabla u|^{2-a} - |u|^{2(2-a)}|\mathcal{Z}|^{2-a}\\ &\ge 2^{a-1}|\nabla u|^{2-a} - |u|^{2(2-a)}\chi_*^{2-a}.\end{aligned}$$

Hence,

$$-I_0 \le -2^{a-1}c_5\chi_*^{-1}\int_U (1+u)^{-2}|\nabla u|^{2-a}|\bar u|^{\alpha-2}dx + c_5\chi_*^{1-a}\int_U (1+u)^{-2}|u|^{2(2-a)}|\bar u|^{\alpha-2}dx.$$

Regarding the last integrand, we note, for $\beta > 0$ and $\gamma \ge 0$, that

$$u^\beta|\bar u|^\gamma \le (|\bar u| + |\Psi|)^\beta|\bar u|^\gamma \le 2^\beta(|\bar u|^\beta + |\Psi|^\beta)|\bar u|^\gamma = 2^\beta(|\bar u|^{\beta+\gamma} + |\bar u|^\gamma|\Psi|^\beta). \tag{3.15}$$

Use $(1+u)^{-2} \le u^{-2}$ and (3.15) with $\beta = 2-2a$ and $\gamma = \alpha - 2$, we have

$$(1+u)^{-2}|u|^{2(2-a)}|\bar u|^{\alpha-2} \le |u|^{2-2a}|\bar u|^{\alpha-2} \le 2^{2-2a}(|\bar u|^{\alpha-2a} + |\Psi|^{2-2a}|\bar u|^{\alpha-2}).$$

Therefore,

$$-I_0 \le -2^{-1}c_4\chi_*^{-1}\int_U (1+u)^{-2}|\nabla u|^{2-a}|\bar u|^{\alpha-2}dx + J_0,$$

where

$$J_0 = 4^{1-a}c_5\chi_*^{1-a}\left\{\int_U |\bar u|^{\alpha-2a}dx + \int_U |\bar u|^{\alpha-2}|\Psi|^{2-2a}dx\right\}.$$

Estimation of I_2. We will use the following estimates.

Let $P \ge 0$ and $\varepsilon > 0$. We write

$$|\nabla u|^{1-a}P = \left(\varepsilon^{(1-a)/(2-a)}|\nabla u|^{1-a}(1+u)^{-2(1-a)/(2-a)}\right) \times \left(\varepsilon^{-(1-a)/(2-a)}(1+u)^{2(1-a)/(2-a)}P\right).$$

Applying Young's inequality (2.4) with powers $(2-a)/(1-a)$ and $2-a$ to the last product, and multiplying the resulting inequality by $|\bar u|^{\alpha-2}$, we obtain

$$|\nabla u|^{1-a}|\bar u|^{\alpha-2}P \le \varepsilon|\nabla u|^{2-a}(1+u)^{-2}|\bar u|^{\alpha-2} + \varepsilon^{-(1-a)}(1+u)^{2(1-a)}|\bar u|^{\alpha-2}P^{2-a}.$$

For the last term, writing $1+u=\bar{u}+1+\Psi$, then using the triangle inequality and (2.7), we have

$$(1+u)^{2(1-a)} \le (|\bar{u}|+1+|\Psi|)^{2(1-a)} \le 2^{2(1-a)}(|\bar{u}|^{2(1-a)}+(1+|\Psi|)^{2(1-a)}).$$

Thus,

$$\begin{aligned} |\nabla u|^{1-a}|\bar{u}|^{\alpha-2}P &\le \varepsilon|\nabla u|^{2-a}(1+u)^{-2}|\bar{u}|^{\alpha-2} \\ &\quad + 4^{1-a}\varepsilon^{-(1-a)}\left(|\bar{u}|^{\alpha-2a}P^{2-a}+|\bar{u}|^{\alpha-2}(1+|\Psi|)^{2(1-a)}P^{2-a}\right). \end{aligned} \tag{3.16}$$

Letting $P=|\bar{u}|^2$ in (3.16), we have

$$\begin{aligned} |\nabla u|^{1-a}|\bar{u}|^{\alpha} &\le \varepsilon|\nabla u|^{2-a}(1+u)^{-2}|\bar{u}|^{\alpha-2} \\ &\quad + 4^{1-a}\varepsilon^{-(1-a)}(|\bar{u}|^{\alpha+4(1-a)}+|\bar{u}|^{\alpha+2(1-a)}(1+|\Psi|)^{2(1-a)}). \end{aligned} \tag{3.17}$$

For the rest of I_2, letting $\beta=2-2a$ and $\gamma=\alpha$ in (3.15), we have

$$u^{2-2a}|\bar{u}|^{\alpha} \le 2^{2-2a}(|\bar{u}|^{\alpha+2-2a}+|\Psi|^{2-2a}|\bar{u}|^{\alpha}). \tag{3.18}$$

Therefore, (3.17) and (3.18) yield

$$\begin{aligned} I_2 &= 2c_3\int_U (\chi_*|\nabla u|^{1-a}|\bar{u}|^{\alpha}+\chi_*^{2-a}|u|^{2(1-a)}|\bar{u}|^{\alpha})dx \\ &\le 2\varepsilon c_3\chi_*\int_U (1+u)^{-2}|\nabla u|^{2-a}|\bar{u}|^{\alpha-2}dx + J_2, \end{aligned}$$

where

$$\begin{aligned} J_2 &= 2\cdot 4^{1-a}\varepsilon^{-(1-a)}c_3\chi_*\left\{\int_U |\bar{u}|^{\alpha+4(1-a)}dx+\int_U |\bar{u}|^{\alpha+2(1-a)}(1+|\Psi|)^{2(1-a)}dx\right\} \\ &\quad + 2\cdot 4^{1-a}c_3\chi_*^{2-a}\int_U (|\bar{u}|^{\alpha+2-2a}+|\bar{u}|^{\alpha}|\Psi|^{2-2a})dx. \end{aligned}$$

Estimation of I_3. Using (3.16) with $P=\Psi^2$, and applying (3.15) to $\beta=2(1-a)$ and $\gamma=\alpha-2$, we obtain

$$\begin{aligned} I_3 &= 2c_3\int_U (\chi_*|\nabla u|^{1-a}|\bar{u}|^{\alpha-2}\Psi^2+\chi_*^{2-a}|u|^{2-2a}|\bar{u}|^{\alpha-2}\Psi^2)dx \\ &\le 2\varepsilon c_3\chi_*\int_U (1+u)^{-2}|\nabla u|^{2-a}|\bar{u}|^{\alpha-2}dx + J_3, \end{aligned}$$

where

$$\begin{aligned} J_3 &= 2\cdot 4^{1-a}\varepsilon^{-(1-a)}c_3\chi_*\left\{\int_U |\bar{u}|^{\alpha-2a}|\Psi|^{2(2-a)}dx+\int_U |\bar{u}|^{\alpha-2}(|1+|\Psi|)^{2(1-a)}|\Psi|^{2(2-a)}dx\right\} \\ &\quad + 2\cdot 4^{1-a}c_3\chi_*^{2-a}\left\{\int_U |\bar{u}|^{\alpha-2a}\Psi^2dx+\int_U |\bar{u}|^{\alpha-2}|\Psi|^{4-2a}dx\right\}. \end{aligned}$$

Estimation of I_4. Let $\delta > 0$. Using (3.16) again with $\varepsilon = \delta$ and $P = |\nabla\Psi|$, and applying inequality (3.15) to $\beta = 2-2a$ and $\gamma = \alpha - 2$, we have

$$I_4 \leq \delta c_3 \int_U (1+u)^{-2}|\nabla u|^{2-a}|\bar{u}|^{\alpha-2}dx + J_4,$$

where

$$J_4 = 4^{1-a}c_3\delta^{-(1-a)}\left\{\int_U |\bar{u}|^{\alpha-2a}|\nabla\Psi|^{2-a}dx + \int_U |\bar{u}|^{\alpha-2}(1+|\Psi|)^{2(1-a)}|\nabla\Psi|^{2-a}dx\right\}$$
$$+ 4^{1-a}c_3\chi_*^{1-a}\left\{\int_U |\bar{u}|^{\alpha-2a}|\nabla\Psi|dx + \int_U |\bar{u}|^{\alpha-2}|\Psi|^{2-2a}|\nabla\Psi|dx\right\}.$$

Combining (3.14) with the above estimates of I_i, for $i = 0,2,3,4$, we have

$$\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha dx \leq -(\alpha-1)C_{\varepsilon,\delta}\int_U (1+u)^{-2}|\nabla u|^{2-a}|\bar{u}|^{\alpha-2}dx$$
$$+ (\alpha-1)(J_0 + I_1 + J_2 + J_3 + J_4) + I_5, \tag{3.19}$$

where $C_{\varepsilon,\delta} = 2^{-1}c_4\chi_*^{-1} - 4c_3\varepsilon\chi_* - \delta c_3$. We select

$$\varepsilon = \frac{c_4}{32c_3}\chi_*^{-2} \text{ and } \delta = \frac{c_4}{8c_3}\chi_*^{-1}, \text{ then } C_{\varepsilon,\delta} = c_{10}\chi_*^{-1} \text{ with } c_{10} = c_4/4. \tag{3.20}$$

We estimate the first integral on the right-hand side of (3.19), using (2.8) and the fact $1+u \geq 1$, by

$$(1+u)^{-2}|\nabla u|^{2-a}|\bar{u}|^{\alpha-2} = (1+u)^{-2}|\nabla\bar{u} + \nabla\Psi|^{2-a}|\bar{u}|^{\alpha-2}$$
$$\geq (1+u)^{-2}\left(2^{a-1}|\nabla\bar{u}|^{2-a} - |\nabla\Psi|^{2-a}\right)|\bar{u}|^{\alpha-2}$$
$$\geq 2^{-1}(1+u)^{-2}|\nabla\bar{u}|^{2-a}|\bar{u}|^{\alpha-2} - |\nabla\Psi|^{2-a}|\bar{u}|^{\alpha-2}.$$

Then

$$\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha dx \leq -\frac{c_{10}(\alpha-1)}{2\chi_*}\int_U (1+u)^{-2}|\nabla\bar{u}|^{2-a}|\bar{u}|^{\alpha-2}dx$$
$$+ (\alpha-1)(J_0 + I_1 + J_2 + J_3 + J_4 + I_6) + I_5, \tag{3.21}$$

where

$$I_6 = c_{10}\chi_*^{-1}\int_U |\bar{u}|^{\alpha-2}|\nabla\Psi|^{2-a}dx.$$

We continue to estimate the right-hand side of (3.21).

