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Blow-up of solutions for wave equation with multiple α(x)-laplacian and variable-exponent nonlinearities

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Abstract: We consider an initial value problem related to the equation

$$u_{tt} - div\left(|\nabla u|^{m(x)-2}\nabla u\right) - div\left(|\nabla u_t|^{r(x)-2}\nabla u_t\right) - \gamma\Delta u_t = |u|^{p(x)-2}u,$$

with homogeneous Dirichlet boundary condition in a bounded domain Ω. Under suitable conditions on variable-exponent $m(.)$, $r(.)$, and $p(.)$, we prove a blow-up of solutions with negative initial energy.

Key words: Wave equation, negative initial energy, variable-exponent, blow-up

1. Introduction

In this paper, we consider the following problem:

$$u_{tt} - div\left(|\nabla u|^{m(x)-2}\nabla u\right) - div\left(|\nabla u_t|^{r(x)-2}\nabla u_t\right) - \gamma\Delta u_t = |u|^{p(x)-2}u, \text{ in } \Omega\times(0,T), \tag{1.1}$$

$$u(x,t) = 0, \quad x\in\partial\Omega, t>0, \tag{1.2}$$

with the initial conditions

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x\in\Omega, \tag{1.3}$$

where $\gamma > 0$, $0 < T < \infty$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. $m(.)$, $r(.)$, and $p(.)$ are given measurable functions on Ω satisfying

$$2 \leq r_1 \leq r(x) \leq r_2 \leq m_1 \leq m(x) \leq m_2 \leq p_1 \leq p(x) \leq p_2 < r^*, \tag{1.4}$$

with

$$\begin{aligned} r_1 \ &: = ess \inf_{x\in\Omega} r(x), \ r_2 := ess \sup_{x\in\Omega} r(x), \\ m_1 \ &: = ess \inf_{x\in\Omega} m(x), \ m_2 := ess \sup_{x\in\Omega} m(x), \\ p_1 \ &: = ess \inf_{x\in\Omega} p(x), \ p_2 := ess \sup_{x\in\Omega} p(x), \end{aligned}$$

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and

$$r^* = \begin{cases} \frac{nm(x)}{ess\sup_{x\in\Omega}(n-m(x))}, & \text{if } m_2 < n \\ +\infty, & \text{if } m_2 \geq n \end{cases}$$

We also assume that $m(.)$ satisfies the log-Hölder continuity conditions:

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e } x, y \in \Omega, \text{ with } |x-y| < \delta, \ A > 0, \ 0 < \delta < 1. \tag{1.5}$$

Problems of this type arise in many different fields, such as physics, acoustics, electromagnetics, fluid mechanics, and so forth.

Many authors have studied problem (1.1) in case of constant and variable exponent nonlinearities see e.g., [9, 12, 13, 18].

In the case where $m(.)$, $r(.)$, and $p(.)$ are constants, many problems similar or related to problem (1.1) have been exhaustively investigated as a result of blow-up, global existence and stability have been established. Chen et al. [5] considered the nonlinear p-Laplacian wave equation:

$$u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + g(x,t) = f(x), \text{ in } \Omega \times (0,T) \tag{1.6}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $2 \leq p < n$ and f, g are given functions. They proved the global existence, uniqueness under suitable conditions on the initial data and the functions f, g, and they also discussed the long-time behavior of the solution. In [20], Erhan studied the following quasilinear hyperbolic equation:

$$u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u, \text{ in } \Omega \times (0,T). \tag{1.7}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ $(n \geq 1)$, $m > 0$, $p, q \geq 1$. He proved the decay estimates of the energy function by using Nakao's inequality and he also obtained the blow-up of solutions and lifespan estimates in three different ranges of the initial energy. In [19], Ouaoua and Maouni considered the following equation:

$$u_{tt} - div\left(\frac{|\nabla u|^{2m-2}\nabla u}{\sqrt{1+|\nabla u|^{2m}}}\right) - \omega\Delta u_t + \mu u_t = |u|^{p-2}u, \text{ in } \Omega \times (0,T) \tag{1.8}$$

where Ω is a bounded regular domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. ω, μ m, and p are real numbers, they proved local existence and uniqueness of the solution by using the Faedo–Galerkin method and that the local solution is globally in time. They also proved that the solutions with some conditions exponentially decay. In [4], Benaissa and Mokeddem looked into the following equation:

$$u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) - \sigma(t)\, div\left(|\nabla u_t|^{m-2}\nabla u_t\right) = 0, \text{ in } \Omega \times (0,T) \tag{1.9}$$

where σ is a positive function, p, $m \geq 2$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, they gave an energy decay estimate for the solution. In [17], the work of Messaoudi and Houari considered the nonlinear wave equation:

$$u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta-2}\nabla u_t\right) + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \text{ in } \Omega \times (0,T) \tag{1.10}$$

where $a,\ b>0,\ \alpha,\ \beta,\ m,\ p>2$ and Ω is a bounded domain in $\mathbb{R}^n\ (n\geq 1)$, with a smooth boundary $\partial\Omega$. They proved under suitable conditions on $\alpha,\ \beta,\ m,\ p>2$ and for negative initial energy, a global nonexistence theorem. Ye in [22], investigated the blow-up property of solutions of a quasilinear hyperbolic system. He proved that certain solutions with positive initial energy blow up in finite time under suitable conditions and gave an estimation for the solution.

In the case of variable exponents nonlinearities, Antonsev, Ferreira and Erhan in [3] considered a nonlinear plate Petrovesky equation:

$$u_{tt}+\Delta^2 u-\Delta u_t+|u_t|^{p(x)-2}u_t=|u|^{q(x)-2}u,\ \text{in } \Omega\times(0,T) \tag{1.11}$$

where Ω is a bounded domain in $\mathbb{R}^n\ (n\geq 1)$ with a smooth boundary $\partial\Omega$. They proved the local weak solutions by using the Banach contraction mapping principle. Then, they showed that the solution is global if $p(.)\geq q(.)$ and they proved that a solution with negative initial energy and $p(.)<q(.)$ blows up in finite time. In [21], Erhan considered the strongly damped nonlinear Klein-Gordon equation:

$$u_{tt}-\Delta u-\Delta u_t+m^2u+|u_t|^{p(x)-2}u_t=|u|^{q(x)-2}u,\ \text{in } \Omega\times(0,T) \tag{1.12}$$

where Ω is a bounded domain in $\mathbb{R}^n$. He obtained nonexistence of solutions if variable exponents $p(.)$, $q(.)$ and initial data satisfy some conditions. In [1, 2], Antontsev considered the equation:

$$u_{tt}-div\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right)-\alpha\Delta u_t=b(x,t)|u|^{\sigma(x,t)-2}u,\ \text{in } \Omega\times(0,T), \tag{1.13}$$

where $\alpha>0$ is a constant, $a,\ b,\ p,\ \sigma$ are given functions and Ω is a bounded domain in $\mathbb{R}^n$. Under appropriate conditions on the initial data and the functions $a,\ b,\ p,\ \sigma$, he proved some blow-up results for certain solutions with nonpositive initial energy and discussed the same equation and proved the local and global existence of a weak solution under suitable conditions on $a,\ b,\ p,\ \sigma$. In [15], Messaoudi and Talahmeh considered the following equation:

$$u_{tt}-div\left(|\nabla u|^{r(x)-2}\nabla u\right)+a|u_t|^{m(x)-2}u_t=b|u|^{p(x)-2}u,\ \text{in } \Omega\times(0,T), \tag{1.14}$$

where $a,\ b$ is a nonnegative constant. They proved a finite-time blow-up result of the solution with negative initial energy as well as for certain solutions with positive initial energy. In [13], the case where $m(x)=2$ and under suitable conditions on the exponents, they established a blow-up result for solutions with arbitrary positive initial energy. In [16], Messaoudi and Al.Smail discuss the case where $b=0$ and $a=1$ of the same equation (1.14). They proved the decay estimates for the solution under suitable assumptions on the variable exponents $m,\ r$, and the initial data. They also gave two numerical applications to illustrate your theoretical results.

