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Discontinuous Galerkin method for blow-up solutions of nonlinear wave equations

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Abstract: We develop and study an explicit time-space discrete discontinuous Galerkin finite element method to approximate the solution of one-dimensional nonlinear wave equations. We show that the numerical scheme is stable if a nonuniform time mesh is considered. We also investigate the blow-up phenomena and we prove that under weak convergence assumptions, the numerical blow-up time tends toward the theoretical one. The validity of our results is confirmed throughout several examples and benchmarks.

Key words: Nonlinear wave equation, discontinuous Galerkin methods, numerical blow-up, numerical analysis

1. Introduction

This paper is concerned with the development of a numerical method, based on discontinuous Galerkin (DG) formulation, in order to approximate the blow-up behaviors of smooth solutions of the semilinear wave equation in one space dimension $\Omega = (a, b) \subset \mathbb{R}$ with periodic boundary conditions

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = |u|^p, & \text{in } \Omega \times (0, \infty) \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \bar{\Omega} \\ u(a, t) = u(b, t), & t \geqslant 0. \end{cases} \tag{1.1}$$

with $p > 1$. The theoretical study of the semilinear wave equation is well developed. In [7] and [8], Cafarelli and Friedman showed the existence of solutions of Cauchy problems for smooth initial data and gave a description of the blow-up set. In [24], Glassey proved that under suitable assumptions on the initial data, the solution u of (1.1) blows up in the following sense: there exists $T_\infty < \infty$, called the blow-up time, such that the solution u exists on $[0, T_\infty)$ and

$$\|u(., t)\|_{L^\infty(\Omega)} \longrightarrow \infty \quad \text{as} \quad t \longrightarrow T_\infty.$$

Recently, Merle and Zaag gave in a series of papers a classification of the blow-up behavior and an exhaustive description of the geometry of the blowup set [36–39]. More theoretical results can also be found in [4, 8, 25, 32, 34].

From a numerical point of view, the approximation of solutions which blow up in finite time is more delicate. Indeed, one of the major difficulties when deriving numerical schemes is related to the standard

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stability criterion which imposes the boundedness of the numerical solution at any finite time. This is clearly in opposition with the sought blow-up behavior. In addition, the numerical solutions may remain bounded though the exact solutions do explode in finite time. These aspects have been observed when using a spectral method or even a finite differences (FD) method for the Constantin-Lax-Majda equation [12, 20]. To overcome such a difficulty, Nakagawa [40] first introduced an adaptive time-stepping strategy to compute the blow-up FD solutions and the blow-up time for the 1D semilinear heat equation $\partial_t u - \partial_{xx} u = u^2$ in $(0,\, 1)$ with homogeneous Dirichlet boundary conditions. To ensure the stability of his numerical scheme, he defined a local time stepping given by

$$\Delta t^n = \tau \min\left(1,\, \frac{1}{\|u_h^n\|_2}\right),$$

where τ is a prescribed parameter. He showed that the numerical solution converges point-wise toward the exact solution. Moreover, by setting the numerical blow-up time

$$T(\tau, \Delta x) = \sum_{n=0}^{\infty} \Delta t^n,$$

he proved that $T(\tau, \Delta x)$ is finite and converges toward the theoretical blow-up time when Δx goes to zero. Since then, many authors have improved Nakagawa's results and showed that the FD schemes with adaptively-defined time mesh give good approximation for the blow-up solution of the nonlinear heat equation [1, 10, 11]. Other methods using different approaches, such as finite elements methods, semidiscretization and line methods, rescaling techniques, for the numerical approximation of blow-up solutions of parabolic equations can also be found in [5, 6, 14, 41] and references therein.

For hyperbolic equations, Cho applied Nakagawa's ideas to the nonlinear wave equation with nonuniform time mesh [12]. Recently, Sasaki and Saito [42] reduced the nonlinear wave equation to a first order system and considered an FD scheme with a local time stepping. They succeeded in proving the convergence of their FD scheme and the numerical blow-up time. It is worth noticing that almost all the methods we found in the literature are essentially based on FD discretizations, and only few use variational (integral) formulations [26, 28, 30]. We propose in this paper to investigate a DG method to numerically solve the semilinear wave equation (1.1) when blow-up phenomena occur.

The organization of this paper is as follows. In Section 2, we present the DG methods and we derive a numerical scheme for the nonlinear wave equation. Section 3 is devoted to the proof of the stability of the proposed numerical scheme. In Section 4, we prove that the numerical blow-up time converges toward the exact blow-up time under weak convergence assumptions. Finally, we provide several numerical examples that illustrate the validity of our proposed method in Section 5.

2. Discontinuous Galerkin method

In this section, we derive a discontinuous Galerkin scheme (DG) for the nonlinear wave equation (1.1). Formally, one may rewrite the D'Alembert operator as $\Box = (\partial_t - \partial_x)\,(\partial_t + \partial_x)$. Based on such a decomposition, we split (1.1) into a first order system as follows:

$$\begin{cases} \partial_t u + \partial_x u = v, & \text{in } (a,b)\times(0,\infty) \\ \partial_t v - \partial_x v = |u|^p, & \text{in } (a,b)\times(0,\infty) \\ u(x,0) = u_0(x), & x\in(a,b) \\ v(x,0) = v_0(x), & x\in(a,b), \\ u(a,t) = u(b,t), & t\geqslant 0 \\ v(a,t) = v(b,t), & t\geqslant 0. \end{cases} \tag{2.1}$$

with $v_0 = u_1 + u_0'$.

2.1. Space discretization

In order to introduce a variational approximation of the system (2.1), we consider a partition for the spatial domain $[a,b] = \bigcup_{i=1}^{I} K_i$ consisting of cells $K_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $1\leqslant i\leqslant I$. The length of the cell K_i is denoted $h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$. For simplicity, we shall assume that $h_i = h > 0$ for all i. Next, we define the finite dimensional space X_h^k consisting of all functions f such that their restriction on a cell K_i is a polynomial of degree at most k, i.e.

$$X_h^k = \{f \;/\; f|_{K_i} \in \mathbb{P}_k[K_i],\ i = 1,\ldots,I\},$$

where $\mathbb{P}_k[K_i]$ denotes the space of polynomials in K_i of degree less than or equal to k. In the sequel, we will consider the Lagrange polynomials, denoted $\langle\varphi_j^i\rangle_{1\leqslant j\leqslant k+1}$, as a basis of $\mathbb{P}_k[K_i]$. Notice that the functions of X_h^k are allowed to be discontinuous across the elements interfaces. The solutions of the numerical method are denoted by u_h and v_h and both belong to X_h^k. We denote by $(u_h)^-_{i+\frac{1}{2}}$ and $(u_h)^+_{i+\frac{1}{2}}$ the left and right limits of u_h at $x_{i+\frac{1}{2}}$, respectively. Moreover, we denote by $[u_h]_{i+\frac{1}{2}} = (u_h)^+_{i+\frac{1}{2}} - (u_h)^-_{i+\frac{1}{2}}$ the jump of u_h at the cell interface $x_{i+\frac{1}{2}}$. The same notations apply also to v_h. Multiplying the system (2.1) by test functions and integrating over the cells yields the following variational formulation: find $(u_h, v_h)\in X_h^k\times X_h^k$ such that for all test functions $(\varphi_h,\psi_h)\in X_h^k\times X_h^k$ and for any $1\leqslant i\leqslant I$

$$\begin{aligned} &\int_{K_i} \partial_t u_h\,\varphi_h dx - \int_{K_i} u_h\,\partial_x\varphi_h dx \\ &\qquad + (\widehat{u}_h\,\varphi_h)_{i+\frac{1}{2}} - (\widehat{u}_h\,\varphi_h)_{i-\frac{1}{2}} = \int_{K_i} v_h\,\varphi_h dx \end{aligned} \tag{2.2a}$$

$$\begin{aligned} &\int_{K_i} \partial_t v_h\,\psi_h dx + \int_{K_i} v_h\,\partial_x\psi_h dx \\ &\qquad - (\widehat{v}_h\,\psi_h)_{i+\frac{1}{2}} + (\widehat{v}_h\,\psi_h)_{i-\frac{1}{2}} = \int_{K_i} \mathcal{I}_h^k(|u_h|^p)\,\psi_h dx, \end{aligned} \tag{2.2b}$$

where $\widehat{u}_h$ and $\widehat{v}_h$ are the numerical fluxes and have to be defined at the cell interfaces, and $\mathcal{I}_h^k : C([a,b]) \to X_h^k$ is the interpolation operator defined by $\mathcal{I}_h^k(f) = \sum_{j=1}^{k+1} f(x_j)\varphi_j$. In general, these numerical fluxes depend on the values of the numerical solution from both sides of the interface. Here, we propose a backward (resp. forward) flux to define the trace of u_h (resp. v_h) at an interface $x_{i\pm\frac{1}{2}}$, i.e.

$$(\widehat{u}_h)_{i\pm\frac{1}{2}} = (u_h)^-_{i\pm\frac{1}{2}}, \quad (\widehat{v}_h)_{i\pm\frac{1}{2}} = (v_h)^+_{i\pm\frac{1}{2}}. \tag{2.3}$$