Estimation of J_0, I_1, J_4, I_6. We combine J_0, I_1, J_4, with δ in (3.20), and I_6, and collect the corresponding terms of $|\bar{u}|^{\alpha-2a}$ and $|\bar{u}|^{\alpha-2}$. All together, they result in

$$J_0 + I_1 + J_4 + I_6 \leq c_{11}\chi_*^{1-a}\int_U (|\bar{u}|^{\alpha-2a}(1+|\nabla\Psi|)^{2-a} + |\bar{u}|^{\alpha-2}(1+|\Psi|)^{2(1-a)}(1+|\nabla\Psi|)^{2-a})dx, \tag{3.22}$$

where $c_{11} = 4^{1-a}c_5 + c_9 + 2c_3(32c_3/c_4)^{1-a} + c_{10}$. In the last integral, applying Young's inequality (2.4) with

powers $\dfrac{\alpha+4-4a}{\alpha-2a}$ and $\dfrac{\alpha+4-4a}{4-2a}$ to the product $|\bar{u}|^{\alpha-2a}(1+|\nabla\Psi|)^{2-a}$, and

powers $\dfrac{\alpha+4-4a}{\alpha-2}$, $\dfrac{\alpha+4-4a}{2-2a}$, $\dfrac{\alpha+4-4a}{4-2a}$ to the product

$|\bar{u}|^{\alpha-2}(1+|\Psi|)^{2(1-a)}(1+|\nabla\Psi|)^{2-a}$, we obtain

$$J_0 + I_1 + J_4 + I_6 \le c_{11}\chi_*^{1-a}\int_U \left\{2|\bar{u}|^{\alpha+4(1-a)} + (1+|\Psi|)^{\alpha+4(1-a)} + 2(1+|\nabla\Psi|)^{\frac{\alpha+4(1-a)}{2}}\right\}dx.$$

We conveniently split the last integral with the use of the fact $\chi_*^{1-a} \le \chi_*^{3-2a}$ for the first two terms, and obtain

$$\begin{aligned} J_0 + I_1 + J_4 + I_6 \le c_{11}\chi_*^{3-2a}&\int_U (2|\bar{u}|^{\alpha+4(1-a)} + (1+|\Psi|)^{\alpha+4(1-a)})dx \\ &+ 2c_{11}\chi_*^{1-a}\int_U (1+|\nabla\Psi|)^{\frac{\alpha+4(1-a)}{2}}dx. \end{aligned} \tag{3.23}$$

Estimation of J_2, J_3. With the value of ε in (3.20), we evaluate and estimate J_2 and J_3. Note that the powers for χ_* in J_2 and J_3 are $3-2a$ and $2-a$ now. Since $3-2a > 2-a$ and $\chi_* \ge 1$, we estimate

$$J_2 \le c_{12}\chi_*^{3-2a}\int_U \left\{|\bar{u}|^{\alpha+4(1-a)} + |\bar{u}|^{\alpha+2-2a}(1+|\Psi|)^{2(1-a)} + |\bar{u}|^{\alpha}|\Psi|^{2-2a}\right\}dx, \tag{3.24}$$

where $c_{12} = 2\cdot 4^{1-a}c_3[1+(32c_3/c_4)^{1-a}]$. Similarly,

$$J_3 \le c_{12}\chi_*^{3-2a}\int_U \left\{|\bar{u}|^{\alpha-2a}(1+|\Psi|)^{2(2-a)} + |\bar{u}|^{\alpha-2}(1+|\Psi|)^{2(3-2a)}\right\}dx. \tag{3.25}$$

For the integrals on the right-hand side (3.24) and (3.25), by applying Young's inequality (2.4) with

powers $\dfrac{\alpha+4-4a}{\alpha+2-2a}$ and $\dfrac{\alpha+4-4a}{2-2a}$ to the product $|\bar{u}|^{\alpha+2-2a}(1+|\Psi|)^{2(1-a)}$,

powers $\dfrac{\alpha+4-4a}{\alpha}$ and $\dfrac{\alpha+4-4a}{4-4a}$ to the product $|\bar{u}|^{\alpha}|\Psi|^{2-2a}$,

powers $\dfrac{\alpha+4-4a}{\alpha-2a}$ and $\dfrac{\alpha+4-4a}{4-2a}$ to the product $|\bar{u}|^{\alpha-2a}(1+|\Psi|)^{2(2-a)}$,

powers $\dfrac{\alpha+4-4a}{\alpha-2}$ and $\dfrac{\alpha+4-4a}{6-4a}$ to the product $|\bar{u}|^{\alpha-2}(1+|\Psi|)^{2(3-2a)}$,

we obtain

$$\begin{aligned} J_2 + J_3 &\le c_{12}\chi_*^{3-2a}\int_U (5|\bar{u}|^{\alpha+4(1-a)} + 3(1+|\Psi|)^{\alpha+4(1-a)} + |\Psi|^{\frac{\alpha+4(1-a)}{2}})dx \\ &\le c_{12}\chi_*^{3-2a}\int_U (5|\bar{u}|^{\alpha+4(1-a)} + 4(1+|\Psi|)^{\alpha+4(1-a)})dx. \end{aligned} \tag{3.26}$$

Estimation of I_5. We write

$$|\bar{u}|^{\alpha-1}|\Psi_t| = \Big(\chi_*^{\frac{(3-2a)(\alpha-1)}{\alpha+4-4a}}|\bar{u}|^{\alpha-1}\Big)\cdot\Big(\chi_*^{-\frac{(3-2a)(\alpha-1)}{\alpha+4-4a}}|\Psi_t|\Big).$$

Applying Young's inequality (2.4) with powers $(\alpha+4-4a)/(\alpha-1)$ and $(\alpha+4-4a)/(5-4a)$ to the last product yields

$$\begin{aligned} I_5 &\le \chi_*^{3-2a}\int_U |\bar{u}|^{\alpha+4(1-a)}dx + \chi_*^{-\frac{(3-2a)(\alpha-1)}{5-4a}}\int_U |\Psi_t|^{\frac{\alpha+4(1-a)}{5-4a}}dx \\ &\le \chi_*^{3-2a}(\alpha-1)\int_U |\bar{u}|^{\alpha+4(1-a)}dx + \chi_*^{-\frac{(3-2a)(\alpha-1)}{5-4a}}\int_U |\Psi_t|^{\frac{\alpha+4(1-a)}{5-4a}}dx. \end{aligned} \tag{3.27}$$

Combining (3.21) with estimates (3.23), (3.26), and (3.27) gives

$$\begin{aligned} \frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha dx &\le -\frac{c_{10}(\alpha-1)}{2\chi_*}\int_U (1+u)^{-2}|\nabla\bar{u}|^{2-a}|\bar{u}|^{\alpha-2}dx \\ &+ c_{13}\chi_*^{3-2a}(\alpha-1)\int_U |\bar{u}|^{\alpha+4(1-a)}dx + c_{14}\chi_*^{3-2a}(\alpha-1)\int_U (1+|\Psi|)^{\alpha+4(1-a)}dx \\ &+ 2c_{11}\chi_*^{1-a}(\alpha-1)\int_U (1+|\nabla\Psi|)^{\frac{\alpha+4(1-a)}{2}}dx + \chi_*^{-\frac{(3-2a)(\alpha-1)}{5-4a}}\int_U |\Psi_t|^{\frac{\alpha+4(1-a)}{5-4a}}dx, \end{aligned} \tag{3.28}$$

where $c_{13} = 5c_{12} + 2c_{11} + 1$ and $c_{14} = 4c_{12} + c_{11}$.

Step 2. We improve inequality (3.28) to obtain (3.8).

We apply Corollary 2.4(ii) to $n = 3$, $p = 2 - a$, $r = 4(1 - a)$, $s = 2$, $\beta = 2$, and functions $u := \bar{u}(\cdot, t)$, $\varphi := \Psi(\cdot, t)$, $v := \bar{u}(\cdot, t) + \Psi(\cdot, t) = u(\cdot, t) \ge 0$. Note, in this case, that the number m in (2.21) is $m = (\alpha - a)/(2 - a)$.

Because $n/p = 3/(2-a) > 1$, then condition (2.19) becomes (3.6). Also, because of (3.12), the conditions on α in (2.20) and condition (2.34) become (3.7). Then it follows inequality (2.35), for any $\varepsilon > 0$, that

$$\begin{aligned} \int_U |\bar{u}|^{\alpha+4(1-a)}dx &\le \varepsilon\int_U (1+u)^{-2}|\nabla\bar{u}|^{2-a}|\bar{u}|^{\alpha-2}dx \\ &+ C_1\varepsilon^{-\frac{\theta}{1-\theta}}\Big(\|\bar{u}\|_{L^\alpha}^{\alpha+\mu+\frac{2\theta}{1-\theta}} + \|\bar{u}\|_{L^\alpha}^{\alpha+\mu}\|1+|\Psi|\|_{L^{\frac{2r_*}{1-r_*}}}^{\frac{2\theta}{1-\theta}}\Big), \end{aligned} \tag{3.29}$$

where

$$C_1 = \Big[2^{\frac{1+r_*}{r_*}}\Big(\frac{\bar{c}(\alpha-a)}{2-a}\Big)^{2-a}(1+|U|)^{\frac{\alpha(1-r_*)-2r_*}{\alpha r_*}}\Big]^{\frac{\theta}{1-\theta}}, \tag{3.30}$$

and, according to (2.23),

$$\theta = \frac{nrr_*}{nr_*(p-s)+\alpha(r_*(n+p)-n)} = \frac{12(1-a)r_*}{\alpha((5-a)r_*-3)-3ar_*},$$

$$\mu = \frac{r+\theta(s-p)}{1-\theta} = \frac{4(1-a)+a\theta}{1-\theta}.$$

The positive constant $\bar{c}$ in (3.30) is the one in (2.25) that corresponds to the domain $U \subset \mathbb{R}^3$, number r_* in (3.6) and $p = 2 - a$. The fact that $\theta \in (0,1)$ in (3.10) comes from (2.23).

We utilize inequality (3.29) in (3.28) with ε chosen to satisfy

$$c_{13}\varepsilon\chi_*^{3-2a} = \frac{c_{10}}{4\chi_*}, \text{ i.e., } \varepsilon = \frac{c_{10}}{4c_{13}}\chi_*^{2a-4}.$$

It results in

$$\begin{aligned} \frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha dx &\le -\frac{c_{10}(\alpha-1)}{4\chi_*}\int_U (1+u)^{-2}|\nabla\bar{u}|^{2-a}|\bar{u}|^{\alpha-2}dx \\ &+ C_2(\alpha-1)\chi_*^{3-2a}\chi_*^{\frac{\theta(4-2a)}{1-\theta}}\left\{\|\bar{u}\|_{L^\alpha}^{\alpha+\mu+\frac{2\theta}{1-\theta}} + \|\bar{u}\|_{L^\alpha}^{\alpha+\mu}\|1+|\Psi|\|_{L^{\frac{2r_*}{1-r_*}}}^{\frac{2\theta}{1-\theta}}\right\} + c_{15}(\alpha-1)M(t), \end{aligned} \tag{3.31}$$

where $C_2 = c_{13}C_1(4c_{13}/c_{10})^{\frac{\theta}{1-\theta}}$ and $c_{15} = \max\{1, 2c_{11}, c_{14}\}$.

Note that the explicit power of χ_* in the second term on the right-hand side of (3.31) is

$$3 - 2a + \frac{\theta(4-2a)}{1-\theta} = \frac{3-2a+\theta}{1-\theta} = \bar{\mu}.$$

Multiplying (3.31) by α, and using the fact $\alpha(\alpha-1) \ge 2$ for the negative term on the right-hand side, we obtain (3.8) with

$$C_0 = \max\{C_2, c_{15}\}\alpha(\alpha-1). \tag{3.32}$$

With $a, \theta \in (0,1)$, it is clear that μ, $\bar{\mu}$, γ, γ' are positive numbers. The proof is complete. □

We are now ready to obtain estimates for $\bar{u}$ in terms of its initial data and the boundary data Ψ, at least for a short time.