Our objective of this paper is to study: In Section 2, some notations, assumptions, and preliminaries are introduced. We also state without proof an existence result. In Section 3, we show the blow-up of solutions.

2. Assumptions and preliminaries

In this section, we present some Lemmas about the Lebesgue and Sobolev space with variable exponents (See [6]-[8],[10]). Let $p:\Omega\to[1,+\infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$.

We define the Lebesgue space with variable exponent $p(.)$ by:

$$L^{p(.)}(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \text{ measurable in } \Omega. \ \varrho_{p(.)}(\lambda v) < +\infty, \text{ for some } \lambda > 0 \right\},$$

where $\varrho_{p(.)}(v) = \int_\Omega |v(x)|^{p(x)} dx$.
The set $L^{p(.)}(\Omega)$ equipped with the norm (Luxemburg's norm)

$$\|v\|_{p(.)} := \inf \left\{ \lambda > 0 :_\Omega \left| \frac{v(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

$L^{p(.)}(\Omega)$ is a Banach space [11].

We next define the variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ as follows:

$$W^{1,p(.)}(\Omega) := \left\{ v \in L^{p(.)}(\Omega) \text{ such that } \nabla v \text{ exists } |\nabla v| \in L^{p(.)}(\Omega) \right\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,p(.)}(\Omega)} = \|v\|_{p(.)} + \|\nabla v\|_{p(.)}$.

Furthermore, we set $W_0^{1,p(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p(.)}(\Omega)$. Let us note that the space $W^{1,p(.)}(\Omega)$ has a different definition in the case of variable exponents. However, under condition (1.5), both definitions are equivalent [11]. The space $W^{-1,p'(.)}(\Omega)$ dual of $W_0^{1,p(.)}(\Omega)$ is defined in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

Lemma 2.1 *([11]) Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(.)$ satisfies (1.5), then*

$$\|u\|_{p(.)} \leq c \|\nabla u\|_{p(.)}, \quad \text{for all } u \in W_0^{1,p(.)}(\Omega),$$

where the positive constant depends on Ω, p_1, p_2. In particular, the space $W_0^{1,p(.)}(\Omega)$ has an equivalent norm given by $\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)}$.

Lemma 2.2 *([11]) If $p\colon \Omega \to [1,\infty)$ is a measurable function and*

$$2 \leq p_1 \leq p(x) \leq p_2 < \frac{2n}{n-2}, \ n \geq 3.$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

Lemma 2.3 *([11]) If $p\colon \Omega \to [1,\infty)$ is a measurable function with $p_2 < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^{p(.)}(\Omega)$.*

Lemma 2.4 *([11]) Let $p, q, s \geq 1$ be measurable function defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \ \textit{for a.e } y \in \Omega.$$

If $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$, then $fg \in L^{s(.)}(\Omega)$ and $\|fg\|_{s(.)} \leq 2 \|f\|_{p(.)} \|g\|_{q(.)}$.

Lemma 2.5 *([11]) Let p be a measurable function on Ω. Then*

$$\|f\|_{p(.)} \leq 1 \text{ if and only if } \varrho_{p(.)}(f) \leq 1.$$

Lemma 2.6 *([11]) If p is a measurable function on Ω satisfying (1.4) , then for a.e $x \in \Omega$, we have*

$$\min\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \leq \varrho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\},$$

for any $u \in L^{p(.)}(\Omega)$.