It follows that (2.2) can be written as: $\forall\, 1 \leqslant i \leqslant I$ and $\forall\, 1 \leqslant j \leqslant k+1$

$$\int_{K_i} \partial_t u_h^i\, \varphi_j^i dx - \int_{K_i} u_h^i\, \partial_x \varphi_j^i dx + (u_h)^-_{i+\frac{1}{2}} \varphi_j^i(x_{i+\frac{1}{2}}) - (u_h)^-_{i-\frac{1}{2}} \varphi_j^i(x_{i-\frac{1}{2}}) = \int_{K_i} v_h^i\, \varphi_j^i dx \tag{2.4a}$$

$$\int_{K_i} \partial_t v_h^i\, \psi_j^i dx + \int_{K_i} v_h^i\, \partial_x \psi_j^i dx - (v_h)^+_{i+\frac{1}{2}} \psi_j^i(x_{i+\frac{1}{2}}) + (v_h)^+_{i-\frac{1}{2}} \psi_j^i(x_{i-\frac{1}{2}}) = \int_{K_i} \mathcal{I}_h^k\left(|u_h^i|^p\right) \psi_j^i dx, \tag{2.4b}$$

with $u_h^i = u_{h|_{K_i}}$ (resp. $v_h^i = v_{h|_{K_i}}$) is the restriction of u_h (resp. v_h) over the cell K_i. Integrating by parts once more, one may write (2.4) as: $\forall\, 1 \leqslant i \leqslant I$ and $\forall\, 1 \leqslant j \leqslant k+1$

$$\int_{K_i} \left(\partial_t u_h^i + \partial_x u_h^i\right) \varphi_j^i\, dx + [u_h]_{i-\frac{1}{2}} \varphi_j^i(x_{i-\frac{1}{2}}) = \int_{K_i} v_h^i\, \varphi_j^i\, dx \tag{2.5a}$$

$$\int_{K_i} \left(\partial_t v_h^i - \partial_x v_h^i\right) \psi_j^i\, dx - [v_h]_{i+\frac{1}{2}} \psi_j^i(x_{i+\frac{1}{2}}) = \int_{K_i} \mathcal{I}_h^k\left(|u_h^i|^p\right) \psi_j^i\, dx, \tag{2.5b}$$

where $[\cdot]$ denotes the jump at the cell interface. Recall that u_h and v_h belong to X_h^k; hence, one can write

$$u_h^i(x,t) = \sum_{\ell=1}^{k+1} u_\ell^i(t) \varphi_\ell^i(x) \quad \text{and} \quad v_h^i(x,t) = \sum_{\ell=1}^{k+1} v_\ell^i(t) \psi_\ell^i(x). \tag{2.6}$$

Moreover, since $\mathcal{I}_h^k\left(|u_h^i|^p\right)$ also belongs to X_h^k, then we have for all $1 \leqslant i \leqslant I$

$$\mathcal{I}_h^k\left(|u_h^i(x,t)|^p\right) = \sum_{\ell=1}^{k+1} |u_\ell^i(t)|^p\, \varphi_\ell^i(x). \tag{2.7}$$

Plugging (2.6) and (2.7) into (2.5) yields the semidiscrete matricial system: $\forall\, t > 0$ and $\forall\, 1 \leqslant i \leqslant I$

$$M^i \partial_t U_h^i(t) + R^i\, U_h^i(t) + A^i\, U_h^i(t) - B^i\, U_h^{i-1}(t) = M^i V_h^i(t), \tag{2.8a}$$

$$M^i \partial_t V_h^i(t) - R^i\, V_h^i(t) - C^i\, V_h^{i+1}(t) + D^i\, V_h^i(t) = M^i |U_h^i(t)|^p, \tag{2.8b}$$

where $U_h^i = \left(u_1^i, \ldots, u_{k+1}^i\right)$, $V_h^i = \left(v_1^i, \ldots, v_{k+1}^i\right)$, $|U_h^i|^p = \left(|u_1^i|^p, \ldots, |u_{k+1}^i|^p\right)$ and $\forall\, 1 \leqslant j, \ell \leqslant k+1$

$$M_{j\ell}^i = \int_{K_i} \varphi_j^i\ \varphi_\ell^i\ dx, \quad R_{j\ell}^i = \int_{K_i} \varphi_j^i\ \partial_x \varphi_\ell^i\ dx,$$

$$A_{j\ell}^i = \varphi_j^i(x_{i-\frac{1}{2}})\ \varphi_\ell^i(x_{i-\frac{1}{2}}), \quad B_{j\ell}^i = \varphi_j^i(x_{i-\frac{1}{2}})\ \varphi_\ell^{i-1}(x_{i-\frac{1}{2}}),$$

and

$$C_{j\ell}^i = \varphi_j^i(x_{i+\frac{1}{2}})\ \varphi_\ell^{i+1}(x_{i+\frac{1}{2}}), \quad D_{j\ell}^i = \varphi_j^i(x_{i+\frac{1}{2}})\ \varphi_\ell^i(x_{i+\frac{1}{2}}).$$

For the boundary conditions, we set $U_h^0(t) := U_h^I(t)$ and $V_h^{I+1}(t) := V_h^1(t)$ for all $t \geqslant 0$.

2.2. Time discretization

A fully discrete scheme of (2.8) can be derived using an approximation of the time derivative $\partial_t U_h$ and $\partial_t V_h$. Here, we used the explicit forward Euler method with nonconstant time step. Let Δt^0, Δt^1, ... be positive constants and set

$$t^0 = 0, \quad t^n = \sum_{\ell=0}^{n-1} \Delta t^\ell = t^{n-1} + \Delta t^{n-1} \quad (n \geqslant 1). \tag{2.9}$$

Then, we approximate the time derivative of U_h and V_h at time t^n as follows

$$\partial_t U_h(t^n) \approx \frac{U_h^{n+1} - U_h^n}{\Delta t^n} \quad \text{and} \quad \partial_t V_h(t^n) \approx \frac{V_h^{n+1} - V_h^n}{\Delta t^n},$$

where U_h^n (resp. V_h^n) is the value of U_h (resp. V_h) at time t^n. The fully discrete DG scheme for the nonlinear wave equation (1.1) is then given by: $\forall\, n \geqslant 0$, $\forall\ 1 \leqslant i \leqslant I$ and $\forall\ 1 \leqslant j \leqslant k+1$

$$\int_{K_i} \left(\frac{u_h^{i,n+1} - u_h^{i,n}}{\Delta t^n} + \partial_x u_h^{i,n} \right) \varphi_j^i \, dx + [u_h^n]_{i-\frac{1}{2}} \varphi_j^i(x_{i-\frac{1}{2}}) = \int_{K_i} v_h^{i,n} \, \varphi_j^i \, dx \tag{2.10a}$$

$$\int_{K_i} \left(\frac{v_h^{i,n+1} - v_h^{i,n}}{\Delta t^n} - \partial_x v_h^{i,n} \right) \psi_j^i \, dx - [v_h^n]_{i+\frac{1}{2}} \psi_j^i(x_{i+\frac{1}{2}}) = \int_{K_i} \mathcal{I}_h^k \left(|u_h^{i,n+1}|^p \right) \psi_j^i \, dx, \tag{2.10b}$$

with the initial conditions $(u_h^{i,0}, v_h^{i,0}) = (\mathcal{I}_h^k u_0^i, \mathcal{I}_h^k v_0^i)$ and the periodic boundary conditions $(u_h^{1,n}, v_h^{1,n}) = (u_h^{I,n}, v_h^{I,n})$. Equivalently, the system (2.10) writes in matricial form: $\forall\ n \geqslant 0$ and $\forall\ 1 \leqslant i \leqslant I$

$$M^i \frac{U_h^{i,n+1} - U_h^{i,n}}{\Delta t^n} + \left(R^i + A^i\right) U_h^{i,n} - B^i\, U_h^{i-1,n} = M^i V_h^{i,n}, \tag{2.11a}$$

$$M^i \frac{V_h^{i,n+1} - V_h^{i,n}}{\Delta t^n} - \left(R^i - D^i\right) V_h^{i,n} - C^i\, V_h^{i+1,n} = M^i |U_h^{i,n+1}|^p, \tag{2.11b}$$

$$U_h^{i,0} = \mathcal{I}_h^k u_0^i, \quad V_h^{i,0} = \mathcal{I}_h^k v_0^i, \tag{2.11c}$$

$$U_h^{0,n} := U_h^{I,n}, \quad V_h^{I+1,n} := V_h^{1,n}. \tag{2.11d}$$

Notice that scheme (2.11) is fully explicit in time. This is of major advantage when performing the calculation of the numerical solution from one time step to another since neither matrix inversions nor implicit nonlinear computations have to be performed. The drawback of such formulations is their lack of convergence results. On the other hand, the implicit DG schemes are proved to be stable and convergent in general. However, the numerical solution is given implicitly and requires tremendous computational costs to be evaluated. Moreover, the existence of the solution is not even guaranteed in some cases. For implicit DG formulations, on may refer, e.g., to [30, 33].

3. Study of the DG scheme

We prove in this section the consistency and the local stability of the DG scheme.

3.1. Consistency

Lemma 3.1 *The DG scheme* (2.10) *is consistent with the system* (2.1)*.*

Proof It is obvious from (2.3) that the numerical fluxes are monotone and thus consistent [19]. Our purpose now is to prove that the approximation of the nonlinear term is also consistent with the original system (2.1). We shall assume that the solution $u \in C^2([0,T_\infty), H^{m+2}(a,b))$ for a given $m \geqslant 1$. Thus, $v \in C^1([0,T_\infty), H^{m+1}(a,b))$ and the jumps $[u]_{i+\frac{1}{2}}$ and $[v]_{i+\frac{1}{2}}$ vanish over the interfaces $x_{i+\frac{1}{2}}$ for all $0 \leqslant i \leqslant I$ and for all time t. Denote

$$\begin{aligned} r^n := &\sum_{i=1}^{I} \int_{K_i} \left(\frac{u(x,t^{n+1}) - u(x,t^n)}{\Delta t^n} + \partial_x u(x,t^n) \right) \varphi^i(x)\, dx \\ &+ \sum_{i=1}^{I} \underbrace{[u(\cdot,t^n)]_{i-\frac{1}{2}}}_{=0} \varphi^i(x_{i-\frac{1}{2}}) - \sum_{i=1}^{I} \int_{K_i} v(x,t^n)\, \varphi^i(x)\, dx \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} s^n := &\sum_{i=1}^{I} \int_{K_i} \left(\frac{v(x,t^{n+1}) - v(x,t^n)}{\Delta t^n} - \partial_x v(x,t^n) \right) \psi^i(x)\, dx \\ &- \sum_{i=1}^{I} \underbrace{[v(\cdot,t^n)]_{i+\frac{1}{2}}}_{=0} \psi^i(x_{i+\frac{1}{2}}) - \sum_{i=1}^{I} \int_{K_i} \mathcal{I}_h^k \left(|u(x,t^{n+1})|^p \right) \psi^i(x)\, dx. \end{aligned} \tag{3.2}$$