Theorem 3.2 *Let r_*, α, θ, γ, $\bar{\mu}$, C_0 be as in Lemma 3.1. Set*

$$C_* = C_0\left(1 + 2^{\frac{2\theta}{1-\theta}} + 2^{\alpha+4(1-a)-1} + 2^{\frac{\alpha+4(1-a)}{2}-1}\right)(1+|U|)^{\max\left\{1, \frac{(1-r_*)\theta}{r_*(1-\theta)}\right\}},$$

$$V_0 = 1 + \int_U |\bar{u}_0(x)|^\alpha dx, \ V(t) = 1 + \int_U |\bar{u}(x,t)|^\alpha dx,$$

$$\begin{aligned} \mathcal{E}(t) &= \chi_*^{\bar{\mu}}\left(1 + \|\Psi(\cdot,t)\|_{L^{\frac{2r_*}{1-r_*}}}^{\frac{2\theta}{1-\theta}}\right) + \chi_*^{3-2a}\int_U |\Psi(x,t)|^{\alpha+4(1-a)}dx \\ &+ \chi_*^{1-a}\int_U |\nabla\Psi(x,t)|^{\frac{\alpha+4(1-a)}{2}}dx + \chi_*^{-\frac{(3-2a)(\alpha-1)}{5-4a}}\int_U |\Psi_t(x,t)|^{\frac{\alpha+4(1-a)}{5-4a}}dx. \end{aligned}$$

Suppose there are numbers $T > 0$ and $B \in (0,1)$ such that

$$\gamma C_* V_0^\gamma \int_0^T \mathcal{E}(\tau)d\tau \le B. \tag{3.33}$$

Then

$$V(t) \le V_0\left(1 - \gamma C_* V_0^\gamma \int_0^t \mathcal{E}(\tau)d\tau\right)^{-1/\gamma} \quad \text{for all } t \in [0,T]. \tag{3.34}$$

Consequently,

$$V(t) \le V_0(1-B)^{-1/\gamma} \text{ for all } t \in [0,T], \tag{3.35}$$

$$\int_0^T \int_U (1+u(x,t))^{-2} |\nabla \bar{u}(x,t)|^{2-a} |\bar{u}(x,t)|^{\alpha-2} dxdt \le \frac{8\chi_*}{c_4} V_0 \left(1 + \frac{B}{\gamma(1-B)^{1+\frac{1}{\gamma}}}\right). \tag{3.36}$$

Proof Let $d_0 = c_4/(8\chi_*)$ and

$$H(t) = \int_U (1+u(x,t))^{-2} |\nabla \bar{u}(x,t)|^{2-a} |\bar{u}(x,t)|^{\alpha-2} dx.$$

Let γ' and $M(t)$ be defined by (3.9) and (3.11) in Lemma 3.1. Noticing from (3.9) that $\gamma' < \gamma$, we estimate

$$\left(\int_U |\bar{u}(x,t)|^\alpha dx\right)^{1+\gamma}, \left(\int_U |\bar{u}(x,t)|^\alpha dx\right)^{1+\gamma'} \le V(t)^{1+\gamma}.$$

Combining this with (3.8), we have, for $t > 0$, that

$$\frac{d}{dt} V(t) + d_0 H(t) \le C_0 \widetilde{M}(t) V(t)^{1+\gamma}, \tag{3.37}$$

where

$$\widetilde{M}(t) = \chi_*^{\bar{\mu}} \left(1 + \|1 + |\Psi(\cdot,t)|\|_{L^{\frac{2r_*}{1-r_*}}}^{\frac{2\theta}{1-\theta}}\right) + M(t).$$

We find an upper bound for $\widetilde{M}(t)$ with a simpler expression in Ψ. Using the triangle inequality for the $L^{\frac{2r_*}{1-r_*}}$-norm and (2.6) and (2.7), we estimate

$$\|1 + |\Psi(\cdot,t)|\|_{L^{\frac{2r_*}{1-r_*}}}^{\frac{2\theta}{1-\theta}} \le 2^{\frac{2\theta}{1-\theta}} \left(|U|^{\frac{(1-r_*)\theta}{r_*(1-\theta)}} + \|\Psi(\cdot,t)\|_{L^{\frac{2r_*}{1-r_*}}}^{\frac{2\theta}{1-\theta}}\right),$$

$$\begin{aligned} M(t) \le \chi_*^{3-2a} \cdot 2^{\alpha+4(1-a)-1} &\left(|U| + \int_U |\Psi(x,t)|^{\alpha+4(1-a)} dx\right) \\ &+ \chi_*^{1-a} \cdot 2^{\frac{\alpha+4(1-a)}{2}-1} \left(|U| + \int_U |\nabla\Psi(x,t)|^{\frac{\alpha+4(1-a)}{2}} dx\right) \\ &+ \chi_*^{-\frac{(3-2a)(\alpha-1)}{5-4a}} \int_U |\Psi_t(x,t)|^{\frac{\alpha+4(1-a)}{5-4a}} dx. \end{aligned}$$

Regarding the powers of χ_*, observe that $\bar{\mu} > 3-2a > 1-a$. Then

$$\widetilde{M}(t) \le \left(1 + 2^{\frac{2\theta}{1-\theta}} + 2^{\alpha+4(1-a)-1} + 2^{\frac{\alpha+4(1-a)}{2}-1}\right)(1+|U|)^{\max\left\{1, \frac{(1-r_*)\theta}{r_*(1-\theta)}\right\}} \mathcal{E}(t). \tag{3.38}$$

Therefore, we obtain from (3.37) and (3.38) that

$$\frac{d}{dt} V(t) + d_0 H(t) \le C_* \mathcal{E}(t) V(t)^{1+\gamma}. \tag{3.39}$$

Neglecting $d_0H(t)$ in (3.39) and solving the remaining differential inequality give (3.34).

As a consequence of (3.34), we have, for $t \in [0,T]$,

$$V(t) \le V_0 \left(1 - \gamma C_* V_0^\gamma \int_0^T \mathcal{E}(\tau)d\tau\right)^{-1/\gamma} \le (1-B)^{-1/\gamma} V_0.$$

Thus, we obtain (3.35).

Integrating inequality (3.39) in t from 0 to T, we have

$$d_0 \int_0^T H(t)dt \le V_0 + C_* \int_0^T V(t)^{1+\gamma}\mathcal{E}(t)dt.$$

Combining this with estimate (3.35) of $V(t)$ and condition (3.33), we obtain

$$\begin{aligned} d_0 \int_0^T H(t)dt &\le V_0 + ((1-B)^{-1/\gamma}V_0)^{1+\gamma} C_* \int_0^T \mathcal{E}(t)dt \\ &\le V_0 + (1-B)^{-1-1/\gamma} V_0 \gamma^{-1} B = V_0 \left(1 + \frac{B}{\gamma(1-B)^{1+\frac{1}{\gamma}}}\right). \end{aligned}$$

Then estimate (3.36) follows. □

Remark 3.3 *Inequality (3.36) gives an indirect estimate for the gradient $\nabla \bar{u}$, or, in other words, for its weighted L^{2-a}-norm with the weight $|\bar{u}|^{\alpha-2}/(1+u)^2$ depending on the solution u. In [10] when $X = X(y)$ and $\alpha = 2$, a similar L^{2-a}-estimate (without a weight) was the starting point for other estimates of higher L^p-norms of ∇u. They were obtained by the use of Ladyženskaja–Ural′ceva's iteration [18]. It is not known whether this method still works for the PDE (1.15) with $X = X(z,y)$.*

4. Estimates for the essential supremum

We establish L^∞-estimates for a solution $\bar{u}$ of (3.3) with possibly unbounded initial data. They will contain some quantities that only involve the boundary data of the following form.

For numbers $q_1, q_2, q_3, q_4 > 0$ and $T > T' \ge 0$, define

$$\begin{aligned} \mathcal{M}_{T',T}(q_1,q_2,q_3,q_4) &= 1 + \chi_*^{1-a}\left(\int_{T'}^T \int_U (1+|\nabla\Psi(x,t)|)^{(2-a)q_1} dxdt\right)^{\frac{1}{q_1}} \\ &+ \chi_*^{1-a}\left(\int_{T'}^T \int_U (1+|\Psi(x,t)|)^{2(1-a)q_2}(1+|\nabla\Psi(x,t)|)^{(2-a)q_2} dxdt\right)^{\frac{1}{q_2}} \\ &+ \chi_*^{3-a}\left(\int_{T'}^T \int_U (1+|\Psi(x,t)|)^{\tilde{q}q_3} dxdt\right)^{\frac{1}{q_3}} + \left(\int_{T'}^T \int_U |\Psi_t|^{q_4} dxdt\right)^{\frac{1}{q_4}}, \end{aligned} \tag{4.1}$$

where $\tilde{q} = 2\max\{2-a, 3-2a\}$.

We use Moser's iteration and have technical preparations with key inequalities in Lemmas 4.1 and 4.2 below. In the following, Q_T denotes the cylinder $U \times (0,T)$ in $\mathbb{R}^4$, and $|Q_T|$ denotes its 4-dimensional Lebesgue measure.

Lemma 4.1 *Assume numbers* $\tilde{\kappa}$ *and* p_i*, for* $i = 1, 2, 3, 4$*, satisfy*

$$\tilde{\kappa} > p_1, p_2, p_3, p_4 > 1. \tag{4.2}$$

For $i = 1, 2, 3, 4$*, let* q_i *be the Hölder conjugate exponent of* p_i*, that is,*

$$1/p_i + 1/q_i = 1 \text{ for } i = 1, 2, 3, 4. \tag{4.3}$$

Let $T > T_2 > T_1 \geq 0$*. If*

$$\alpha \geq \max\left\{2, \frac{2p_3(1-a)}{\tilde{\kappa} - p_3}, \frac{4(1-a)}{\tilde{\kappa} - 1}\right\}, \tag{4.4}$$

then one has

$$\sup_{t\in[T_2,T]} \int |\bar{u}|^\alpha(x,t)dx \leq \alpha^2 K\mathcal{M}_0(\|\bar{u}\|^{\alpha-2}_{L^{\tilde{\kappa}\alpha}(U\times(T_1,T))} + \|\bar{u}\|^{\alpha+4(1-a)}_{L^{\tilde{\kappa}\alpha}(U\times(T_1,T))}), \tag{4.5}$$

$$\int_{T_2}^T \int_U (1+u)^{-2}|\nabla\bar{u}|^{2-a}|\bar{u}|^{\alpha-2}dxdt \leq \frac{2\chi_*}{c_{10}} K\mathcal{M}_0(\|\bar{u}\|^{\alpha-2}_{L^{\tilde{\kappa}\alpha}(U\times(T_1,T))} + \|\bar{u}\|^{\alpha+4(1-a)}_{L^{\tilde{\kappa}\alpha}(U\times(T_1,T))}), \tag{4.6}$$

where $\mathcal{M}_0 = \mathcal{M}_{T_1,T}(q_1, q_2, q_3, q_4)$ *and*

$$K = c_{16}(1 + |Q_T|)\Big(1 + \frac{1}{T_2 - T_1}\Big) \text{ with } c_{16} = 9(1 + c_{11} + c_{12}). \tag{4.7}$$

Proof Let $\zeta = \zeta(t)$ be a C^1-function on $[0, T]$ with

$$\begin{aligned} &\zeta(t) = 0 \text{ for } 0 \leq t \leq T_1,\ \zeta(t) = 1 \text{ for } T_2 \leq t \leq T, \\ &0 \leq \zeta(t) \leq 1 \text{ and } 0 \leq \zeta'(t) \leq \frac{2}{T_2 - T_1} \text{ for } 0 \leq t \leq T. \end{aligned} \tag{4.8}$$

Multiplying the PDE in (3.3) by $|\bar{u}|^{\alpha-2}\bar{u}\zeta^2(t)$, integrating over U, and integrating by parts give

$$\begin{aligned} &\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha\zeta^2 dx - \frac{1}{\alpha}\int_U 2|\bar{u}|^\alpha\zeta\zeta' dx \\ &= -(\alpha-1)\int_U X(u, \Phi(x,t))|\bar{u}|^{\alpha-2}\nabla\bar{u}\zeta^2 dx - \int_U \Psi_t|\bar{u}|^{\alpha-2}\bar{u}\zeta^2 dx. \end{aligned}$$

Noticing that the function $\zeta = \zeta(t) \geq 0$ is independent of x, we have, the same as inequality (3.14),

$$\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha\zeta^2 dx - \frac{2}{\alpha}\int_U |\bar{u}|^\alpha\zeta\zeta' dx \leq (\alpha-1)(-\tilde{I}_0 + \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4) + \tilde{I}_5,$$

where $\tilde{I}_i = I_i\zeta^2$ for $i = 0, 1, \ldots, 5$. Then, similar to (3.21),

$$\begin{aligned} \frac{1}{\alpha}\frac{d}{dt}\int_U |\bar{u}|^\alpha\zeta^2 dx - \frac{2}{\alpha}\int_U |\bar{u}|^\alpha\zeta\zeta' dx &\leq -\frac{c_{10}}{2}\chi_*^{-1}(\alpha-1)\int_U (1+u)^{-2}|\nabla\bar{u}|^{2-a}|\bar{u}|^{\alpha-2}\zeta^2 dx \\ &\quad + (\alpha-1)(\tilde{J}_0 + \tilde{I}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{I}_6) + \tilde{I}_5, \end{aligned} \tag{4.9}$$

where $\tilde{J}_i = J_i\zeta^2$ for $i = 0, 2, 3, 4$, with ε and δ particularly chosen in (3.20), and $\tilde{I}_6 = I_6\zeta^2$.