Theorem 2.7 *([2]) Let $u_0 \in W_0^{1,m(.)}(\Omega)$, $u_1 \in L^2(\Omega)$ and assume that the exponents m, r, p satisfy conditions (1.4) and (1.5). Then problem (1.1) has a unique weak solution such that*

$$\begin{aligned} u &\in L^{\infty}\left((0,T), W_0^{1,m(.)}(\Omega)\right), \\ u_t &\in L^{\infty}\left((0,T), L^2(\Omega)\right), \\ u_{tt} &\in L^{\infty}\left((0,T), W_0^{-1,m'(.)}(\Omega)\right), \end{aligned}$$

where $\frac{1}{m(.)} + \frac{1}{m'(.)} = 1$.

3. Blow-up

In order to state and prove our result, we define the potential energy function by the following:

$$E(t) = \frac{1}{2}\int_{\Omega} u_t^2 dx + \int_{\Omega} \frac{1}{m(x)} |\nabla u|^{m(x)} dx - \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx. \tag{3.1}$$

Lemma 3.1 *Assume that u be a solution of (1.1). Then, we have*

$$E'(t) = -\int_{\Omega} |\nabla u_t|^{r(x)} dx - \gamma \int_{\Omega} |\nabla u_t|^2 dx \leq 0, \ t \in [0,T] \tag{3.2}$$

and

$$E(t) \leq E(0). \tag{3.3}$$

Proof We multiply the first equation (1.1) by u_t and integrate over the domain Ω to get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega} u_t^2 dx + \int_{\Omega} \frac{1}{m(x)} |\nabla u|^{m(x)} dx - \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx\right) \\ = &\ -\int_{\Omega} |\nabla u_t|^{r(x)} dx - \gamma \int_{\Omega} |\nabla u_t|^2 dx. \end{aligned}$$

Then

$$E'(t) = -\int_{\Omega} |\nabla u_t|^{r(x)} dx - \gamma \int_{\Omega} |\nabla u_t|^2 dx \leq 0.$$

Integrating (3.2) over $(0,t)$, we obtain

$$E(t) \le E(0).$$

□

Let $H(t) = -E(t)$, using equation (3.1) and (3.3), we have

$$H(0) \le H(t) \le \frac{1}{p_1}\int_\Omega |u|^{p(x)}\,dx, \tag{3.4}$$

for any $t \ge 0$. Let

$$L(t) = H^{(1-\sigma)}(t) + \varepsilon\int_\Omega u u_t dx, \tag{3.5}$$

where ϵ and σ are constants.

Lemma 3.2 *Suppose that $u(x,t)$ is a regular solution of (1.1) under conditions (1.2)-(1.3), and the initial energy satisfies $E(0) < 0$. If $\sigma < \min\left\{\frac{m_1-2}{p_1}, \frac{m_1-2}{p_2}, \frac{m_1-r_1}{p_1(r_2-1)}, \frac{m_1-r_2}{p_1(r_2-1)}, \frac{m_1-r_1}{p_2(r_2-1)}, \frac{m_1-r_2}{p_2(r_2-1)}, 1\right\}$, then there exists a positive constant c such that*

$$L'(t) \ge c\varepsilon\left(H(t) + \int_\Omega |\nabla u|^{m(x)}\,dx + \int_\Omega u_t^2 dx\right). \tag{3.6}$$

Proof Differentiating (3.5), the following equality can be obtained:

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t^2 dx + \varepsilon\int_\Omega u u_{tt} dx. \tag{3.7}$$

Using Equations (1.1)-(1.3) and Green's first formula, we obtain

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t^2 dx \\ &\quad +\varepsilon\int_\Omega u\left[div\left(|\nabla u|^{m(x)-2}\nabla u\right) + div\left(|\nabla u_t|^{r(x)-2}\nabla u_t\right) + \gamma\Delta u_t + |u|^{p(x)-2}u\right]dx. \end{aligned} \tag{3.8}$$

$$\begin{aligned} &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\textstyle\int_\Omega u_t^2 dx - \varepsilon\int_\Omega |\nabla u|^{m(x)}\,dx \\ &\quad -\varepsilon\textstyle\int_\Omega |\nabla u_t|^{r(x)-2}\nabla u_t\nabla u dx - \varepsilon\gamma\int_\Omega \nabla u\nabla u_t dx + \varepsilon\int_\Omega |u|^{p(x)}\,dx. \end{aligned} \tag{3.9}$$