It follows by (2.1) and using a first order Taylor expansion in (3.1) that

$$|r^n| \leqslant C_1 \Delta t^n \quad \forall\, n \geqslant 0$$

with $C_1 > 0$ is independent of Δt^n. Similarly, we have using a second order Taylor series in (3.2)

$$s^n = \sum_{i=1}^{I} \int_{K_i} \Delta t^n \partial_{tt} v(x,\xi^n) \psi^i(x)\, dx + \sum_{i=1}^{I} \int_{K_i} \left(|u(x,t^{n+1})|^p - \mathcal{I}_h^k \left(|u(x,t^{n+1})|^p \right) \right) \psi^i(x)\, dx.$$

Using the classical estimate (see, e.g., [22, Theorem 1.103])

$$\|v - \mathcal{I}_h^k v\|_{L^\infty(K)} \leqslant \tilde{C} h^m \quad \text{for any } v \in H^{m+1}(K) \tag{3.3}$$

we deduce

$$|s^n| \leqslant C_2 \Delta t^n + C_3 h^m \quad \forall\, n \geqslant 0.$$

with C_2 and C_3 are positive constants independent of Δt^n and h. This concludes the consistency of the proposed DG scheme. □

3.2. Positivity and local stability

For $u_h \in X_h^k$, we define the norm

$$\|u_h\|_\infty := \|U_h\|_\infty = \max_{1 \leqslant i \leqslant I} \|U_h^i\|_\infty = \max_{1 \leqslant i \leqslant I} \max_{1 \leqslant j \leqslant k+1} |u_j^i|,$$

where the u_j^i are the coordinates of u_h in the Lagrange polynomial basis.

Proposition 3.2 *Let $\sigma > 0$ and $\nu > 0$ be arbitrary real numbers and set*

$$\Delta t^n = h^{1+\sigma} \min\left(1, \frac{1}{\|u_h^n\|_\infty^{1+\nu}}\right). \tag{3.4}$$

Suppose the initial conditions satisfy $\min(u_0, v_0) > \mu \geqslant 0$. *Then, for any* $N \in \mathbb{N}$, *there exists a constant* $h_N > 0$ *depending on* N, u_0 *and* v_0 *such that for all* $h \in (0, h_N]$,

$$U_h^n > \mu \quad \text{and} \quad V_h^n > \mu \quad \forall\, 1 \leqslant n \leqslant N. \tag{3.5}$$

(the inequalities are element-wise). In addition, if $\sum_{n\geqslant 0} \frac{1}{\|u_h^n\|_\infty^\nu} + \frac{1}{\|v_h^n\|_\infty^\nu} < \infty$, *then* (3.5) *holds for* $h_N = h_*$ *independent of* N.

Proof We proceed by induction on n. Since $u_0 > \mu \geqslant 0$ (resp. $v_0 > \mu \geqslant 0$) then $u_j^{i,0} = u_{0|_{K_i}}(x_j^i) > \mu$ (resp. $v_j^{i,0} = v_{0|_{K_i}}(x_j^i) > \mu$); hence, (3.5) holds true for $n = 0$. Let $N \in \mathbb{N}$ and suppose (3.5) is valid for all $0 \leqslant n \leqslant N-1$, then $u_j^{i,n} > \mu$ and $v_j^{i,n} > \mu$ for all $1 \leqslant i \leqslant I$ and all $1 \leqslant j \leqslant k+1$. Moreover, equation (2.11) reads

$$U_h^{i,n+1} = U_h^{i,n} + \frac{\Delta t^n}{h}\left(E\,U_h^{i,n} + F\,U_h^{i-1,n}\right) + \Delta t^n V_h^{i,n}$$

with $E = h(M^i)^{-1}(R^i + A^i)$ and $F = -h(M^i)^{-1}B^i$ are constant matrices that do not depend on h and satisfy

$$\sum_{\ell=1}^{k+1} E_{j\ell} + F_{j\ell} = 0 \quad \forall\, 1 \leqslant j \leqslant k+1. \tag{3.6}$$

(see Appendix A for details). Denote $x^+ = \max(x, 0)$ and $x^- = \min(x, 0)$ for any $x \in \mathbb{R}$, then we obtain for $1 \leqslant i \leqslant I$ and $1 \leqslant j \leqslant k+1$

$$\begin{aligned} u_j^{i,n+1} &= u_j^{i,n} + \frac{\Delta t^n}{h}\sum_{\ell=1}^{k+1}\left(E_{j\ell}\,u_\ell^{i,n} + F_{j\ell}\,u_\ell^{i-1,n}\right) + \Delta t^n v_j^{i,n} \\ &= u_j^{i,n} + \frac{\Delta t^n}{h}\left(\sum_{\ell=1}^{k+1}\left(E_{j\ell}^+\,u_\ell^{i,n} + F_{j\ell}^+\,u_\ell^{i-1,n}\right) + \sum_{\ell=1}^{k+1}\left(E_{j\ell}^-\,u_\ell^{i,n} + F_{j\ell}^-\,u_\ell^{i-1,n}\right)\right) + \Delta t^n v_j^{i,n}. \end{aligned}$$

Denote $\underline{u}^n = \min_{i,j} u_j^{i,n}$ and $\overline{u}^n = \max_{i,j} u_j^{i,n}$ and use (3.6), one obtains

$$\begin{aligned} u_j^{i,n+1} &\geqslant \underline{u}^n + \frac{\Delta t^n}{h}\left(\sum_{\ell=1}^{k+1}\left(E_{j\ell}^+ + F_{j\ell}^+\right)\underline{u}^n + \sum_{\ell=1}^{k+1}\left(E_{j\ell}^- + F_{j\ell}^-\right)\overline{u}^n\right) \\ &= \underline{u}^n + \frac{\Delta t^n}{h}\left(\sum_{\ell=1}^{k+1}\left(E_{j\ell}^+ + F_{j\ell}^+\right)\right)\left(\underline{u}^n - \overline{u}^n\right). \end{aligned}$$

Let $\alpha_n = \rho\frac{\Delta t^n}{h}$ with

$$\rho = \min_{1\leqslant j\leqslant k+1}\sum_{\ell=1}^{k+1}\left(E_{j\ell}^+ + F_{j\ell}^+\right), \tag{3.7}$$

then we have

$$\underline{u}^{n+1} \geqslant \underline{u}^n + \alpha_n(\underline{u}^n - \overline{u}^n). \tag{3.8}$$

A straightforward induction on n shows that

$$\begin{aligned}\underline{u}^{n+1} &\geqslant \left(\prod_{m=0}^{n}(1+\alpha_m)\right)\underline{u}^0 - \sum_{\ell=0}^{n}\left(\prod_{m=\ell+1}^{n}(1+\alpha_m)\right)\alpha_\ell \overline{u}^\ell \\ &\geqslant \left(\prod_{m=0}^{n}(1+\alpha_m)\right)\left(\underline{u}^0 - \frac{\rho}{h}\sum_{\ell=0}^{n}\Delta t^\ell \,\|u^\ell\|_\infty\right). \end{aligned} \tag{3.9}$$

Now, if $\Delta t^\ell \leqslant \frac{h^{1+\sigma}}{\|u_h^\ell\|_\infty^{1+\nu}} \leqslant \frac{h^{1+\sigma}}{\|u_h^\ell\|_\infty}$ then the inequality (3.9) implies that for all $0 \leqslant n \leqslant N$

$$\underline{u}^{n+1} > \underline{u}^0 - N\rho h^\sigma.$$

Hence, if $h \leqslant h_N := \left(\frac{\underline{u}^0-\mu}{N\rho}\right)^{1/\sigma}$, then $\underline{u}^{n+1} > \mu$; thus, $U_h^{n+1} > \mu$. Moreover, if $\sum_{n\geqslant 0}\frac{1}{\|u_h^n\|_\infty^\nu} < \infty$ then $S = \sum_{n\geqslant 0}\Delta t^n \|u_h^n\|_\infty < \infty$ and (3.9) implies $\underline{u}^{n+1} > \underline{u}^0 - S\rho h^\sigma$. Taking $h_* = \left(\frac{\underline{u}^0-\mu}{S\rho}\right)^{1/\sigma}$ yields the result. The proof for V_h^n is similar. □

Theorem 3.3 *Let Δt^n be given by (3.4), and let $\Lambda_\infty = \|u_h^0\|_\infty + \|v_h^0\|_\infty$. Then, for any $N \in \mathbb{N}$, there exists a constant $h_{N,\Lambda_\infty} > 0$ depending only on N and Λ_∞ such that if $h \in (0, h_{N,\Lambda_\infty}]$, then*

$$\sup_{1\leqslant n\leqslant N}(\|u_h^n\|_\infty + \|v_h^n\|_\infty) \leqslant 2\Lambda_\infty. \tag{3.10}$$

Proof Firstly, we rewrite the scheme (2.11) as

$$\begin{cases} U_h^{n+1} = & M_n U_h^n + \Delta t^n V_h^n \\ V_h^{n+1} = & N_n V_h^n + \Delta t^n f(U_h^{n+1}) \end{cases} \tag{3.11}$$

where

$$M_n = \begin{pmatrix} \mathcal{M}_A & 0 & \dots & 0 & \mathcal{M}_B \\ \mathcal{M}_B & \mathcal{M}_A & 0 & \dots & 0 \\ 0 & \mathcal{M}_B & \mathcal{M}_A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathcal{M}_B & \mathcal{M}_A \end{pmatrix} \quad \text{and} \quad N_n = \begin{pmatrix} \mathcal{N}_D & \mathcal{N}_C & 0 & \dots & 0 \\ 0 & \mathcal{N}_D & \mathcal{N}_C & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathcal{N}_D & \mathcal{N}_C \\ \mathcal{N}_C & 0 & \dots & 0 & \mathcal{N}_D \end{pmatrix}$$

with

$$\mathcal{M}_A = I_{k+1} - \Delta t^n M^{-1}(R+A), \quad \mathcal{M}_B = \Delta t^n M^{-1}B,$$

$$\mathcal{N}_D = I_{k+1} - \Delta t^n M^{-1}(D-R), \quad \mathcal{N}_C = \Delta t^n M^{-1}C,$$

and

$$f(v) = (|v_1|^p, \dots, |v_I|^p)^T \quad \text{for} \quad v = (v_1, \dots, v_I)^T.$$

Now, we prove (3.10) by induction on n. Let $N \in \mathbb{N}$ and assume that

$$\|U_h^n\|_\infty + \|V_h^n\|_\infty \leqslant 2\Lambda_\infty \quad \forall\, 0 \leqslant n \leqslant N-1.$$

Using (3.11), we may rewrite U^{n+1} and V^{n+1} as

$$U_h^{n+1} = M_n \dots M_0\ U_h^0 + \sum_{j=0}^{n} \Delta t^{n-j} M_n \dots M_{n-j+1} V_h^{n-j}, \tag{3.12}$$

$$V_h^{n+1} = N_n \dots N_0\ V_h^0 + \sum_{j=0}^{n} \Delta t^{n-j} N_n \dots N_{n-j+1}\ f(U_h^{n-j+1}). \tag{3.13}$$

At this stage, we need the following result.