On the one hand, neglecting the negative term on the right-hand side of (4.9) and integrating the resulting inequality in time from 0 to t, for $t \in [T_2, T]$, with the use of the fact $\zeta(0) = 0$, and then taking the supremum in t over $[T_2, T]$, we obtain

$$\frac{1}{\alpha} \sup_{t\in[T_2,T]} \int_U |\bar{u}(x,t)|^\alpha dx = \frac{1}{\alpha} \sup_{t\in[T_2,T]} \int_U |\bar{u}(x,t)|^\alpha \zeta^2(t) dx \le \mathcal{J}, \tag{4.10}$$

where

$$\mathcal{J} = (\alpha - 1) \int_0^T (\tilde{J}_0 + \tilde{I}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{I}_6) dt + \int_0^T \tilde{I}_5 dt + \frac{2}{\alpha} \iint_{Q_T} |\bar{u}|^\alpha \zeta\zeta' dxdt. \tag{4.11}$$

On the other hand, integrating (4.9) in t from 0 to T gives

$$\frac{c_{10}(\alpha-1)}{2\chi_*} \iint_{Q_T} (1+u)^{-2} |\nabla \bar{u}|^{2-a} |\bar{u}|^{\alpha-2} \zeta^2 dxdt \le \mathcal{J}.$$

Hence,

$$\begin{aligned} &\int_{T_2}^T \int_U (1+u)^{-2} |\nabla \bar{u}|^{2-a} |\bar{u}|^{\alpha-2} dxdt = \int_{T_2}^T \int_U (1+u)^{-2} |\nabla \bar{u}|^{2-a} |\bar{u}|^{\alpha-2} \zeta^2 dxdt \\ &\le \iint_{Q_T} (1+u)^{-2} |\nabla \bar{u}|^{2-a} |\bar{u}|^{\alpha-2} \zeta^2 dxdt \le \frac{2\chi_*}{c_{10}(\alpha-1)} \mathcal{J}. \end{aligned} \tag{4.12}$$

We focus on estimating the quantity $\mathcal{J}$ now. Define $Y(\alpha) = \iint_{Q_T} |\bar{u}|^\alpha \zeta dxdt$.

Using the fact $0 \le \zeta^2 \le \zeta$ and previous estimate (3.22), we have

$$\begin{aligned} \int_0^T (\tilde{J}_0 + \tilde{I}_1 + \tilde{J}_4 + \tilde{I}_6) dt &\le \int_0^T (J_0 + I_1 + J_4 + I_6) \zeta dt \\ &\le c_{11} \chi_*^{1-a} \iint_{Q_T} |\bar{u}|^{\alpha-2a} (1 + |\nabla \Psi|)^{2-a} \zeta dxdt \\ &\quad + c_{11} \chi_*^{1-a} \iint_{Q_T} |\bar{u}|^{\alpha-2} (1 + |\Psi|)^{2(1-a)} (1 + |\nabla \Psi|)^{2-a} \zeta dxdt. \end{aligned}$$

On the right-hand side of the preceding inequality, by applying Hölder's inequality with powers p_1, q_1 to the first integral on the right-hand side, and with powers p_2, q_2 to the second integral, we obtain

$$\int_0^T (\tilde{J}_0 + \tilde{I}_1 + \tilde{J}_4 + \tilde{I}_6) dt \le c_{11} \chi_*^{1-a} [Y(p_1(\alpha - 2a))^{\frac{1}{p_1}} E_1 + Y(p_2(\alpha-2))^{\frac{1}{p_2}} E_2], \tag{4.13}$$

where

$$E_1 = \left(\iint_{Q_T} (1 + |\nabla \Psi|)^{(2-a)q_1} \zeta dxdt \right)^{\frac{1}{q_1}},$$

$$E_2 = \left(\iint_{Q_T} (1 + |\Psi|)^{2(1-a)q_2} (1 + |\nabla \Psi|)^{(2-a)q_2} \zeta dxdt \right)^{\frac{1}{q_2}}.$$

Similarly, by the fact $0 \le \zeta^2 \le \zeta$ and estimates (3.24) and (3.25), we have

$$\begin{aligned}
&\int_0^T (\tilde{J}_2 + \tilde{J}_3)dt \le \int_0^T J_2\zeta dt + \int_0^T J_3\zeta dt \\
&\le c_{12}\chi_*^{3-2a} \iint_{Q_T} \left\{|\bar{u}|^{\alpha+4(1-a)} + |\bar{u}|^{\alpha}|\Psi|^{2(1-a)} + |\bar{u}|^{\alpha+2-2a}(1+|\Psi|)^{2(1-a)}\right\} \zeta dxdt \\
&\quad + c_{12}\chi_*^{3-2a} \iint_{Q_T} \left\{|\bar{u}|^{\alpha-2a}(1+|\Psi|)^{2(2-a)} + |\bar{u}|^{\alpha-2}(1+|\Psi|)^{2(3-2a)}\right\} \zeta dxdt.
\end{aligned} \tag{4.14}$$

Note that each power of $|\Psi|$ or $(1+|\Psi|)$ in (4.14) are less than or equal to $\tilde{q}$. Then

$$\begin{aligned}
\int_0^T (\tilde{J}_2 + \tilde{J}_3)dt &\le c_{12}\chi_*^{3-2a}\Big\{ \iint_{Q_T} |\bar{u}|^{\alpha+4(1-a)}\zeta dxdt + \iint_{Q_T} |\bar{u}|^{\alpha}(1+|\Psi|)^{\tilde{q}}\zeta dxdt \\
&\quad + \iint_{Q_T} |\bar{u}|^{\alpha+2-2a}(1+|\Psi|)^{\tilde{q}}\zeta dxdt + \iint_{Q_T} |\bar{u}|^{\alpha-2a}(1+|\Psi|)^{\tilde{q}}\zeta dxdt \\
&\quad + \iint_{Q_T} |\bar{u}|^{\alpha-2}(1+|\Psi|)^{\tilde{q}}\zeta dxdt\Big\}.
\end{aligned}$$

Applying Hölder's inequality with powers p_3 and q_3 to the last four integrals yields

$$\begin{aligned}
\int_0^T (\tilde{J}_2 + \tilde{J}_3)dt \le c_{12}\chi_*^{3-2a}\Big\{&Y(\alpha+4(1-a)) + Y(p_3\alpha)^{\frac{1}{p_3}}E_3 + Y(p_3(\alpha+2-2a))^{\frac{1}{p_3}}E_3 \\
&+ Y(p_3(\alpha-2a))^{\frac{1}{p_3}}E_3 + Y(p_3(\alpha-2))^{\frac{1}{p_3}}E_3\Big\},
\end{aligned} \tag{4.15}$$

where

$$E_3 = \left(\iint_{Q_T} (1+|\Psi|)^{\tilde{q}q_3}\zeta dxdt\right)^{\frac{1}{q_3}}.$$

Next, by the fact $0 \le \zeta^2 \le \zeta$ and Hölder's inequality with powers p_4 and q_4,

$$\int_0^T \tilde{I}_5 dt \le \iint_{Q_T} |\bar{u}|^{\alpha-1}|\Psi_t|\zeta dxdt \le Y(p_4(\alpha-1))^{\frac{1}{p_4}}E_4, \tag{4.16}$$

where

$$E_4 = \left(\iint_{Q_T} |\Psi_t|^{q_4}\zeta dxdt\right)^{\frac{1}{q_4}}.$$

Finally, for the last term of $\mathcal{J}$ in (4.11), using the second property of (4.8), we have

$$\frac{2}{\alpha}\iint_{Q_T} |\bar{u}|^{\alpha}\zeta\zeta' dxdt \le \frac{4}{\alpha(T_2-T_1)}Y(\alpha) \le \frac{2}{T_2-T_1}Y(\alpha). \tag{4.17}$$

Combining formula (4.11) with the above estimates (4.13), (4.15), (4.16), and (4.17) yields

$$\begin{aligned}\mathcal{J} \leq (\alpha-1)c_{11}\chi_*^{1-a}\Big[Y(p_1(\alpha-2a))^{\frac{1}{p_1}}E_1 + Y(p_2(\alpha-2))^{\frac{1}{p_2}}E_2\Big] \\ + (\alpha-1)c_{12}\chi_*^{3-2a}\Big\{Y(\alpha+4(1-a)) + \Big[Y(p_3\alpha)^{\frac{1}{p_3}} + Y(p_3(\alpha+2-2a))^{\frac{1}{p_3}} \\ + Y(p_3(\alpha-2a))^{\frac{1}{p_3}} + Y(p_3(\alpha-2))^{\frac{1}{p_3}}\Big]E_3\Big\} + Y(p_4(\alpha-1))^{\frac{1}{p_4}}E_4 + \frac{2}{T_2-T_1}Y(\alpha).\end{aligned} \tag{4.18}$$

Define $Y_* = \|\bar{u}\|_{L^{\tilde{\kappa}\alpha}(U\times(T_1,T))}$. If $0<\beta\leq\tilde{\kappa}\alpha$, then, by Hölder's inequality,

$$Y(\beta) = \iint_{Q_T}|\bar{u}|^\beta\zeta dxdt \leq \int_{T_1}^{T}\int_U|\bar{u}|^\beta dxdt \leq Y_*^\beta|Q_T|^{1-\frac{\beta}{\tilde{\kappa}\alpha}} \leq Y_*^\beta(1+|Q_T|). \tag{4.19}$$

Under conditions (4.2) and (4.4), one has

$$p_1(\alpha-2a), p_2(\alpha-2), p_4(\alpha-1) < \tilde{\kappa}\alpha \text{ and } \alpha+4(1-a), p_3(\alpha+2-2a) \leq \tilde{\kappa}\alpha.$$

Thus, we have from (4.18) and (4.19) that

$$\begin{aligned}\mathcal{J} \leq (\alpha-1)(1+|Q_T|)c_{11}\chi_*^{1-a}\left[Y_*^{\alpha-2a}E_1 + Y_*^{\alpha-2}E_2\right] \\ + (\alpha-1)(1+|Q_T|)c_{12}\chi_*^{3-2a}\{Y_*^{\alpha+4(1-a)} + [Y_*^\alpha + Y_*^{\alpha+2-2a} + Y_*^{\alpha-2a} + Y_*^{\alpha-2}]E_3\} \\ + (1+|Q_T|)Y_*^{\alpha-1}E_4 + \frac{2}{T_2-T_1}(1+|Q_T|)Y_*^\alpha.\end{aligned}$$

It follows that

$$\mathcal{J} \leq (1+c_{11}+c_{12})(1+|Q_T|)(\alpha-1)\Big(1+\frac{1}{T_2-T_1}\Big)M_0 J,$$

where

$$\begin{aligned}M_0 = 1 + \chi_*^{1-a}(E_1+E_2) + \chi_*^{3-a}E_3 + E_4, \\ J = 2Y_*^{\alpha-2} + 2Y_*^{\alpha-2a} + Y_*^{\alpha-1} + 3Y_*^\alpha + Y_*^{\alpha+2(1-a)} + Y_*^{\alpha+4(1-a)}.\end{aligned}$$

Because $0\leq\zeta\leq 1$ on $[0,T]$ and $\zeta=0$ on $[0,T_1]$, we have $M_0\leq\mathcal{M}_0$.