We can obtain the following inequalities from Young's inequality and Hölder inequality:

$$\int_\Omega \nabla u\nabla u_t dx \le \frac{1}{4k}\int_\Omega |\nabla u|^2\,dx + k\int_\Omega |\nabla u_t|^2\,dx. \tag{3.10}$$

$$\begin{aligned} \int_\Omega |\nabla u_t|^{r(x)-2}\nabla u_t\nabla u dx &\le \frac{1}{r_1}\int_\Omega \delta^{r(x)}|\nabla u|^{r(x)}\,dx \\ &\quad +\frac{r_2-1}{r_2}\int_\Omega \delta^{-\frac{r(x)}{r(x)-1}}|\nabla u_t|^{r(x)}\,dx. \end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned} L^{'}(t) &\geq (1-\sigma)H^{-\sigma}(t)H^{'}(t)+\varepsilon\int_{\Omega}u_t^2dx-\varepsilon\int_{\Omega}|\nabla u|^{m(x)}dx \\ &-\frac{\varepsilon}{r_1}\int_{\Omega}\delta^{r(x)}|\nabla u|^{r(x)}dx-\varepsilon\frac{r_2-1}{r_2}\int_{\Omega}\delta^{-\frac{r(x)}{r(x)-1}}|\nabla u_t|^{r(x)}dx \\ &-\frac{\varepsilon\gamma}{4k}\int_{\Omega}|\nabla u|^2dx-\varepsilon\gamma k\int_{\Omega}|\nabla u_t|^2dx+\varepsilon\int_{\Omega}|u|^{p(x)}dx. \end{aligned} \tag{3.12}$$

Taking $k=M_1H^{-\sigma}(t)$ and $\delta^{-\frac{r(x)}{r(x)-1}}=M_2H^{-\sigma}(t)$, and using $H(t)>0$, we obtain

$$\begin{aligned} L^{'}(t) &\geq (1-\sigma)H^{-\sigma}(t)H^{'}(t)+\varepsilon\int_{\Omega}u_t^2dx-\varepsilon\int_{\Omega}|\nabla u|^{m(x)}dx \\ &-\frac{\varepsilon M_2^{1-r_1}}{r_1}H^{\sigma(r_2-1)}(t)\int_{\Omega}|\nabla u|^{r(x)}dx \\ &-\varepsilon\frac{r_2-1}{r_2}M_2H^{-\sigma}(t)\int_{\Omega}|\nabla u_t|^{r(x)}dx \\ &-\frac{\varepsilon\gamma}{4M_1}H^{\sigma}(t)\int_{\Omega}|\nabla u|^2dx-\varepsilon\gamma M_1H^{-\sigma}(t)\int_{\Omega}|\nabla u_t|^2dx \\ &+\varepsilon\int_{\Omega}|u|^{p(x)}dx. \end{aligned} \tag{3.13}$$

Let $M=\max\left\{M_1,\frac{r_2-1}{r_2}M_2\right\}$, using the energy functional (3.2), we have

$$\begin{aligned} L^{'}(t) &\geq (1-\sigma)H^{-\sigma}(t)H^{'}(t)+\varepsilon\int_{\Omega}u_t^2dx-\varepsilon\int_{\Omega}|\nabla u|^{m(x)}dx \\ &-\frac{\varepsilon M_2^{1-r_1}}{r_1}H^{\sigma(r_2-1)}(t)\int_{\Omega}|\nabla u|^{r(x)}dx \\ &-\varepsilon MH^{-\sigma}(t)\left[\int_{\Omega}|\nabla u_t|^{r(x)}dx+\gamma\int_{\Omega}|\nabla u_t|^2dx\right] \\ &-\frac{\varepsilon\gamma}{4M_1}H^{\sigma}(t)\int_{\Omega}|\nabla u|^2dx+\varepsilon\int_{\Omega}|u|^{p(x)}dx. \end{aligned}$$