Lemma 3.4 $\|M_n\|_\infty = \|N_n\|_\infty \leqslant 1 + 2\rho \dfrac{\Delta t^n}{h}$.

Proof See Appendix B. □

It follows by the induction hypothesis

$$\begin{aligned}
\|U_h^{n+1}\|_\infty &\leqslant \prod_{\ell=0}^{n} \left(1 + 2\rho \frac{\Delta t^\ell}{h}\right) \|U_h^0\|_\infty + h^{1+\sigma} \sum_{j=0}^{n} \prod_{\ell=0}^{j-1} \left(1 + 2\rho \frac{\Delta t^{n-\ell}}{h}\right) \|V_h^{n-j}\|_\infty \\
&\leqslant \prod_{\ell=0}^{n} (1 + 2\rho h^\sigma) \left(\|U_h^0\|_\infty + 2\Lambda_\infty (n+1) h^{1+\sigma}\right) \\
&= (1 + 2\rho h^\sigma)^{n+1} \left(\|U_h^0\|_\infty + 2\Lambda_\infty (n+1) h^{1+\sigma}\right)
\end{aligned}$$

where ρ is given by (3.7). It follows that $\forall\ 0 \leqslant n \leqslant N-1$

$$\|U_h^{n+1}\|_\infty \leqslant (1 + 2\rho h^\sigma)^N \left(\|U_h^0\|_\infty + 2\Lambda_\infty N h^{1+\sigma}\right). \tag{3.14}$$

Similarly, we obtain from (3.13)

$$\begin{aligned}
\|V_h^{n+1}\|_\infty &\leqslant \prod_{\ell=0}^{n} \left(1 + 2\rho \frac{\Delta t^\ell}{h}\right) \|V_h^0\|_\infty + h^{1+\sigma} \sum_{j=1}^{n} \prod_{\ell=0}^{j-1} \left(1 + 2\rho \frac{\Delta t^{n-\ell}}{h}\right) \|U_h^{n-j+1}\|_\infty^p + h^{1+\sigma} \|U_h^{n+1}\|_\infty^p \\
&\leqslant (1 + 2\rho h^\sigma)^{n+1} \left(\|V_h^0\|_\infty + (2\Lambda_\infty)^p n h^{1+\sigma}\right) + h^{1+\sigma} (1 + 2\rho h^\sigma)^{p(n+1)} \left(\|U_h^0\|_\infty + 2\Lambda_\infty (n+1) h^{1+\sigma}\right)^p.
\end{aligned}$$

Using the identity $(x+y)^r \leqslant 2^{r-1}(x^r + y^r)$ for any nonnegative reals x and y and any $r \geqslant 1$, we obtain $\forall\ 0 \leqslant n \leqslant N-1$

$$\begin{aligned}
\|V_h^{n+1}\|_\infty &\leqslant (1 + 2\rho h^\sigma)^N \left(\|V_h^0\|_\infty + (2\Lambda_\infty)^p N h^{1+\sigma}\right) \\
&\quad + 2^{p-1} h^{1+\sigma} (1 + 2\rho h^\sigma)^{pN} \left(\|U_h^0\|_\infty^p + (2\Lambda_\infty N h^{1+\sigma})^p\right).
\end{aligned} \tag{3.15}$$

It follows by (3.14) and (3.15)

$$\begin{aligned}
\|U_h^{n+1}\|_\infty + \|V_h^{n+1}\|_\infty &\leqslant (1 + 2\rho h^\sigma)^N \Lambda_\infty + N h^{1+\sigma} (1 + 2\rho h^\sigma)^N (2\Lambda_\infty + (2\Lambda_\infty)^p) \\
&\quad + 2^{p-1} h^{1+\sigma} (1 + 2\rho h^\sigma)^{pN} \Lambda_\infty^p \left(1 + (2N h^{1+\sigma})^p\right).
\end{aligned}$$

Set

$$h_{N,\Lambda_\infty} = \min\left\{ \left(\frac{(\frac{3}{2})^{\frac{1}{N}} - 1}{2\rho} \right)^{\frac{1}{\sigma}},\ \frac{\Lambda_\infty}{[12N\Lambda_\infty(1+(2\Lambda_\infty)^{p-1})]^{\frac{1}{1+\sigma}}},\ \frac{\Lambda_\infty}{\left[4\,(3\Lambda_\infty)^p\left(1+\frac{\Lambda_\infty^{p\sigma}}{6^p(1+(2\Lambda_\infty)^{p-1})^p}\right)\right]^{\frac{1}{1+\sigma}}} \right\},$$

then one can check that $\forall\ h \in (0, h_{N,\Lambda_\infty}]$, we have

$$\|U_h^{n+1}\|_\infty + \|V_h^{n+1}\|_\infty \leqslant \frac{3\Lambda_\infty}{2} + \frac{\Lambda_\infty}{4} + \frac{\Lambda_\infty}{4} = 2\Lambda_\infty.$$

□

4. Numerical blow-up

In this section, we prove that the numerical blow-up time converges toward the exact blow-up time if the discrete solution u_h weakly converges toward the exact solution u. The following functional will be useful.

$$K(u(t)) := \frac{1}{b-a}\int_a^b u(x,t)dx. \tag{4.1}$$

Proposition 4.1 *[42] Assume that*

$$\alpha = K(u_0) \geqslant 0,\ \ \beta = K(u_1) > 0.$$

Then, the solution u of (1.1) blows up in finite time $T_\infty \in (0,\infty)$.

Definition 4.2 *We define the numerical blow-up time by*

$$T(h) = \lim_{n\longrightarrow\infty} t^n = \sum_{n=0}^{\infty}\Delta t^n.$$

We say that the numerical solution blows up if

$$\lim_{n\to\infty}\|u_h^n\|_{L^\infty(a,b)} = \lim_{t^n\to T(h)}\|u_h^n\|_{L^\infty(a,b)} = \infty.$$

Moreover, we say that the numerical solution blows up in finite time if $T(h) < \infty$.

Proposition 4.3 *Let (u_h^n, v_h^n) be the solution of (2.10). Define*

$$K_h(u_h^n) = \frac{1}{b-a}\sum_{i=1}^{I}\int_{K_i} u_h^{i,n}(x)dx, \tag{4.2}$$

and suppose $\beta_h := K_h(u_h^1) > 0$ and $\alpha_h := K_h(u_h^0) \geqslant 0$. Then $(K_h(u_h^n))_n$ is a strictly increasing unbounded sequence and for all $n \geqslant 0$

$$\left(\frac{K_h(u_h^{n+1}) - K_h(u_h^n)}{\Delta t^n}\right)^2 \geqslant \frac{\lambda}{p+1}\,(K_h(u_h^n))^{p+1} + \gamma_h \geqslant 0$$

where

$$\gamma_h = \left(\frac{\beta_h - \alpha_h}{\Delta t^0}\right)^2 - \frac{\lambda}{p+1}\alpha_h^{p+1}$$

and $\lambda > 0$ *is a constant independent of* h*.*

Proof Recall that the scheme (2.10a)-(2.10b) is equivalent to equations (2.4a)-(2.4b). Then, take $\varphi_j^i \equiv 1$ in (2.4a) yields

$$\int_{K_i} \frac{u_h^{i,n+1} - u_h^{i,n}}{\Delta t^n} dx + u_h^{i,n}(x_{i+\frac{1}{2}}) - u_h^{i-1,n}(x_{i-\frac{1}{2}}) = \int_{K_i} v_h^{i,n} dx.$$

Sum up over $i = 1, \dots I$ and use the periodic boundary condition,

$$\frac{K_h(u_h^{n+1}) - K_h(u_h^n)}{\Delta t^n} = K_h(v_h^n) \quad \forall\ n \geqslant 0. \tag{4.3}$$

In particular

$$\frac{K_h(u_h^1) - K_h(u_h^0)}{\Delta t^0} = K_h(v_h^0) > 0. \tag{4.4}$$

Similarly, we have by (2.4b)

$$\int_{K_i} \frac{v_h^{i,n+1} - v_h^{i,n}}{\Delta t^n} dx - v_h^{i+1,n}(x_{i+\frac{1}{2}}) + v_h^{i,n}(x_{i-\frac{1}{2}}) = \int_{K_i} \mathcal{I}_h^k \left(|u_h^{i,n+1}|^p\right) dx;$$

hence,

$$\frac{K_h(v_h^{n+1}) - K_h(v_h^n)}{\Delta t^n} = K_h(\mathcal{I}_h^k(|u_h^{n+1}|^p)) \quad \forall\ n \geqslant 0.$$

At this stage, we need the following technical lemma.