Thanks to inequality (2.5), one has

$$Y_*^{\alpha-2a}, Y_*^{\alpha-1}, Y_*^\alpha, Y_*^{\alpha+2(1-a)} \leq Y_*^{\alpha-2} + Y_*^{\alpha+4(1-a)};$$

hence, $J\leq 9(Y_*^{\alpha-2}+Y_*^{\alpha+4(1-a)})$. Therefore,

$$\mathcal{J} \leq (\alpha-1)K\mathcal{M}_0(Y_*^{\alpha-2}+Y_*^{\alpha+4(1-a)}). \tag{4.20}$$

Then estimate (4.5) follows (4.10) and (4.20), while estimate (4.6) follows (4.12) and (4.20). The proof is complete. □

Lemma 4.2 *Let* r_* *satisfy* (3.6) *and set* $\lambda_0 = (r_*(5-a)-3)/(3r_*)$. *Assume* (4.2), (4.3),

$$\alpha \ge \max\left\{2, \frac{2p_3(1-a)}{\tilde\kappa - p_3}, \frac{4(1-a)}{\tilde\kappa - 1}\right\} \text{ and } \alpha > \frac{a}{\lambda_0}. \tag{4.21}$$

Let

$$\kappa = 1 + \lambda_0 - \frac{a}{\alpha}, \quad \tilde\theta = \frac{1}{1+\frac{\lambda_0\alpha}{\alpha-a}}, \quad \mu_1 = 1 + \frac{\tilde\theta a}{\alpha - a}. \tag{4.22}$$

If $T > T_2 > T_1 \ge 0$, *then*

$$\|\bar u\|_{L^{\kappa\alpha}(U\times(T_2,T))} \le (A_\alpha B_\alpha)^{\frac{1}{\alpha}} \Big(\|\bar u\|^{\nu_1}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))} + \|\bar u\|^{\nu_2}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}\Big)^{\frac{1}{\alpha}}, \tag{4.23}$$

where $\nu_1 = (\alpha-2)\mu_1$, $\nu_2 = (\alpha + 4(1-a))\mu_1$,

$$A_\alpha = 2^{\mu_1 - 1 + \frac{1}{\kappa}(2+\frac{1}{r_*})} \left[\frac{1}{c_{10}}\left(\frac{\bar c}{2-a}\right)^{2-a}\right]^{\frac{1}{\kappa}} c_{16}^{\mu_1}, \tag{4.24}$$

$$B_\alpha = \chi_*^{\frac{1}{\kappa}} \alpha^{2\mu_1 - \frac{a}{\kappa}} \hat E^{\frac{1}{\kappa}} \left[(1+|Q_T|)\Big(1 + \frac{1}{T_2 - T_1}\Big)\mathcal{M}_0\right]^{\mu_1} \tag{4.25}$$

with $\mathcal{M}_0 = \mathcal{M}_{T_1,T}(q_1,q_2,q_3,q_4)$ *and*

$$\hat E = \operatorname*{ess\,sup}_{t\in(T_2,T)} \|1 + |\Psi(\cdot,t)|\|^2_{L^{\frac{2r_*}{1-r_*}}} + \operatorname*{ess\,sup}_{t\in(T_2,T)} \|\bar u(\cdot,t)\|^2_{L^{\frac{2r_*}{1-r_*}}}.$$

Proof We apply Lemma 2.5(ii) to $n=3$, $p = 2-a$, $s=2$, $\beta=2$, and functions $u := \bar u$, $\varphi := \Psi$, $v := \bar u + \Psi = u$, and the interval (T_2,T) in place of $(0,T)$. Note, from (2.21), that $m = (\alpha-a)/(2-a)$. The same as in Step 2 in the proof of Lemma 3.1, condition (2.19) becomes (3.6). Clearly, condition (2.37) becomes

$$\alpha \ge 2 \text{ and } \alpha > \frac{a}{\lambda_0},$$

which is satisfied thanks to (4.21). Then, by inequality (2.40), one has

$$\|\bar u\|_{L^{\kappa\alpha}(U\times(T_2,T))} \le \hat C^{\frac{1}{\kappa\alpha}} \hat E^{\frac{1}{\kappa\alpha}} \operatorname*{ess\,sup}_{t\in(T_2,T)} \|\bar u(\cdot,t)\|^{1-\tilde\theta}_{L^\alpha} \left(\int_{T_2}^T\int_U |\bar u|^{\alpha-2}|\nabla\bar u|^{2-a}(1+|u|)^{-2}dxdt\right)^{\frac{1}{\kappa\alpha}}, \tag{4.26}$$

where $\hat C = 2^{2+\frac{1-r_*}{r_*}}(\bar c\cdot\frac{\alpha-a}{2-a})^{2-a}$, the numbers κ and $\tilde\theta$ are given in (2.39), which assume the values in (4.22) now.

We estimate the right-hand side of (4.26) by Lemma 4.1. Note that condition (4.4) is the first part of (4.21). Recalling that K is defined in (4.7), we denote

$$Y_* = \|\bar u\|_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}, \quad M_1 = \alpha^2 K\mathcal{M}_0, \quad S = M_1(Y_*^{\alpha-2} + Y_*^{\alpha+4(1-a)}).$$

By estimates (4.5) and (4.6) in Lemma 4.1, we have

$$\operatorname*{ess\,sup}_{t\in(T_2,T)} \int_U |\bar u|^\alpha dx \le S, \quad \int_{T_2}^T\int_U |\bar u|^{\alpha-2}|\nabla\bar u|^{2-a}(1+|u|)^{-2}dxdt \le \frac{2\chi_*}{c_{10}\alpha^2}S. \tag{4.27}$$

Then combining (4.26) and (4.27) yields

$$\|\bar u\|_{L^{\kappa\alpha}(U\times(T_2,T))} \le \left(\frac{2\chi_*}{c_{10}\alpha^2}\hat C\hat E\right)^{\frac{1}{\kappa\alpha}} S^{\frac{1}{\alpha}(1-\tilde\theta+\frac{1}{\kappa})}$$
$$= \left\{\left(\frac{2\chi_*}{c_{10}\alpha^2}\hat C\hat E\right)^{\frac{1}{\kappa}} M_1^{1-\tilde\theta+\frac{1}{\kappa}}\left(Y_*^{\alpha-2}+Y_*^{\alpha+4(1-a)}\right)^{1-\tilde\theta+\frac{1}{\kappa}}\right\}^{\frac{1}{\alpha}}. \tag{4.28}$$

Using the formula of κ in (2.43), we have

$$\frac{1}{\kappa}-\tilde\theta=\frac{\tilde\theta\alpha}{mp}-\tilde\theta=\frac{\tilde\theta\alpha}{\alpha-s+p}-\tilde\theta=\frac{\tilde\theta(s-p)}{\alpha-s+p}=\frac{\tilde\theta a}{\alpha-a}.$$

It follows that the power $1-\tilde\theta+\frac{1}{\kappa}$ in (4.28) is exactly the number $\mu_1>1$ defined in (4.22). Applying inequality (2.6) to $(Y_*^{\alpha-2}+Y_*^{\alpha+4(1-a)})^{\mu_1}$ in (4.28), we obtain

$$\|\bar u\|_{L^{\kappa\alpha}(U\times(T_2,T))} \le \left\{\left(\frac{2\chi_*}{c_{10}\alpha^2}\hat C\hat E\right)^{\frac{1}{\kappa}} M_1^{\mu_1}2^{\mu_1-1}(Y_*^{(\alpha-2)\mu_1}+Y_*^{(\alpha+4(1-a))\mu_1})\right\}^{\frac{1}{\alpha}}$$
$$= M_2^{\frac{1}{\alpha}}(Y_*^{\nu_1}+Y_*^{\nu_2})^{\frac{1}{\alpha}},\ \text{where } M_2=2^{\mu_1-1}\left(\frac{2\chi_*}{c_{10}\alpha^2}\hat C\hat E\right)^{\frac{1}{\kappa}}M_1^{\mu_1}. \tag{4.29}$$

Simple bound $\alpha-a<\alpha$ in the formula of $\hat C$, and elementary calculations give

$$M_2 \le 2^{\mu_1-1}\left[\frac{2^{3+\frac{1-r_*}{r_*}}\chi_*}{c_{10}\alpha^2}\left(\frac{\bar c\alpha}{2-a}\right)^{2-a}\hat E\right]^{\frac{1}{\kappa}}$$
$$\times\left[c_{16}(1+|Q_T|)\Big(1+\frac{1}{T_2-T_1}\Big)\alpha^2\mathcal{M}_0\right]^{\mu_1}=A_\alpha B_\alpha. \tag{4.30}$$

Therefore, we obtain (4.23) from (4.29) and (4.30). □

We simplify inequality (4.23) to make it more suitable to the Moser iteration below.

Firstly, observe that $1<\mu_1<1+a/(\alpha-a)$; hence, the powers ν_1 and ν_2 in (4.23) can be simply bounded by $\nu_3<\nu_1<\nu_2<\nu_4$, where

$$\nu_3=\nu_{3,\alpha}\overset{\text{def}}{=}\alpha-2 \text{ and } \nu_4=\nu_{4,\alpha}\overset{\text{def}}{=}(\alpha+4(1-a))\left(1+\frac{a}{\alpha-a}\right). \tag{4.31}$$

Then, thanks to (2.5),

$$\|\bar u\|^{\nu_1}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}+\|\bar u\|^{\nu_2}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}\le 2\left(\|\bar u\|^{\nu_3}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}+\|\bar u\|^{\nu_4}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}\right). \tag{4.32}$$

Secondly, using the facts $\mu_1\le 2$ and $\kappa\ge 1$, we estimate A_α in (4.24) by

$$A_\alpha\le 2^{3+\frac{1}{r_*}}\bar c_1 c_{16}^2,\ \text{where } \bar c_1=\max\left\{1,\frac{1}{c_{10}}\left(\frac{\bar c}{2-a}\right)^{2-a}\right\}. \tag{4.33}$$

Thirdly, we estimate B_α given by formula (4.25). Regarding the powers in that formula, note that $\tilde\theta \le \frac{1}{1+\lambda_0}$, then

$$\mu_1 \le 1 + \tilde\theta a \le \mu_2 \overset{\text{def}}{=} 1 + \frac{a}{1+\lambda_0}. \tag{4.34}$$

Property (4.34) and the fact $\kappa \le 1+\lambda_0$ yield that the power of α satisfies

$$2\mu_1 - \frac{a}{\kappa} \le 2\mu_2 - \frac{a}{1+\lambda_0} = \mu_3 \overset{\text{def}}{=} 2 + \frac{a}{1+\lambda_0}. \tag{4.35}$$

Concerning the remaining power $1/\kappa$, one has, thanks to the fact $\alpha \ge 2$, that

$$\kappa \ge \hat\kappa \overset{\text{def}}{=} 1 + \lambda_0 - \frac{a}{2}. \tag{4.36}$$

Therefore,

$$B_\alpha \le \chi_*^{1/\hat\kappa}\alpha^{\mu_3}\bar E^{1/\hat\kappa}\left[(1+|Q_T|)\Big(1+\frac{1}{T_2-T_1}\Big)\mathcal{M}_0\right]^{\mu_2}, \text{ where } \bar E = \max\{1,\hat E\}. \tag{4.37}$$

Fourthly, assume

$$\kappa \ge \kappa' > 1. \tag{4.38}$$

By Hölder's inequality,

$$\|\bar u\|_{L^{\kappa'\alpha}(U\times(T_2,T))} \le |Q_T|^{(\frac{1}{\kappa'}-\frac{1}{\kappa})\frac{1}{\alpha}}\|\bar u\|_{L^{\kappa\alpha}(U\times(T_2,T))} \le (1+|Q_T|)^{\frac{1}{\alpha}}\|\bar u\|_{L^{\kappa\alpha}(U\times(T_2,T))}. \tag{4.39}$$

Combining (4.39) with (4.23), and making use of estimates (4.32), (4.33), and (4.37) yield

$$\|\bar u\|_{L^{\kappa'\alpha}(U\times(T_2,T))} \le \left\{\bar A\bar B\alpha^{\mu_3}\left(\|\bar u\|^{\nu_3}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))} + \|\bar u\|^{\nu_4}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}\right)\right\}^{\frac{1}{\alpha}}, \tag{4.40}$$

where

$$\bar A = 2^{4+\frac{1}{r_*}}\bar c_1 c_{16}^2, \quad \bar B = \chi_*^{1/\hat\kappa}\bar E^{1/\hat\kappa}(1+|Q_T|)^{1+\mu_2}\Big(1+\frac{1}{T_2-T_1}\Big)^{\mu_2}\mathcal{M}_0^{\mu_2}. \tag{4.41}$$

Obviously, (4.40) is only useful when $\kappa' > \tilde\kappa$, which will be satisfied in Theorem 4.5 below.