Then,

$$\begin{aligned} L^{'}(t) &\geq (1-\sigma-\varepsilon M)H^{-\sigma}(t)H^{'}(t)+\varepsilon\int_{\Omega}u_t^2dx \\ &-\varepsilon\int_{\Omega}|\nabla u|^{m(x)}dx-\frac{\varepsilon M_2^{1-r_1}}{r_1}H^{\sigma(r_2-1)}(t)\int_{\Omega}|\nabla u|^{r(x)}dx \\ &-\frac{\varepsilon\gamma}{4M_1}H^{\sigma}(t)\int_{\Omega}|\nabla u|^2dx+\varepsilon\int_{\Omega}|u|^{p(x)}dx. \end{aligned} \tag{3.14}$$

From the definition of $H(t)$, it follows that there is a constant k such that

$$\begin{aligned} L^{'}(t) \geq\; & (1-\sigma-\varepsilon M) H^{-\sigma}(t) H^{'}(t) + kH(t) + \varepsilon \int_\Omega u_t^2 dx \\ & -\varepsilon \int_\Omega |\nabla u|^{m(x)} dx - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & -\frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega |u|^{p(x)} dx \\ & -k\left[-\frac{1}{2}\int_\Omega u_t^2 dx - \int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)} dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx\right]. \end{aligned} \tag{3.15}$$

So

$$\begin{aligned} L^{'}(t) \geq\; & (1-\sigma-\varepsilon M) H^{-\sigma}(t) H^{'}(t) + kH(t) + \left(\varepsilon + \frac{k}{2}\right) \int_\Omega u_t^2 dx \\ & +\left(\frac{k}{m_2} - \varepsilon\right) \int_\Omega |\nabla u|^{m(x)} dx + \left(\varepsilon - \frac{k}{p_1}\right) \int_\Omega |u|^{p(x)} dx \\ & -\frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & -\frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx. \end{aligned} \tag{3.16}$$

Using $W^{1,m(.)}(\Omega) \hookrightarrow H^1(\Omega)$, $W^{1,m(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, $m(x) \geq r(x)$, and inequality (3.4), after simple the calculation, it can be concluded

$$\begin{aligned} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx \leq\; & \left(\frac{1}{p_1}\right)^{\sigma} \left(\int_\Omega |u|^{p(x)} dx\right)^{\sigma} \int_\Omega |\nabla u|^2 dx \\ \leq\; & C \left(\frac{1}{p_1}\right)^{\sigma} \left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{\sigma p_1+2}{m_1}} \\ & + C \left(\frac{1}{p_1}\right)^{\sigma} \left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{\sigma p_2+2}{m_1}}. \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \leq\; & C \left(\frac{1}{p_1}\right)^{\sigma(r_2-1)} \left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{\sigma p_1(r_2-1)+r_1}{m_1}} \\ & + C \left(\frac{1}{p_1}\right)^{\sigma(r_2-1)} \left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{\sigma p_2(r_2-1)+r_1}{m_1}} \\ & + C \left(\frac{1}{p_1}\right)^{\sigma(r_2-1)} \left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{\sigma p_1(r_2-1)+r_2}{m_1}} \\ & + C \left(\frac{1}{p_1}\right)^{\sigma(r_2-1)} \left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{\sigma p_2(r_2-1)+r_2}{m_1}}. \end{aligned} \tag{3.18}$$