Lemma 4.4 *There exists* $\lambda > 0$ *independent of* h *such that*

$$K_h\left(\mathcal{I}_h^k(|u_h^{n+1}|^p)\right) \geqslant \lambda \left(K_h(u_h^{n+1})\right)^p.$$

Proof See Appendix C. □

Thus, we have

$$\frac{K_h(v_h^{n+1}) - K_h(v_h^n)}{\Delta t^n} \geqslant \lambda \left(K_h(u_h^{n+1})\right)^p. \tag{4.5}$$

Using (4.3) and (4.5), one can easily show by induction on n that $K_h(u_h^n)$ and $K_h(v_h^n)$ are nonnegative for all n. Now, combining (4.3), (4.4), and (4.5) yields for all $n \geqslant 0$

$$\frac{K_h(u_h^{n+2}) - K_h(u_h^{n+1})}{\Delta t^{n+1}} \geqslant \frac{K_h(u_h^{n+1}) - K_h(u_h^n)}{\Delta t^n} + \lambda \Delta t^n (K_h(u_h^{n+1}))^p \tag{4.6}$$

$$\geqslant \frac{K_h(u_h^1) - K_h(u_h^0)}{\Delta t^0} + \lambda \sum_{k=0}^{n} \Delta t^k (K_h(u_h^{k+1}))^p \tag{4.7}$$

$$> 0.$$

Consequently, $(K_h(u_h^n))_n$ is a strictly increasing sequence. Now, we again make use of (4.6) to obtain

$$\left(\frac{K_h(u_h^{n+2}) - K_h(u_h^{n+1})}{\Delta t^{n+1}}\right)^2 \geqslant \frac{K_h(u_h^{n+1}) - K_h(u_h^n)}{\Delta t^n}\left(\frac{K_h(u_h^{n+1}) - K_h(u_h^n)}{\Delta t^n} + \lambda \Delta t^n (K_h(u_h^{n+1}))^p\right)$$
$$= \left(\frac{K_h(u_h^{n+1}) - K_h(u_h^n)}{\Delta t^n}\right)^2 + \lambda\big(K_h(u_h^{n+1}) - K_h(u_h^n)\big)(K_h(u_h^{n+1}))^p.$$

A straightforward induction implies

$$\left(\frac{K_h(u_h^{n+2}) - K_h(u_h^{n+1})}{\Delta t^{n+1}}\right)^2 \geqslant \lambda \sum_{k=0}^{n}\Big(K_h(u_h^{k+1}) - K_h(u_h^k)\Big)\left(K_h(u_h^{k+1})\right)^p + \left(\frac{K_h(u_h^1) - K_h(u_h^0)}{\Delta t^0}\right)^2$$
$$\geqslant \lambda \int_{\alpha_h}^{K_h(u_h^{n+1})} z^p dz + \left(\frac{\beta_h - \alpha_h}{\Delta t^0}\right)^2$$
$$= \frac{\lambda}{p+1}\Big((K_h(u_h^{n+1}))^{p+1} - \alpha_h^{p+1}\Big) + \left(\frac{\beta_h - \alpha_h}{\Delta t^0}\right)^2.$$

Moreover, since $K_h(u_h^n)$ is increasing in n, then $\frac{\lambda}{p+1}\Big((K_h(u_h^{n+1}))^{p+1} - \alpha_h^{p+1}\Big) + \left(\frac{\beta_h - \alpha_h}{\Delta t^0}\right)^2$ is nonnegative. Finally, assume $(K_h(u_h^n))_n$ is bounded, then it is convergent. Hence, we can extract a subsequence $(u_h^{n_\ell})_{n_\ell}$ of $(u_h^n)_n$ which converges a.e.; thus, it is bounded. We deduce from (3.4) that $\Delta t^{n_\ell} \not\to 0$ as n_ℓ goes to infinity, and using (4.4) and (4.7), we obtain

$$0 < \Delta t^{n_\ell+1} K_h(v_h^0) \leqslant K_h(u_h^{n_\ell+2}) - K_h(u_h^{n_\ell+1}).$$

Take the limit when n_ℓ tends to infinity gives a contradiction with (4.4). Thus, $(K_h(u_h^n))_n$ is unbounded and the proof is completed. □

Lemma 4.5 *Let $0 \leqslant k \leqslant 7$ and let (u_h^n, v_h^n) be the solution of (2.10). Then $(u_h^n)_n$ blows up.*

Proof If $0 \leqslant k \leqslant 7$, then $\alpha_j := \int_{-1}^1 \varphi_j \, dx > 0$ for all $1 \leqslant j \leqslant k+1$ (see table 3). Consequently, one may deduce from Proposition 3.2 that if the initial data (u_0, v_0) are positives, then $K_h(u_h^n) = \|u_h^n\|_1$ for h small enough, where

$$\|u_h\|_1 = \frac{1}{b-a}\sum_{i=1}^{I}\frac{h}{2}\sum_{j=1}^{k+1}\alpha_j |u_j^i|.$$

It follows that $\|u_h^n\|_1 \xrightarrow[n\to\infty]{} \infty$; thus, $\|u_h^n\|_{L^\infty(a,b)} \xrightarrow[n\to\infty]{} \infty$. □

Define

$$G(z) = \sqrt{\frac{\lambda}{p+1} z^{p+1} + \gamma_h},$$

then G is a strictly increasing function in $[\alpha_h, \infty)$. In view of Proposition 4.3, we can proceed the same as in [12] to prove the following.

Lemma 4.6 *There exists a constant $C > 0$ independent of h such that*

$$T(h) \leqslant 2\Big(\int_{\alpha_h}^{\infty} \frac{dz}{G(z)} + Ch\Big).$$

In particular, $(u_h^n)_n$ blows up in a finite time $T(h)$.

Proof See [12, Lemma 5.3] □

Theorem 4.7 *Let (u,v) and (u_h,v_h) be the solutions of (2.1) and (2.10), respectively. Assume that $u_0 > 0$ and $u_1 > 0$ are large enough and $v_0 > 0$. If u_h weakly converges towards u, then u_h^n blows up in finite time $T(h)$ and*

$$\lim_{h\to 0} T(h) = T_\infty. \tag{4.8}$$

Proof We follow the strategy of [42]. According to Lemma 4.6, u_h^n blows up in finite time $T(h)$. To establish (4.8), we will prove the following inequalities:

$$T_\infty \leqslant \liminf_{h\to 0} T(h) = T_*, \tag{4.9}$$

and

$$T_\infty \geqslant \limsup_{h\to 0} T(h) = T^*. \tag{4.10}$$

Suppose that $T_* < T_\infty$ and let $\varepsilon = \frac{T_\infty - T_*}{2} > 0$. Then there exists $h_\varepsilon > 0$ sufficiently small such that

$$T(h_\varepsilon) \leqslant T_* + \varepsilon < T_\infty.$$

On the one hand, we have $\sup_{0\leqslant t\leqslant T_*+\varepsilon} \|u(\cdot,t)\|_{L^\infty(a,b)} < \infty$; hence,

$$K_1 := \sup_{0\leqslant t\leqslant T_*+\varepsilon} K(u(t)) < \infty.$$

On the other hand, if $u_h^n \underset{h\to 0}{\longrightarrow} u(t^n)$ then $K_h(u_h^n) \underset{h\to 0}{\longrightarrow} K(u(t^n))$. Hence, if h_ε is sufficiently small, then $K_{h_\varepsilon}(u_{h_\varepsilon}^n) \leqslant K(u(t^n)) + \varepsilon$ for all n such that $t^n < T_\infty$. It follows

$$\lim_{n\to\infty} K_{h_\varepsilon}(u_{h_\varepsilon}^n) = \lim_{t^n\to T(h_\varepsilon)} K_{h_\varepsilon}(u_{h_\varepsilon}^n) \leqslant \lim_{t^n\to T(h_\varepsilon)} K(u(t^n)) + \varepsilon \leqslant K_1 + \varepsilon,$$

which contradicts Proposition 4.3; hence, (4.9) holds. Next, suppose that $T^* > T_\infty$ and let $N > 0$ be the number of iterations to reach the time T_∞, i.e. $T_\infty = t^N = \sum_{n=0}^{N-1} \Delta t^n$. Let $h_1 = \min(h_N, h_{N,\Lambda})$ with h_N given in Proposition 3.2 and $h_{N,\Lambda}$ given in Theorem 3.3, with $\Lambda_\infty = \|u_h^0\|_\infty + \|v_h^0\|_\infty$. Then $\forall\, h \in (0,h_1]$ and $0 \leqslant n \leqslant N$, we have

$$\begin{aligned}
\Big|\sum_{i=1}^{I} \int_{K_i} u_h^{i,n}(x)\,dx\Big| &\leqslant \sum_{i=1}^{I}\sum_{j=1}^{k+1} |u_j^{i,n}| \Big|\int_{K_i} \varphi_j^i(x)\,dx\Big| \\
&\leqslant \sum_{i=1}^{I} \frac{h}{2}\|u_h^n\|_\infty \sum_{j=1}^{k+1} \Big|\int_{-1}^{1} \varphi_j(x)\,dx\Big| \\
&\leqslant K_2\Lambda_\infty,
\end{aligned}$$

with $K_2 = \frac{b-a}{2}\sum_{j=1}^{k+1}\left|\int_{-1}^{1}\varphi_j(x)\,dx\right|$. Let $\varepsilon = \frac{T^*-T_\infty}{4}$. Using Lemma 4.6, there exist $h_\varepsilon > 0$ and $R \geqslant \frac{1}{b-a}\left(\|u\|_{L^\infty([a,b]\times[0,t^{N-1}])} + K_2\Lambda_\infty\right)$ such that

$$\int_R^\infty \frac{dz}{G(z)} + Ch_\varepsilon < \frac{\varepsilon}{2}. \tag{4.11}$$