Next, we recall a lemma on numeric sequences that will be used in our version of Moser's iteration.

Lemma 4.3 ([9], Lemma A.2) *Let $y_j \ge 0$, $\kappa_j > 0$, $s_j \ge r_j > 0$ and $\omega_j \ge 1$ for all $j \ge 0$. Suppose there is $A \ge 1$ such that*

$$y_{j+1} \le A^{\frac{\omega_j}{\kappa_j}}(y_j^{r_j} + y_j^{s_j})^{\frac{1}{\kappa_j}} \quad \forall j \ge 0. \tag{4.42}$$

Denote $\beta_j = r_j/\kappa_j$ and $\gamma_j = s_j/\kappa_j$. Assume

$$\bar\alpha \overset{\text{def}}{=} \sum_{j=0}^\infty \frac{\omega_j}{\kappa_j} < \infty \text{ and the products } \prod_{j=0}^\infty \beta_j, \prod_{j=0}^\infty \gamma_j \text{ converge to positive numbers } \bar\beta, \bar\gamma, \text{ resp.}$$

Then

$$y_j \le (2A)^{G_j\bar\alpha}\max\left\{y_0^{\gamma_0\cdots\gamma_{j-1}}, y_0^{\beta_0\cdots\beta_{j-1}}\right\} \quad \forall j \ge 1,$$

where $G_j = \max\{1, \gamma_m\gamma_{m+1}\dots\gamma_n : 1 \le m \le n < j\}$. *Consequently,*

$$\limsup_{j\to\infty} y_j \le (2A)^{G\bar{\alpha}} \max\{y_0^{\bar{\gamma}}, y_0^{\bar{\beta}}\}, \quad \textit{where } G = \limsup_{j\to\infty} G_j. \tag{4.43}$$

Some conditions on involved parameters will be imposed and are summarized here.

Assumption 4.4 *Let number* r_* *satisfy* (3.6) *and set* $\lambda_0 = (r_*(5-a)-3)/(3r_*)$. *Fix a number* $\tilde{\kappa} \in (1, \sqrt{1+\lambda_0})$ *and let* $p_i > 1$, $q_i > 1$, *for* $i = 1,2,3,4$, *satisfy* (4.2) *and* (4.3).

We obtain the first estimate for the essential supremum of $\bar{u}(x,t)$.

Theorem 4.5 *Under Assumption 4.4, let* α_0 *be a positive number such that*

$$\alpha_0 \ge \max\left\{\frac{2p_3(1-a)}{\tilde{\kappa}-p_3}, \frac{4(1-a)}{\tilde{\kappa}-1}, \frac{a}{1+\lambda_0-\tilde{\kappa}^2}\right\} \text{ and } \alpha_0 > \max\left\{2, \frac{a}{\lambda_0}\right\}. \tag{4.44}$$

There are positive constants $\tilde{\mu} < \tilde{\nu}$ *and* ω, *which can be identified by* (4.55) *and* (4.57) *below, such that if* $T > 0$ *and* $\sigma \in (0,1)$, *then*

$$\begin{aligned} \|\bar{u}\|_{L^\infty(U\times(\sigma T,T))} \le \Big[\bar{c}_2 \alpha_0^{\mu_3} \chi_*^{1/\hat{\kappa}} \Big(1+\frac{1}{\sigma T}\Big)^{\mu_2} (1+|Q_T|)^{\mu_3} \mathcal{M}_1^{\mu_2} E_*^{1/\hat{\kappa}}\Big]^{\omega} \\ \times \max\left\{\|\bar{u}\|^{\tilde{\mu}}_{L^{\tilde{\kappa}\alpha_0}(U\times(0,T))}, \|\bar{u}\|^{\tilde{\nu}}_{L^{\tilde{\kappa}\alpha_0}(U\times(0,T))}\right\}, \end{aligned} \tag{4.45}$$

where $\bar{c}_2 = (2\tilde{\kappa})^{\mu_3}\bar{A}$, *numbers* μ_2, μ_3, $\hat{\kappa}$, $\bar{A}$ *are respectively given in* (4.34), (4.35), (4.36), (4.41), $\mathcal{M}_1 = \mathcal{M}_{0,T}(q_1,q_2,q_3,q_4)$ *and*

$$E_* = \max\left\{1, \operatorname*{ess\,sup}_{t\in(\sigma T/2,T)} \|1+|\Psi(\cdot,t)|\|^2_{L^{\frac{2r_*}{1-r_*}}} + \operatorname*{ess\,sup}_{t\in(\sigma T/2,T)} \|\bar{u}(\cdot,t)\|^2_{L^{\frac{2r_*}{1-r_*}}}\right\}. \tag{4.46}$$

Proof We prove (4.45) by adapting Moser's iteration. We iterate inequality (4.40) with suitable parameters. For our convenience, redenote κ defined in (4.22) by κ_α. Then κ_α is increasing in α.

Set $\beta_j = \tilde{\kappa}^j\alpha_0$ for $j \ge 0$. Since $\tilde{\kappa} > 1$, the sequence $(\beta_j)_{j=0}^\infty$ is increasing. In particular,

$$\beta_j \ge \beta_0 = \alpha_0 \text{ for all } j \ge 0. \tag{4.47}$$

This relation and (4.44) imply that $\alpha = \beta_j$ satisfies condition (4.21), and

$$\kappa_{\beta_j} \ge \kappa_{\alpha_0} = 1 + \lambda_0 - \frac{a}{\alpha_0} \ge \tilde{\kappa}^2. \tag{4.48}$$

Set $\kappa' = \tilde{\kappa}^2 > 1$. Then property (4.48) implies that (4.38) holds for $\alpha = \beta_j$.

For $j \ge 0$, let $t_j = \sigma T(1 - \frac{1}{2^j})$. Then $t_0 = 0$, $t_1 = \sigma T/2$, t_j is strictly increasing, and $t_j \to \sigma T$ as $j \to \infty$.

For $j \geq 0$, applying inequality (4.40) to $\alpha = \beta_j$, $T_1 = t_j$ and $T_2 = t_{j+1}$, we have

$$\|\bar{u}\|_{L^{\kappa'\beta_j}(U\times(t_{j+1},T))} \leq (\bar{A}\bar{B}_j\beta_j^{\mu_3})^{\frac{1}{\beta_j}}\Big(\|\bar{u}\|^{\tilde{r}_j}_{L^{\beta_{j+1}}(U\times(t_j,T))} + \|\bar{u}\|^{\tilde{s}_j}_{L^{\beta_{j+1}}(U\times(t_j,T))}\Big)^{\frac{1}{\beta_j}}, \tag{4.49}$$

where $\tilde{r}_j = \nu_{3,\beta_j}$, $\tilde{s}_j = \nu_{4,\beta_j}$, see (4.31), number $\bar{A}$ is given in (4.41), and

$$\bar{B}_j = \chi_*^{1/\hat{\kappa}}\bar{E}_j^{1/\hat{\kappa}}(1+|Q_T|)^{1+\mu_2}\Big(1+\frac{1}{t_{j+1}-t_j}\Big)^{\mu_2}\mathcal{M}_{t_j,T}(q_1,q_2,q_3,q_4)^{\mu_2}$$

with $\bar{E}_j = \max\Big\{1, \operatorname{ess\,sup}_{t\in(t_{j+1},T)}\|1+|\Psi(\cdot,t)|\|^2_{L^{\frac{2r_*}{1-r_*}}} + \operatorname{ess\,sup}_{t\in(t_{j+1},T)}\|\bar{u}(\cdot,t)\|^2_{L^{\frac{2r_*}{1-r_*}}}\Big\}$.

Note from (4.34) and (4.35) that $1+\mu_2 = \mu_3$. Clearly, $\mathcal{M}_{t_j,T}(q_1,q_2,q_3,q_4) \leq \mathcal{M}_1$, and comparing $\bar{E}_j$ with E_* in (4.46) gives $\bar{E}_j \leq E_*$. Let

$$M_3 = \chi_*^{1/\hat{\kappa}}E_*^{1/\hat{\kappa}}(1+|Q_T|)^{\mu_3}\mathcal{M}_1^{\mu_2}. \tag{4.50}$$

Then one can estimate

$$\begin{aligned}\bar{A}\bar{B}_j\beta_j^{\mu_3} &\leq \bar{A}M_3\Big(1+\frac{1}{t_{j+1}-t_j}\Big)^{\mu_2}\beta_j^{\mu_3} = \bar{A}M_3\Big(1+\frac{2^{j+1}}{\sigma T}\Big)^{\mu_2}(\tilde{\kappa}^j\alpha_0)^{\mu_3}\\ &\leq \bar{A}M_3 2^{\mu_2(j+1)}\Big(1+\frac{1}{\sigma T}\Big)^{\mu_2}(\tilde{\kappa}^j\alpha_0)^{\mu_3}.\end{aligned}$$

This yields

$$\bar{A}\bar{B}_j\beta_j^{\mu_3} \leq A^{j+1}_{T,\sigma,\alpha_0} \text{ for all } j \geq 0, \tag{4.51}$$

where

$$A_{T,\sigma,\alpha_0} = 2^{\mu_2}\tilde{\kappa}^{\mu_3}\alpha_0^{\mu_3}\bar{A}M_3\Big(1+\frac{1}{\sigma T}\Big)^{\mu_2} > 1. \tag{4.52}$$

For $j \geq 0$, define $Y_j = \|\bar{u}\|_{L^{\beta_{j+1}}(U\times(t_j,T))}$. Note that $\kappa'\beta_j = \tilde{\kappa}^2\beta_j = \beta_{j+2}$. Then we have, by (4.49) and (4.51),

$$Y_{j+1} \leq A^{\frac{j+1}{\beta_j}}_{T,\sigma,\alpha_0}\big(Y_j^{\tilde{r}_j} + Y_j^{\tilde{s}_j}\big)^{\frac{1}{\beta_j}}. \tag{4.53}$$

Hence, we obtain inequality (4.42) in Lemma 4.3 for the sequence $(y_j)_{j=0}^\infty = (Y_j)_{j=0}^\infty$.