For any constants $z \geq 0$ and $M > 0$, the algebraic inequality

$$z^{\upsilon} \leq z + 1 \leq \left(1 + \frac{1}{M}\right)(z + M), \quad (0 < \upsilon < 1) \tag{3.19}$$

holds. Using known condition $\sigma < \min\left\{\frac{m_1-2}{p_1}, \frac{m_1-2}{p_2}, \frac{m_1-r_1}{p_1(r_2-1)}, \frac{m_1-r_2}{p_1(r_2-1)}, \frac{m_1-r_1}{p_2(r_2-1)}, \frac{m_1-r_2}{p_2(r_2-1)}\right\}$. Hence from (3.19) the following inequalities can be acquired:

$$\begin{aligned}\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1+2}{m_1}} &\leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(0)\right) \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right)\end{aligned} \tag{3.20}$$

and the same for the other inequalities

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2+2}{m_1}} \leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.21}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1(r_2-1)+r_1}{m_1}} \leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.22}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2(r_2-1)+r_1}{m_1}} \leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.23}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1(r_2-1)+r_2}{m_1}} \leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.24}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2(r_2-1)+r_2}{m_1}} \leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.25}$$

From inequalities (3.17) and (3.18), we can obtain that there exist two positive constants N_1 and N_2 such that

$$\frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2\,dx \leq \varepsilon N_1 \left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right) \tag{3.26}$$

and

$$\frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)}\,dx \leq \varepsilon N_2 \left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.27}$$

Consequently, taking $k = \frac{1}{2}(N_1 + N_2 + m_2 N_1 + m_2 N_2 + m_2 + p_1)\varepsilon$, we obtain from (3.16) that there exists a positive constant c such that

$$\begin{aligned}L'(t) \quad \geq \quad & (1 - \sigma - \varepsilon M) H^{-\sigma}(t) H'(t) + (k - \varepsilon N_1 - \varepsilon N_2) H(t) \\ & + \left(\varepsilon + \frac{k}{2}\right) \int_\Omega u_t^2 dx + \left(\varepsilon - \frac{k}{p_1}\right) \int_\Omega |u|^{p(x)}\,dx \\ & + \left(\frac{k}{m_2} - \varepsilon - \varepsilon N_1 - \varepsilon N_2\right) \int_\Omega |\nabla u|^{m(x)}\,dx.\end{aligned}$$

$$L^{'}(t) \geq (1-\sigma-\varepsilon M) H^{-\sigma}(t) H^{'}(t) + c\varepsilon\left[H(t)+\int_{\Omega} u_t^2 dx+\int_{\Omega}|\nabla u|^{m(x)} dx\right]. \tag{3.28}$$

Taking $0<\varepsilon<\frac{1-\sigma}{M}$, we can get the following formula from (3.5):

$$L(0) = H^{1-\sigma}(0)+\varepsilon \int_{\Omega} u_0(x) u_1(x) dx > 0. \tag{3.29}$$

Using inequality (3.28), it holds

$$L^{'}(t) \geq c\varepsilon\left[H(t)+\int_{\Omega} u_t^2 dx+\int_{\Omega}|\nabla u|^{m(x)} dx\right]. \tag{3.30}$$

After integral, we can get $L(t) \geq L(0)>0, \ (\forall t \geq 0)$. □

Theorem 3.3 *Suppose $\gamma>0$ and $2 \leq r_1 \leq r(x) \leq r_2 \leq m_1 \leq m(x) \leq m_2 \leq p_1 \leq p(x) \leq p_2<r^*$, where r^* is the critical Sobolev index in $W^{1, m(\cdot)}(\Omega)$. If the initial energy $E(0)<0$, then any regular solutions of equation (1.1)-(1.3) must blow up in finite time.*

Proof Firstly, it is proved that when $0<\sigma<\frac{m_1-2}{2 m_1}$, there exists a positive constant C such that

$$L^{\frac{1}{1-\sigma}}(t) \leq C\left[H(t)+\int_{\Omega} u_t^2 dx+\int_{\Omega}|\nabla u|^{m(x)} dx\right]. \tag{3.31}$$