It is shown in [24] that if the initial conditions are sufficiently large, then the solution u of (2.1) blows up in L^p norms, for any $1 \leqslant p \leqslant \infty$ (see [24, Theorem 2.1]). We deduce that if the initial conditions are large enough, then there exists $t' = t'_R < T_\infty$ such that

$$K(u(t)) \geqslant 2R \quad \forall\, t \in [t', T_\infty). \tag{4.12}$$

Set

$$T = t' + \frac{T_\infty - t'}{2} = \frac{t' + T_\infty}{2} < T_\infty, \quad h_* = \min\left\{h_1, \left(\frac{T_\infty - t'}{2}\right)^{\frac{1}{1+\sigma}}\right\}$$

and let $h \in (0, h_*]$. Then, we have for all $n \geqslant 0$ such that $t^n < T_\infty$

$$\begin{aligned}|K(u(t^n)) - K_h(u_h^n)| &= \frac{1}{b-a}\left|\sum_{i=1}^{I}\int_{K_i}\left(u(x,t^n) - u_h^{i,n}(x)\right)dx\right| \\ &\leqslant \frac{1}{b-a}\left(\|u(t^n)\|_{L^\infty([a,b])} + \left|\sum_{i=1}^{I}\int_{K_i} u_h^{i,n}(x)dx\right|\right).\end{aligned}$$

In particular, we obtain for all $0 \leqslant n \leqslant N-1$

$$|K(u(t^n)) - K_h(u_h^n)| \leqslant \frac{1}{b-a}\left(\|u\|_{L^\infty([a,b]\times[0,t^{N-1}])} + K_2\Lambda_\infty\right) \leqslant R.$$

It follows

$$K_h(u_h^n) \geqslant K(u(t^n)) - R \quad \forall\, 0 \leqslant n \leqslant N-1.$$

Recall that $\Delta t^n \leqslant h^{1+\sigma} \leqslant T - t' < T_\infty - t'$. Since $T^* > T_\infty$, then there exists $n_1 \leqslant N-1$ such that $t' \leqslant t^{n_1} < T_\infty$. We deduce from (4.12)

$$K_h(u_h^{n_1}) \geqslant K(u(t^{n_1})) - R \geqslant R. \tag{4.13}$$

Now, using $\limsup_{h\to 0} T(h) = T^* > T_\infty$, one may choose $h_\varepsilon \leqslant h_*$ sufficiently small such that

$$T(h_\varepsilon) \geqslant T_\infty + \varepsilon.$$

However, in view of Lemma 4.6 and equations (4.13) and (4.11), we have

$$\begin{aligned}T(h_\varepsilon) = t^{n_1} + \sum_{n=n_1}^{\infty}\Delta t^n &< T_\infty + 2\left(\int_{K_{h_\varepsilon}(u_{h_\varepsilon}^{n_1})}^{\infty}\frac{dz}{G(z)} + Ch_\varepsilon\right) \\ &\leqslant T_\infty + 2\left(\int_R^\infty \frac{dz}{G(z)} + Ch_\varepsilon\right) \\ &< T_\infty + \varepsilon,\end{aligned}$$

which is a contradiction. This achieves the proof. □

5. Numerical examples

In this section, we present some numerical tests in order to illustrate our method. For all the examples, we consider the DG scheme (2.11) with $\mathbb{P}_1$ approximation.

Example 5.1 *In this example, we consider constant initial conditions so that the solution of (1.1) is space-independent. The exact solution we consider is*

$$u(t) = \mu(T-t)^{\frac{2}{1-p}}$$

with $\mu = \left(2\frac{p+1}{(p-1)^2}\right)^{\frac{1}{p-1}}$. *We perform two test cases with* $p=2$ *and* $p=3$. *The blow-up time for both cases is set to* $T = 0.1\,s$. *Figure 1 shows a comparison between the exact solution and the numerical solution functions of time. One can notice a very good superposition of the solutions (with relative errors less than* 1% *in both* L^2 *and* L^∞ *norms), which can justify the validity of the explicit Euler scheme as an appropriate choice for the time discretization of the DG method.*

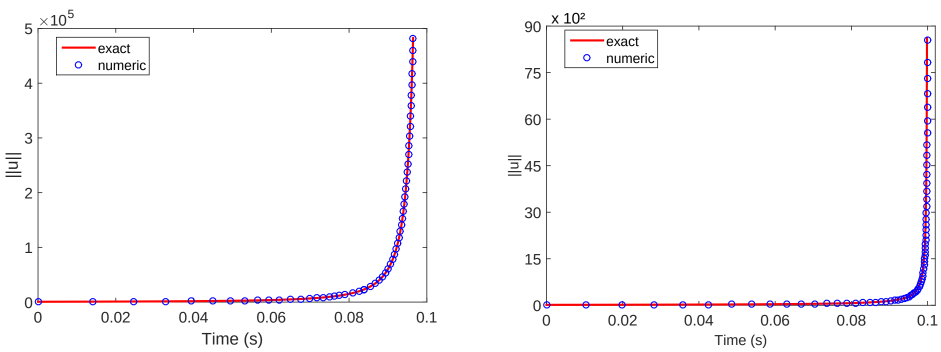

Figure 1. Comparison between the numerical solution (blue circles) and the exact solution (red line) for p = 2 (left) and p = 3 (right).

Example 5.2 *We consider an exact solution of (1.1) given by*

$$u(x,t) = \mu(T-t+d\,x)^{\frac{2}{1-p}} \tag{5.1}$$

with $\mu = \left(2(1-d^2)\frac{p+1}{(p-1)^2}\right)^{\frac{1}{p-1}}$ *and* $d \in (0,1)$ *is an arbitrary parameter. Figures 2 and 3 show a comparison between the exact solution and the numerical solution at various times, for* $p=2$ *and* $p=3$, *respectively. The parameters used are* $T = 0.5\,s$ *and* $d = 0.01$.

One can notice that the numerical solutions fit very well with the exact solutions at all the recorded times. The relative errors in L^∞ *norms is less than* 1% *if a refined mesh is used. We also investigate the blow-up curve in the following sense. Let* $R \geqslant \min_{x\in[0,1]} u(x,0) = \mu(T+d)^{\frac{2}{1-p}}$, *and let* ξ_R *the function defined by* $u(x,\xi_R(x)) = R$. *It is obvious from (5.1) that* ξ_R *is a straight line given by* $\xi_R(x) = T - \left(\frac{\mu}{R}\right)^{\frac{p-1}{2}} + d\,x$. *When* R *goes to infinity,* $\xi_R(x)$ *tends to the blow-up time* $T_\infty(x) = T + d\,x$, *for any* $x \in [0,1]$. *Thus, one can approximate numerically the blow-up curve* T_∞ *by computing* ξ_R *for large values of* R. *In practice, we define* ξ_R *as*

$$\xi_R(x) = \inf\{t \geqslant 0,\ |u(x,t)| \geqslant R\}.$$

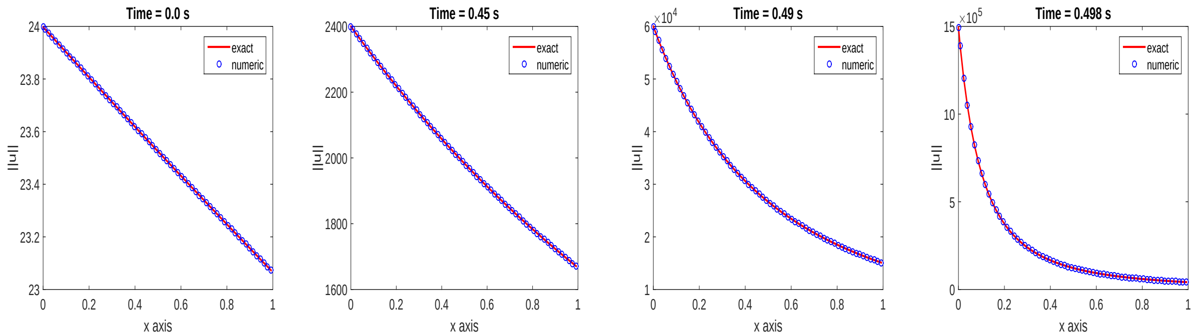

Figure 2. Comparison between the numerical solution (blue circles) and the exact solution (red line) at various times. Case $p = 2$.

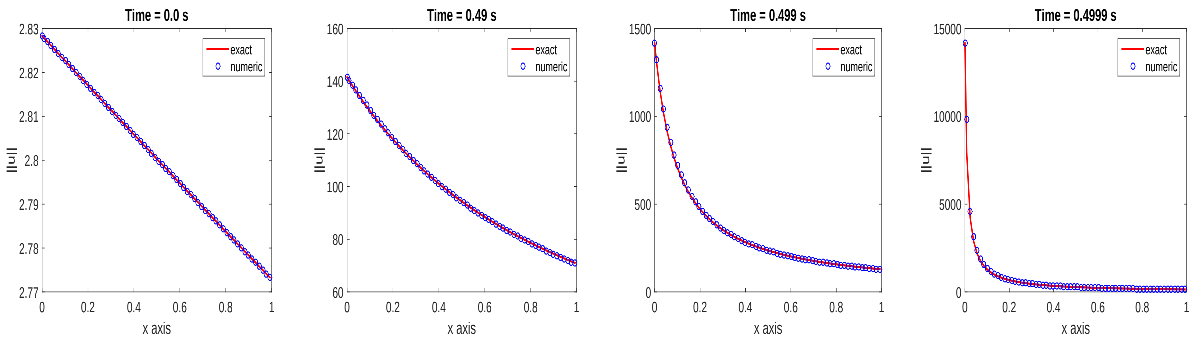

Figure 3. Comparison between the numerical solution (blue circles) and the exact solution (red line) at various times. Case $p = 3$.

Figure 4 shows ξ_R function of x for various values of R. We notice that ξ_R is a straight line with slope equal to d for all values of R, which is in accordance with the theory. Furthermore, as the parameter R gets bigger, one can notice that ξ_R converges (pointwise and uniformly) to the theoretical blow-up curve T_∞.

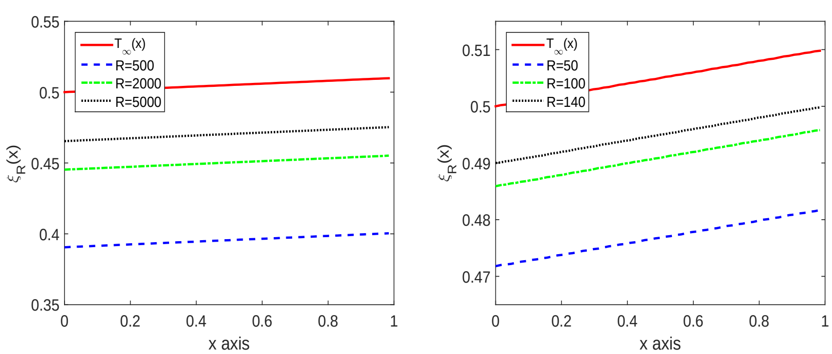

Figure 4. ξ_R for various values of R. Case $p = 2$ (left) and $p = 3$ (right). As R increases, ξ_R converges (pointwise and uniformly) toward the blow-up curve (red line).