We check other conditions in Lemma 4.3. Because $\tilde{\kappa} > 1$, we have

$$\sum_{j=0}^\infty \frac{j+1}{\beta_j} = \frac{1}{\alpha_0}\sum_{j=0}^\infty\frac{j+1}{\tilde{\kappa}^j} = \frac{\tilde{\kappa}^2}{\alpha_0(\tilde{\kappa}-1)^2} \stackrel{\text{def}}{=} \ell_1 \in (0,\infty). \tag{4.54}$$

Using the definitions in (4.31) and the fact $\beta_j \geq \alpha_0 > 2$, see (4.47) and (4.44), we have

$$\frac{\tilde{r}_j}{\beta_j} = \frac{\beta_j - 2}{\beta_j} \in (0,1) \text{ and } \frac{\tilde{s}_j}{\beta_j} = \Big(1+\frac{4(1-a)}{\beta_j}\Big)\Big(1+\frac{a}{\beta_j - a}\Big) \in (1,\infty).$$

Then

$$0 < -\sum_{j=0}^\infty \ln\frac{\tilde{r}_j}{\beta_j} = \sum_{j=0}^\infty \ln\frac{\beta_j}{\tilde{r}_j} = \sum_{j=0}^\infty \ln\Big(1+\frac{2}{\tilde{\kappa}^j\alpha_0 - 2}\Big) \leq \sum_{j=0}^\infty\frac{2}{\tilde{\kappa}^j\alpha_0 - 2} < \infty,$$

$$0<\sum_{j=0}^{\infty}\ln\frac{\tilde{s}_j}{\beta_j}=\sum_{j=0}^{\infty}\ln\left(1+\frac{4(1-a)}{\tilde{\kappa}^j\alpha_0}\right)+\sum_{j=0}^{\infty}\ln\left(1+\frac{a}{\tilde{\kappa}^j\alpha_0-a}\right)$$
$$\le\sum_{j=0}^{\infty}\frac{4(1-a)}{\tilde{\kappa}^j\alpha_0}+\sum_{j=0}^{\infty}\frac{a}{\tilde{\kappa}^j\alpha_0-a}<\infty.$$

Therefore, $\sum_{j=0}^{\infty}\ln(\tilde{r}_j/\beta_j)=\ell_2\in\mathbb{R}$ and $\sum_{j=0}^{\infty}\ln(\tilde{s}_j/\beta_j)=\ell_3\in\mathbb{R}$. Consequently,

$$\tilde{\mu}\overset{\text{def}}{=}\prod_{j=0}^{\infty}\frac{\tilde{r}_j}{\beta_j}=e^{\ell_2}\text{ and }\tilde{\nu}\overset{\text{def}}{=}\prod_{j=0}^{\infty}\frac{\tilde{s}_j}{\beta_j}=e^{\ell_3}\text{ are positive numbers.}\tag{4.55}$$

By (4.53), (4.54), and (4.55), we can apply Lemma 4.3 to the sequence $(Y_j)_{j=0}^{\infty}$, and obtain from (4.43) that

$$\limsup_{j\to\infty}Y_j\le(2A_{T,\sigma,\alpha_0})^{\omega}\max\{Y_0^{\tilde{\mu}},Y_0^{\tilde{\nu}}\},\tag{4.56}$$

where

$$\omega=\ell_4\ell_1\text{ with }\ell_4=\limsup_{j\to\infty}\left(\max\left\{1,\frac{\tilde{s}_m}{\beta_m}\cdot\frac{\tilde{s}_{m+1}}{\beta_{m+1}}\cdots\frac{\tilde{s}_{m'}}{\beta_{m'}}:1\le m\le m'<j\right\}\right).\tag{4.57}$$

In fact, we have, thanks to the property $\tilde{s}_j/\beta_j>1$, that

$$\ell_4=\prod_{k=1}^{\infty}\frac{\tilde{s}_k}{\beta_k}=\frac{\tilde{\nu}\beta_0}{\tilde{s}_0}\in(0,\infty).$$

Note that $\limsup_{j\to\infty}Y_j=\|\bar{u}\|_{L^\infty(U\times(\sigma T,T))}$, $Y_0=\|\bar{u}\|_{L^{\beta_1}(U\times(0,T))}$ and, by (4.52) and (4.50),

$$\begin{aligned}2A_{T,\sigma,\varphi}&=2^{1+\mu_2}\tilde{\kappa}^{\mu_3}\alpha_0^{\mu_3}\bar{A}(1+|Q_T|)^{\mu_3}\left(1+\frac{1}{\sigma T}\right)^{\mu_2}\chi_*^{1/\hat{\kappa}}E_*^{1/\hat{\kappa}}\mathcal{M}_1^{\mu_2}\\&=\bar{c}_2\alpha_0^{\mu_3}(1+|Q_T|)^{\mu_3}\left(1+\frac{1}{\sigma T}\right)^{\mu_2}\chi_*^{1/\hat{\kappa}}E_*^{1/\hat{\kappa}}\mathcal{M}_1^{\mu_2}.\end{aligned}\tag{4.58}$$

Then estimate (4.45) follows (4.56) and (4.58). □

Combining Theorem 4.5 with the L^α-estimate in Theorem 3.2, we have the following L^∞-estimate in terms of initial and boundary data, at least for small time.

Theorem 4.6 *Under Assumption 4.4, let α_0 be a positive number such that*

$$\alpha_0\ge\max\left\{\frac{2p_3(1-a)}{\tilde{\kappa}-p_3},\frac{4(1-a)}{\tilde{\kappa}-1},\frac{a}{1+\lambda_0-\tilde{\kappa}^2},\frac{2r_*}{\tilde{\kappa}(1-r_*)}\right\}\text{ and }\alpha_0>\max\left\{2,\frac{a}{\lambda_0},\frac{4-3a}{\lambda_0\tilde{\kappa}}\right\}.\tag{4.59}$$

Let ω and $\tilde{\nu}$ be the same constants as in Theorem 4.5. Denote

$$\begin{aligned}&\beta_1=\tilde{\kappa}\alpha_0,\ \omega_1=\omega/\hat{\kappa},\ \omega_2=\mu_2\omega,\ \omega_3=\mu_3\omega,\\&\omega_4=\omega_3+\frac{\tilde{\nu}}{\beta_1}\text{ and }\omega_5=\omega_3+\frac{\omega_1(1-r_*)}{r_*}.\end{aligned}\tag{4.60}$$

Let $\mathcal{M}_2(t) = \mathcal{M}_{0,t}(q_1, q_2, q_3, q_4)$ *be defined as in* (4.1)*, and* $\gamma, C_*, V_0, \mathcal{E}(\cdot)$ *be defined as in Theorem 3.2 for* $\alpha = \beta_1$*.*

Suppose $T > 0$ *satisfies* (3.33) *for* $\alpha = \beta_1$ *and some number* $B \in (0,1)$*. For* $t \in [0,T]$*, let*

$$\mathcal{V}(t) = V_0 \left(1 - \gamma C_* V_0^\gamma \int_0^t \mathcal{E}(\tau)d\tau\right)^{-1/\gamma} \text{ and } \mathscr{B}_\sigma(t) = 1 + \operatorname*{ess\,sup}_{\tau\in(\sigma t/2,t)} \|\Psi(\cdot,\tau)\|_{L^{\frac{2r_*}{1-r_*}}}. \tag{4.61}$$

Then one has, for any $t \in (0,T]$ *and* $\sigma \in (0,1)$*, that*

$$\|\bar{u}\|_{L^\infty(U\times(\sigma t,t))} \le \bar{C}_1 \chi_*^{\omega_1}(1+\sigma^{-1}t^{-1})^{\omega_2}(1+t)^{\omega_4}\mathcal{M}_2(t)^{\omega_2}\mathscr{B}_\sigma(t)^{2\omega_1}\mathcal{V}(t)^{\frac{2\omega_1+\tilde{\nu}}{\beta_1}}, \tag{4.62}$$

where $\bar{C}_1 = 2^{\omega_1}\bar{c}_2^{\omega}\alpha_0^{\omega_3}(1+|U|)^{\omega_5}$*.*

Proof Thanks to condition (4.59), α_0 satisfies (4.44), and $\alpha = \beta_1$ satisfies (3.7). Let $\tilde{\mu}$ be the number as in Theorem 4.5. Applying estimate (4.45) to $T := t$ and using definitions of constants in (4.60), we have

$$\begin{aligned}\|\bar{u}\|_{L^\infty(U\times(\sigma t,t))} &\le C_3\chi_*^{\omega_1}(1+\sigma^{-1}t^{-1})^{\omega_2}(1+|Q_t|)^{\omega_3}\mathcal{M}_2^{\omega_2}E_*(t)^{\omega_1}\\ &\quad\times \max\left\{\|\bar{u}\|^{\tilde{\mu}}_{L^{\beta_1}(U\times(0,t))}, \|\bar{u}\|^{\tilde{\nu}}_{L^{\beta_1}(U\times(0,t))}\right\},\end{aligned} \tag{4.63}$$

where $C_3 = (\bar{c}_2\alpha_0^{\mu_3})^\omega$ and

$$E_*(t) = \max\left\{1, \operatorname*{ess\,sup}_{\tau\in(\sigma t/2,t)} \|1+|\Psi(\cdot,\tau)|\|^2_{L^{\frac{2r_*}{1-r_*}}} + \operatorname*{ess\,sup}_{\tau\in(\sigma t/2,t)} \|\bar{u}(\cdot,\tau)\|^2_{L^{\frac{2r_*}{1-r_*}}}\right\}.$$

Note that $\mathcal{V}(t)$ is increasing in $t \in [0,T]$. By (3.35), we have, for all $\tau \in [0,t]$,

$$\int_U |\bar{u}(x,\tau)|^{\beta_1}dx \le \mathcal{V}(\tau) \le \mathcal{V}(t). \tag{4.64}$$

Hence,

$$\int_0^t\int_U |\bar{u}(x,\tau)|^{\beta_1}dxd\tau \le t\mathcal{V}(t).$$

Combining this estimate with the facts $\tilde{\nu} > \tilde{\mu}$ and $\mathcal{V}(t) \ge 1$ yields

$$\begin{aligned}\max\left\{\|\bar{u}\|^{\tilde{\mu}}_{L^{\beta_1}(U\times(0,t))}, \|\bar{u}\|^{\tilde{\nu}}_{L^{\beta_1}(U\times(0,t))}\right\} &\le \max\left\{(t\mathcal{V}(t))^{\frac{\tilde{\mu}}{\beta_1}}, (t\mathcal{V}(t))^{\frac{\tilde{\nu}}{\beta_1}}\right\}\\ &\le (1+t)^{\frac{\tilde{\nu}}{\beta_1}}\mathcal{V}(t)^{\frac{\tilde{\nu}}{\beta_1}}.\end{aligned} \tag{4.65}$$

For $E_*(t)$, we, on the one hand, use the triangle inequality to estimate

$$\operatorname*{ess\,sup}_{\tau\in(\sigma t/2,t)} \|1+|\Psi(\cdot,\tau)|\|^2_{L^{\frac{2r_*}{1-r_*}}} \le \operatorname*{ess\,sup}_{\tau\in(\sigma t/2,t)} \left(|U|^{\frac{1-r_*}{2r_*}} + \|\Psi(\cdot,\tau)\|_{L^{\frac{2r_*}{1-r_*}}}\right)^2 \le (1+|U|)^{\frac{1}{r_*}-1}\mathscr{B}_\sigma(t)^2.$$

On the other hand, we use Hölder's inequality and (4.64) to obtain

$$\operatorname*{ess\,sup}_{\tau\in(\sigma t/2,t)} \|\bar{u}(\cdot,\tau)\|^2_{L^{\frac{2r_*}{1-r_*}}} \le |U|^{\frac{1}{r_*}-1-\frac{2}{\beta_1}} \operatorname*{ess\,sup}_{\tau\in(\sigma t/2,t)} \|\bar{u}(\cdot,\tau)\|^2_{L^{\beta_1}} \le (1+|U|)^{\frac{1}{r_*}-1}\mathcal{V}(t)^{2/\beta_1}.$$

Hence,

$$E_*(t) \le 2(1+|U|)^{\frac{1}{r_*}-1}\mathscr{B}_\sigma(t)^2\mathcal{V}(t)^{2/\beta_1}. \tag{4.66}$$