In fact, using (3.5), we get

$$L^{\frac{1}{1-\sigma}}(t) \leq C(\varepsilon, \sigma)\left[H(t)+\left(\int_{\Omega} u u_t dx\right)^{\frac{1}{1-\sigma}}\right]. \tag{3.32}$$

Furthermore, by using Hölder's inequality and Young's inequality, we can get

$$\begin{aligned}
\left(\int_{\Omega} u u_t dx\right)^{\frac{1}{1-\sigma}} & \leq \left(\int_{\Omega} u^2 dx\right)^{\frac{1}{2(1-\sigma)}}\left(\int_{\Omega} u_t^2 dx\right)^{\frac{1}{2(1-\sigma)}} \\
& \leq C\left(\int_{\Omega}|u|^{m(x)} dx\right)^{\frac{1}{m_1(1-\sigma)}}\left(\int_{\Omega} u_t^2 dx\right)^{\frac{1}{2(1-\sigma)}} \\
& \leq C\left[\left(\int_{\Omega}|u|^{m(x)} dx\right)^{\frac{\mu}{m_1(1-\sigma)}}+\left(\int_{\Omega} u_t^2 dx\right)^{\frac{\theta}{2(1-\sigma)}}\right],
\end{aligned}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. Taking $\theta=2(1-\sigma)$, we have $\mu=\frac{2(1-\sigma)}{1-2\sigma}$, by Poincare's inequality, it follows

$$\left(\int_{\Omega} u u_t dx\right)^{\frac{1}{1-\sigma}} \leq C\left[\left(\int_{\Omega}|\nabla u|^{m(x)} dx\right)^{\frac{2}{m_1(1-\sigma)}}+\int_{\Omega} u_t^2 dx\right]. \tag{3.33}$$

If $0<\sigma<\frac{m_1-2}{2m_1}$, then we get $0<\frac{2}{m_1(1-\sigma)}<1$. From (3.4), (3.19), and (3.33), we obtain

$$\begin{aligned}\left(\int_\Omega uu_t dx\right)^{\frac{1}{1-\sigma}} &\leq C\left(\int_\Omega |\nabla u|^{m(x)}dx + H(0) + \int_\Omega u_t^2 dx\right)\\ &\leq C\left(\int_\Omega |\nabla u|^{m(x)}dx + H(t) + \int_\Omega u_t^2 dx\right). \end{aligned} \tag{3.34}$$

From (3.32), (3.34), we get

$$L^{\frac{1}{1-\sigma}}(t) \leq C\left(\int_\Omega |\nabla u|^{m(x)}dx + H(t) + \int_\Omega u_t^2 dx\right), \tag{3.35}$$

where C is only related to σ, ε. Taking $\sigma<\min\left\{\frac{m_1-2}{p_1}, \frac{m_1-2}{p_2}, \frac{m_1-r_1}{p_1(r_2-1)}, \frac{m_1-r_2}{p_1(r_2-1)}, \frac{m_1-r_1}{p_2(r_2-1)}, \frac{m_1-r_2}{p_2(r_2-1)}, \frac{m_1-2}{2m_1}, 1\right\}$, by inequality (3.31) and (3.6) in Lemma 3.2, it follows that there exists a constant $\zeta>0$ such that

$$L^{'}(t) \geq \zeta L^{\frac{1}{1-\sigma}}(t) \tag{3.36}$$

for any $t\geq 0$. Integrating the above formula with respect to t on $[0,t]$, we get

$$L^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{\sigma}{1-\sigma}}(0) - \frac{\sigma\zeta}{1-\sigma}t} \tag{3.37}$$

for any $t\geq 0$. Hence there exists $T^*\leq \frac{1-\sigma}{\sigma\zeta L^{\frac{\sigma}{1-\sigma}}(0)}$ such that $\lim_{t\to T^*} L(t)=\infty$, that means the regular solution $u(x,t)$ must blow up in finite time. □

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