Example 5.3 *In this example, we consider the system (1.1) with initial conditions*

$$u_0(x) = 5(\sin(4\pi x) + 2),$$
$$u_1(x) = 5(\sin(4\pi x) - 4\pi\cos(4\pi x) + 2).$$

With such conditions, we have $u_0 = v_0 > 0$, $\alpha = K(u_0) = 10 > 0$, and $\beta = K(u_1) = 10 > 0$, so that all the hypotheses of Propositions 3.2 and 4.3 are satisfied. Accordingly, we expect the solution to blow up in a finite time. Figures 5 and 6 show the evolution of the numerical solutions in time-space axes for $p = 2$ and $p = 3$, respectively. One can effectively notice that the numerical solutions do blow up as time evolves.

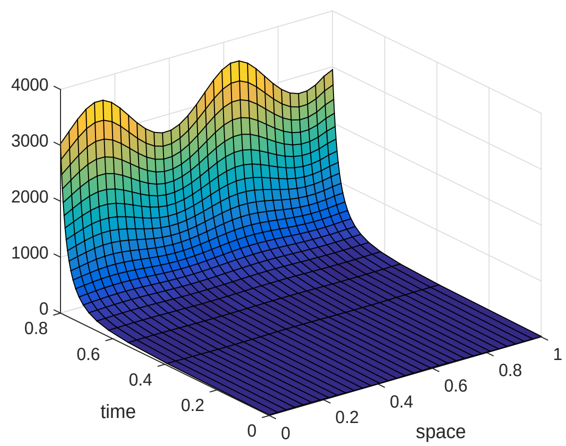

Figure 5. Numerical solution of example 5.3 with $p = 2$.

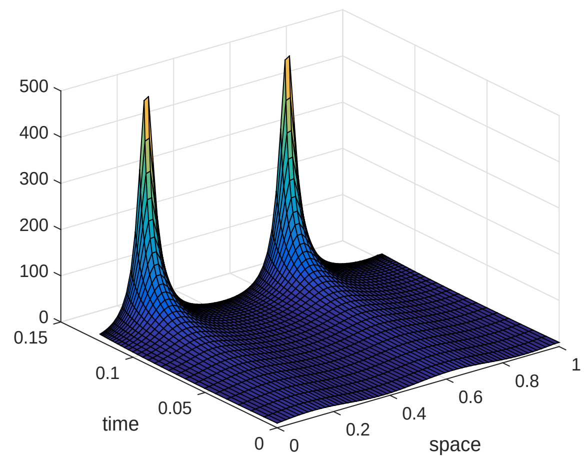

Figure 6. Numerical solution of example 5.3 with $p = 3$.

Example 5.4 *In this example, we compare our DG method to a finite difference (FD) method developed in [42]. Let us mention that the authors also proved that their FD scheme is convergent, as well as the numerical blow-up time, toward the exact solution. We use a very refined grid mesh for the FD algorithm in order to obtain results as accurate as possible*. The initial conditions used are $u_0(x) = 5(\sin(4\pi x) + 2)$ and $u_1(x) = 20\pi + 5$. Figure 7 shows a comparison between the numerical solutions in various time for $p = 2$ and $p = 3$. One can notice a very good superposition between the solutions in all recorded times.*

In Table 1, we report the relative L^2 and L^∞ errors between the FD and the DG solutions at the different times. Moreover, we checked the convergence of the numerical blow-up time when the space path h goes to zero.

Table 2 and Figure 8 show the blow-up times of the DG method versus the FD method function of h for $p = 3$. Since the blow-up time cannot be reached in finite steps (see Definition 4.2), we fix $\|u_h^n\|_\infty \geqslant 10^9$ as a threshold criterion in order to stop the iterations. One can notice that both the DG and the FD algorithms seem to converge toward the same limit, which is $T_\infty \simeq 1.14\,s$ in this case. This confirms the efficiency of our proposed method.

*The FD grid is 16 times finer than the DG grid.

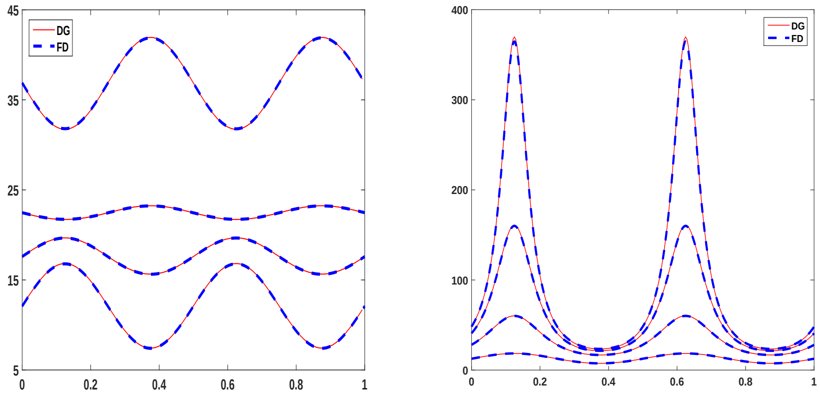

Figure 7. Comparison between the DG solution (red line) and the FD solution [42] (blue dash) of example 5.4 at various times. Case $p = 2$ (left) and $p = 3$ (right).

Table 1. Relative errors of the DG solutions u_h^{DG} versus the FD solutions u_h^{FD} for various times.

$p = 2$		
Time (s)	$\frac{\|u_h^{FD}-u_h^{DG}\|_2}{\|u_h^{FD}\|_2}$	$\frac{\|u_h^{FD}-u_h^{DG}\|_\infty}{\|u_h^{FD}\|_\infty}$
0.03	2.16×10^{-3}	1.95×10^{-3}
0.10	9.15×10^{-4}	9.32×10^{-4}
0.15	5.97×10^{-4}	9.34×10^{-4}
0.25	1.11×10^{-3}	1.66×10^{-3}

$p = 3$		
Time (s)	$\frac{\|u_h^{FD}-u_h^{DG}\|_2}{\|u_h^{FD}\|_2}$	$\frac{\|u_h^{FD}-u_h^{DG}\|_\infty}{\|u_h^{FD}\|_\infty}$
0.03	2.58×10^{-4}	3.46×10^{-4}
0.09	1.25×10^{-3}	1.95×10^{-3}
0.105	4.05×10^{-3}	5.96×10^{-3}
0.110	9.98×10^{-3}	1.39×10^{-2}

Table 2. Blow-up time function of h. Case p=3.

	$T(h)$	
h	DG	FD
$1/2^5$	1.1671	1.1675
$1/2^6$	1.1527	1.1538
$1/2^7$	1.1455	1.1463
$1/2^8$	1.1419	1.1423
$1/2^9$	1.1401	1.1403

6. Conclusion

In this paper, we developed a numerical scheme based on discontinuous Galerkin (DG) formulation for the approximation of the nonlinear wave equation in one dimensional space. We showed that the DG scheme is consistent and stable (in the sense that the numerical solution do not blows up in a finite number of iterations, i.e. before the exact blow-up time). For the time update, we used an explicit Euler scheme. Since blow-up phenomena can occur, one may not expect a constant time increment[†]. Instead, we used a refined time meshing, with time step inversely proportional to the solution's amplitude. Since we are dealing with transport equations,

[†]Otherwise, the numerical solution could be computed beyond the blow-up time. Actually, the author in [13] showed that a constant time step remains also applicable if an appropriate stopping criterion is specified.

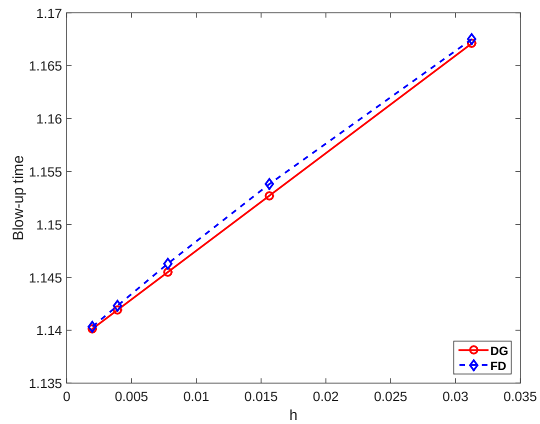

Figure 8. Comparison between the numerical blow-up time of the DG method (red line) and the FD method (dashed blue line) of example 5.4. Case $p = 3$.

the CFL condition is more constrained in case of DG methods. Indeed, the classical theory of the DG methods shows that Δt should be of order $(\Delta x)^{3/2}$ (rather than the standard Δx) to ensure the stability of the method [9, 18][‡]. This condition is obviously fulfilled in case the solution blows up. We also proved that the numerical solution blows up in a finite time $T(h)$, and that $T(h)$ converges toward the theoretical blow-up time as h gets smaller. We illustrate the performance of our method throughout several numerical tests and benchmarks.