Combining (4.63), (4.65), and (4.66) with the fact $1+|Q_t| \le (1+|U|)(1+t)$, we have

$$\begin{aligned}\|\bar u\|_{L^\infty(U\times(\sigma t,t))} &\le C_3\chi_*^{\omega_1}(1+\sigma^{-1}t^{-1})^{\omega_2}(1+t)^{\omega_3}(1+|U|)^{\omega_3}\mathcal{M}_2(t)^{\omega_2}\\ &\quad\times\left[2(1+|U|)^{\frac{1}{r_*}-1}\mathscr{B}_\sigma(t)^2\mathcal{V}(t)^{2/\beta_1}\right]^{\omega_1}(1+t)^{\frac{\tilde\nu}{\beta_1}}\mathcal{V}(t)^{\frac{\tilde\nu}{\beta_1}}\\ &= 2^{\omega_1}C_3(1+|U|)^{\omega_5}\chi_*^{\omega_1}(1+\sigma^{-1}t^{-1})^{\omega_2}(1+t)^{\omega_4}\mathcal{M}_2(t)^{\omega_2}\mathscr{B}_\sigma(t)^{2\omega_1}\mathcal{V}(t)^{\frac{2\omega_1+\tilde\nu}{\beta_1}}.\end{aligned}$$

Then inequality (4.62) follows. □

5. Maximum principle

In this section, we estimate the classical solutions of (1.17) by the maximum principle. Recall that the functions $X(z,y)$, $\mathcal{Z}(x,t)$, and $\Phi(x,t)$ are defined by (1.14), (1.16), and (3.2), respectively. We rewrite equation (1.15) in the nondivergence form as

$$u_t = D_yX(u,\Phi) : (D^2u + u^2\Omega^2\mathbf{J}^2 + 2u\nabla u\mathcal{Z}^{\mathrm{T}}) + D_zX(u,\Phi)\cdot\nabla u. \tag{5.1}$$

For $T>0$, denote $U_T = U\times(0,T]$, its closure $\overline{U_T} = \overline{U}\times[0,T]$ and its parabolic boundary $\partial_pU_T = \overline{U_T}\setminus U_T = U\times\{0\}\cup\Gamma\times[0,T]$.

Theorem 5.1 *Assume $u\in C(\overline{U_T})\cap C^{2,1}_{x,t}(U_T)$, $u\ge 0$ on $\overline{U_T}$ and u satisfies (1.15) in U_T. Then one has*

$$\max_{\overline{U_T}} u = \max_{\partial_pU_T} u. \tag{5.2}$$

Proof Given any $\varepsilon>0$, let $u^\varepsilon(x,t) = e^{-\varepsilon t}u(x,t)$ and $M_\varepsilon = \max_{\overline{U_T}} u^\varepsilon$. We claim that

$$M_\varepsilon = \max_{\partial_pU_T} u^\varepsilon. \tag{5.3}$$

Suppose (5.3) is false. Then $M_\varepsilon>0$ and there exists a point $(x_0,t_0)\in U_T$ such that $u^\varepsilon(x_0,t_0) = M_\varepsilon$. At this maximum point (x_0,t_0), we have

$$u^\varepsilon_t(x_0,t_0)\ge 0 \text{ and } Du^\varepsilon(x_0,t_0) = 0. \tag{5.4}$$

It is proved in [10, Theorem 3.1], based mainly on property (2.15) in Lemma 2.2, that

$$D_yX(u,\Phi):(D^2u+u^2\Omega^2\mathbf{J}^2)\Big|_{(x,t)=(x_0,t_0)} \le 0. \tag{5.5}$$

The second property of (5.4) deduces $Du(x_0,t_0)=0$. This fact, (5.5) and (5.1) imply $u_t(x_0,t_0)\le 0$. Therefore,

$$u^\varepsilon_t(x_0,t_0) = -\varepsilon u^\varepsilon(x_0,t_0) + e^{-\varepsilon t_0}u_t(x_0,t_0) \le -\varepsilon M_\varepsilon < 0,$$

which contradicts the first inequality in (5.4). Thus, (5.3) holds true. Note that

$$e^{-\varepsilon T}\max_{\overline{U_T}} u \le M_\varepsilon = \max_{\partial_p U_T} u^\varepsilon \le \max_{\partial_p U_T} u \le \max_{\overline{U_T}} u.$$

Then passing $\varepsilon \to 0$, we obtain (5.2). □

In the following, $T_* \in (0,\infty]$ is fixed.

Clearly, if $u \in C(\overline{U}\times[0,T_*)) \cap C^{2,1}_{x,t}(U\times(0,T_*))$ is a nonnegative solution of problem (1.17), then, by the virtue of Theorem 5.1, we have the maximum estimates in terms of the initial and boundary data:

$$\sup_{x\in U} u(x,t) \le \max\left\{\sup_{x\in U} u_0(x), \sup_{(x,\tau)\in\Gamma\times(0,t]} \psi(x,\tau)\right\} \text{ for all } t\in(0,T_*). \tag{5.6}$$

In case the solution u belongs to $C(\overline{U}\times(0,T_*))$ but not $C(\overline{U}\times[0,T_*))$, estimate (5.6) is not applicable. For instance, initial data u_0 is unbounded. However, under certain weaker conditions, the maximum estimates can still be established by combining Theorems 4.6 and 5.1.

Under Assumption 4.4, let α_0 satisfy (4.59). We use the same notation as in Theorem 4.6. Assume further that

(i) $\mathcal{M}_2(t)$ is finite for all $t\in(0,T_*)$ and $\mathcal{E}\in L^1_{\rm loc}([0,T_*))$,

(ii) $\Psi \in C(\overline{U}\times(0,T_*)) \cap C([0,T_*),L^{\beta_1}(U))$.

Because of the second property in (i), we can find a number t_0 such that $0<t_0\le 1$, $t_0<T_*$, and (3.33) is satisfied for $T=t_0$, $\alpha=\beta_1$, and some number $B\in(0,1)$.

Theorem 5.2 *Let $u \in C(\overline{U}\times(0,T_*)) \cap C^{2,1}_{x,t}(U\times(0,T_*)) \cap C([0,T_*),L^{\beta_1}(U))$ be a nonnegative solution of problem* (1.17).

If $t\in(0,t_0]$, then

$$\sup_{x\in U} u(x,t) \le \bar C_2 \chi_*^{\omega_1} t^{-\omega_2}\mathcal{M}_2(t)^{\omega_2}\mathscr{B}_*(t)^{2\omega_1}\mathcal{V}(t)^{\frac{2\omega_1+\tilde\nu}{\beta_1}} + \sup_{x\in U}|\Psi(x,t)|, \tag{5.7}$$

where $\bar C_2 = 3^{\omega_2}2^{\omega_4}\bar C_1$ and

$$\mathscr{B}_*(t) = 1 + \operatorname*{ess\,sup}_{\tau\in(t/4,t)} \|\Psi(\cdot,\tau)\|_{L^{\frac{2r_*}{1-r_*}}}. \tag{5.8}$$

If $t\in(t_0,T_*)$, then

$$\sup_{x\in U} u(x,t) \le \max\left\{\bar C_3\chi_*^{\omega_1}\mathcal{M}_3(t_0)\left(1+\|\bar u_0\|_{L^{\beta_1}}\right)^{2\omega_1+\tilde\nu} + \sup_{x\in U}|\Psi(x,t_0)|, \sup_{(x,\tau)\in\Gamma\times[t_0,t]}\psi(x,\tau)\right\}, \tag{5.9}$$

where $\bar C_3 = \bar C_2(1-B)^{-\frac{2\omega_1+\tilde\nu}{\beta_1\gamma}}$ and $\mathcal{M}_3(t_0) = t_0^{-\omega_2}\mathcal{M}_2(t_0)^{\omega_2}\mathscr{B}_*(t_0)^{2\omega_1}$.

Proof First, note that $\mathscr{B}_*(t)$ in (5.8) is, in fact, $\mathscr{B}_{1/2}(t)$ in (4.61). Let $t \in (0, t_0]$. By estimate (4.62) applied to $\sigma = 1/2$, we have

$$\begin{aligned}\|\bar{u}\|_{L^\infty(U\times(t/2,t))} &\le \bar{C}_1 \chi_*^{\omega_1}(1+2t^{-1})^{\omega_2}(1+t)^{\omega_4}\mathcal{M}_2(t)^{\omega_2}\mathscr{B}_*(t)^{2\omega_1}\mathcal{V}(t)^{\frac{2\omega_1+\tilde{\nu}}{\beta_1}}\\ &\le \bar{C}_1 \chi_*^{\omega_1}(3t^{-1})^{\omega_2}2^{\omega_4}\mathcal{M}_2(t)^{\omega_2}\mathscr{B}_*(t)^{2\omega_1}\mathcal{V}(t)^{\frac{2\omega_1+\tilde{\nu}}{\beta_1}}\\ &= \bar{C}_2 \chi_*^{\omega_1}t^{-\omega_2}\mathcal{M}_2(t)^{\omega_2}\mathscr{B}_*(t)^{2\omega_1}\mathcal{V}(t)^{\frac{2\omega_1+\tilde{\nu}}{\beta_1}}.\end{aligned}$$

By the continuity of $\bar{u}$ on $\overline{U} \times [t/2, t]$,

$$\sup_{x\in U}|\bar{u}(x,t)| \le \|\bar{u}\|_{L^\infty(U\times(t/2,t))} \le \bar{C}_2 \chi_*^{\omega_1}t^{-\omega_2}\mathcal{M}_2(t)^{\omega_2}\mathscr{B}_*(t)^{2\omega_1}\mathcal{V}(t)^{\frac{2\omega_1+\tilde{\nu}}{\beta_1}}.$$

Combining this estimate with the triangle inequality $u(x,t) \le |\bar{u}(x,t)| + |\Psi(x,t)|$ gives (5.7).

Let $t \in (t_0, T_*)$ now. Applying the maximum principle in Theorem 5.1 for the interval $[t_0, t]$ in place of $[0, T]$, we have

$$\sup_{x\in U} u(x,t) \le \max\left\{\sup_{x\in U} u(x,t_0), \sup_{(x,\tau)\in\Gamma\times[t_0,t]}\psi(x,\tau)\right\}. \tag{5.10}$$

Estimating $u(x, t_0)$ by using (5.7) for $t = t_0$ yields

$$\sup_{x\in U} u(x,t_0) \le \bar{C}_2 \chi_*^{\omega_1}t_0^{-\omega_2}\mathcal{M}_2(t_0)^{\omega_2}\mathscr{B}_*(t_0)^{2\omega_1}\mathcal{V}(t_0)^{\frac{2\omega_1+\tilde{\nu}}{\beta_1}} + \sup_{x\in U}|\Psi(x,t_0)|. \tag{5.11}$$

Thanks to estimate (3.35) for $t = t_0$, we have

$$\mathcal{V}(t_0)^{1/\beta_1} \le V_0^{1/\beta_1}(1-B)^{-1/(\beta_1\gamma)}. \tag{5.12}$$

Applying inequality (2.6) to $x = 1$, $y = \int_U |u_0(x)|^{\beta_1}dx$ and $p = 1/\beta_1 < 1$ gives

$$V_0^{1/\beta_1} \le 1 + \|u_0\|_{L^{\beta_1}}. \tag{5.13}$$

Then estimate (5.9) follows from (5.10), (5.11), (5.12), and (5.13). □

Remark 5.3 *The following final remarks are in order.*

(a) *As a sequel of our previous work [10], the current paper only considers slightly compressible fluids. Nonetheless, the methods developed here and in [8, 9] can be applied to analyze other types of (compressible) gaseous flows in rotating porous media.*

(b) *Our analysis can be easily adapted for more general PDE of type (1.15) in space $\mathbb{R}^n$ not just $\mathbb{R}^3$. The function X is only required to have similar properties to those in Lemmas 2.1 and 2.2.*

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