Appendix

A. Matrices properties

Since $(\varphi^i_j)_{1\leqslant j\leqslant k+1}$ is a Lagrange polynomial basis of $\mathbb{P}_k[K_i]$, then for any $x \in K_i$, we have

$$\sum_{j=1}^{k+1} \varphi^i_j(x) = 1 \quad \text{and} \quad \sum_{j=1}^{k+1} (\varphi^i_j)'(x) = 0.$$

On the other hand, using the transform $\varphi^i_j = \varphi_j \circ (\gamma^i)^{-1}$ where φ_j is the j^{th} Lagrange polynomial over $[-1, 1]$ and $\gamma^i : [-1, 1] \to K_i$, $x \mapsto \frac{1}{2}(h_i\, x + x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}})$, one can easily show $M^i = h_i M$, $R^i = R$, $A^i = A$, $B^i = B$, $C^i = C$ and $D^i = D$ for all i with

$$M_{j\ell} = \frac{1}{2}\int_{-1}^{1} \varphi_j\ \varphi_\ell\ dx, \quad R_{j\ell} = \int_{-1}^{1} \varphi_j\ \varphi'_\ell\ dx,$$

$$A_{j\ell} = \varphi_j(-1)\ \varphi_\ell(-1), \quad B_{j\ell} = \varphi_j(-1)\ \varphi_\ell(1),$$

and

$$C_{j\ell} = \varphi_j(1)\ \varphi_\ell(-1), \quad D_{j\ell} = \varphi_j(1)\ \varphi_\ell(1).$$

[‡]While the order $3/2$ has been theoretically established for the linear problems, it has been observed numerically that the order one, i.e. $\Delta t = O(\Delta x)$, is sufficient for the stability of nonlinear problems [9].

It follows $\forall\ 1 \leqslant j \leqslant k+1$,

$$\sum_{\ell=1}^{k+1} R_{j\ell} = \sum_{\ell=1}^{k+1} \int_{-1}^{1} \varphi_j(x)\varphi'_\ell(x)dx = \int_{-1}^{1} \varphi_j(x) \left(\sum_{\ell=1}^{k+1} \varphi'_\ell(x) \right) dx = 0,$$

$$\sum_{\ell=1}^{k+1} A_{j\ell} = \sum_{\ell=1}^{k+1} \varphi_j(-1)\varphi_\ell(-1) = \varphi_j(-1) \sum_{\ell=1}^{k+1} \varphi_\ell(-1) = \varphi_j(-1),$$

$$\sum_{\ell=1}^{k+1} B_{j\ell} = \sum_{\ell=1}^{k+1} \varphi_j(-1)\varphi_\ell(1) = \varphi_j(-1) \sum_{\ell=1}^{k+1} \varphi_\ell(1) = \varphi_j(-1).$$

Therefore, $\forall\ 1 \leqslant j \leqslant k+1$,

$$\sum_{\ell=1}^{k+1} (R_{j\ell} + A_{j\ell} - B_{j\ell}) = 0. \tag{A.1}$$

Now, we have

$$E = h_i(M^i)^{-1}(R^i + A^i) = M^{-1}(R + A) \quad \text{and} \quad F = -h_i(M^i)^{-1}B^i = -M^{-1}B$$

are constant matrices, and $\forall\ 1 \leqslant j \leqslant k+1$

$$\begin{aligned}
\sum_{\ell=1}^{k+1} E_{j\ell} + F_{j\ell} &= \sum_{\ell=1}^{k+1} M^{-1}(R + A - B)_{j\ell} \\
&= \sum_{\ell=1}^{k+1} \sum_{s=1}^{k+1} (M^{-1})_{js}(R + A - B)_{s\ell} \\
&= \sum_{s=1}^{k+1} (M^{-1})_{js} \sum_{\ell=1}^{k+1} (R_{s\ell} + A_{s\ell} - B_{s\ell}) \\
&= 0
\end{aligned} \tag{A.2}$$

where the last equality follows from (A.1).

B. Proof of Lemma 3.4

We rewrite M_n as $M_n = I - \widetilde{M}_n$ with

$$\widetilde{M}_n = \begin{pmatrix} \tilde{\mathcal{M}}_A^1 & 0 & \dots & 0 & \mathcal{M}_B^1 \\ \mathcal{M}_B^2 & \tilde{\mathcal{M}}_A^2 & 0 & \dots & 0 \\ 0 & \mathcal{M}_B^3 & \tilde{\mathcal{M}}_A^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathcal{M}_B^I & \tilde{\mathcal{M}}_A^I \end{pmatrix}$$

with

$$\tilde{\mathcal{M}}_A^i = \Delta t^n (M^i)^{-1}(R^i + A^i) = \frac{\Delta t^n}{h_i} E$$

and

$$\mathcal{M}_B^i = -\Delta t^n (M^i)^{-1} B^i = \frac{\Delta t^n}{h_i} F.$$

Let $\left[\widetilde{M}_n\right]^i$ be the i^{th} block-row of $\widetilde{M}_n$ and denote $x^+ = \max(x,0)$ and $x^- = \min(x,0)$ for any $x \in \mathbb{R}$. Then, using (A.2), we obtain for any $1 \leqslant j \leqslant k+1$

$$\begin{aligned}
\sum_{\ell=1}^{k+1} \left| \left[\widetilde{M}_n\right]^i_{j\ell} \right| &= \frac{\Delta t^n}{h_i} \sum_{\ell=1}^{k+1} |E_{j\ell} + F_{j\ell}| \\
&= \frac{\Delta t^n}{h_i} \left(\sum_{\ell=1}^{k+1} \left(E_{j\ell}^+ + F_{j\ell}^+ \right) - \sum_{\ell=1}^{k+1} \left(E_{j\ell}^- + F_{j\ell}^- \right) \right) \\
&= 2\frac{\Delta t^n}{h_i} \sum_{\ell=1}^{k+1} \left(E_{j\ell}^+ + F_{j\ell}^+ \right).
\end{aligned}$$

It follows

$$\begin{aligned}
\|\widetilde{M}_n\|_\infty &= \max_{1\leqslant i\leqslant I} \| \left[\widetilde{M}_n\right]^i \|_\infty \\
&= \max_{1\leqslant i\leqslant I} \left(\max_{1\leqslant j\leqslant k+1} \sum_{\ell=1}^{k+1} \left| \left[\widetilde{M}_n\right]^i_{j\ell} \right| \right) \\
&= \max_{1\leqslant i\leqslant I} \max_{1\leqslant j\leqslant k+1} 2\frac{\Delta t^n}{h_i} \sum_{\ell=1}^{k+1} \left(E_{j\ell}^+ + F_{j\ell}^+ \right).
\end{aligned}$$

In particular, if $h_i = h$ for all i, and if we denote $\rho = \sum_{\ell=1}^{k+1} \left(E_{j\ell}^+ + F_{j\ell}^+ \right)$, then we obtain

$$\|M_n\|_\infty \leqslant \|I\|_\infty + \|\widetilde{M}_n\|_\infty = 1 + 2\rho\frac{\Delta t^n}{h}$$

The same reasoning can be applied to the matrix N_n.

C. Proof of Lemma 4.4

Let $1 \leqslant i \leqslant I$, then using the classical inequality $\left|\sum_{j=1}^m a_j\right|^p \leqslant m^{p-1} \sum_{j=1}^m |a_j|^p$, we obtain

$$\begin{aligned}
\left| \int_{K_i} u_h^{i,n+1}(x)\, dx \right|^p &= \left| \int_{K_i} \sum_{j=1}^{k+1} u_j^{i,n+1} \varphi_j^i(x)\, dx \right|^p \\
&\leqslant (k+1)^{p-1} \sum_{j=1}^{k+1} \left| u_j^{i,n+1} \right|^p \left| \int_{K_i} \varphi_j^i(x)\, dx \right|^p \\
&= (k+1)^{p-1} \sum_{j=1}^{k+1} \left| u_j^{i,n+1} \right|^p \left| \frac{h}{2} \int_{-1}^{1} \varphi_j(x)\, dx \right|^p.
\end{aligned}$$

Denote

$$\lambda = \left(\frac{k+1}{2} \max_{1 \leqslant j \leqslant k+1} \left| \int_{-1}^{1} \varphi_j(x)\, dx \right| \right)^{1-p},$$

then we have

$$\begin{aligned}
\big(K_h(u_h^{n+1})\big)^p &= \frac{1}{(b-a)^p} \left(\sum_{i=1}^{I} \int_{K_i} u_h^{i,n+1}(x)\, dx \right)^p \\
&\leqslant \frac{I^{p-1}}{(b-a)^p} \sum_{i=1}^{I} \left| \int_{K_i} u_h^{i,n+1}(x)\, dx \right|^p \\
&\leqslant \frac{(Ih)^{p-1}}{\lambda (b-a)^p} \sum_{i=1}^{I} \sum_{j=1}^{k+1} \left| u_j^{i,n+1} \right|^p \left| \frac{h}{2} \int_{-1}^{1} \varphi_j(x)\, dx \right|
\end{aligned}$$

Now, if $0 \leqslant k \leqslant 7$, then the integrals $\int_{-1}^{1} \varphi_j(x)\, dx$ are positives for all $1 \leqslant j \leqslant k+1$ (see table 3). It follows

$$\begin{aligned}
\big(K_h(u_h^{n+1})\big)^p &\leqslant \frac{1}{\lambda} \sum_{j=1}^{k+1} \left| u_j^{i,n+1} \right|^p \int_{K_i} \varphi_j^i(x)\, dx \\
&= \frac{1}{\lambda} K_h\big(\mathcal{I}_h^k(|u_h^{n+1}|^p)\big).
\end{aligned}$$

Table 3. Values of $\alpha_j := \int_{-1}^{1} \varphi_j(x) dx$ where φ_j is the j^{th} Lagrange polynomial of degree k over $[-1, 1]$.

$k\alpha_j$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
0	2							
1	1	1						
2	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$					
3	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$				
4	$\frac{7}{45}$	$\frac{32}{45}$	$\frac{12}{45}$	$\frac{32}{45}$	$\frac{7}{45}$			
5	$\frac{19}{144}$	$\frac{75}{144}$	$\frac{50}{144}$	$\frac{50}{144}$	$\frac{75}{144}$	$\frac{19}{144}$		
6	$\frac{41}{420}$	$\frac{216}{420}$	$\frac{27}{420}$	$\frac{272}{420}$	$\frac{27}{420}$	$\frac{216}{420}$	$\frac{41}{420}$	
7	$\frac{751}{8640}$	$\frac{3577}{8640}$	$\frac{1323}{8640}$	$\frac{2989}{8640}$	$\frac{2989}{8640}$	$\frac{1323}{8640}$	$\frac{3577}{8640}$	$\frac{751}{8640}$

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