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Configuration Spaces and Imbedding Invariants

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1. Introduction

For the past two years I have been trying to understand the new physics-inspired invariants of three-manifolds and knots in terms of concepts more congenial to the classical topologist.

In [K] Kontsevich indicates a method for defining a set of invariants for 3-manifolds M, which presumably recreates the asymptotic invariants of the Chern-Simons theory at the trivial representation of $\pi_1(M)$, entirely in terms of the De Rham Theory of Configuration spaces of M. It therefore seemed plausible, $\mathbb{R}^3$ being simply connected, that a similar approach would work for knots in $\mathbb{R}^3$, and in our recent paper [B-T] Cliff Taubes and I showed that this is indeed the case. Since then we have noted that this method yields potential invariants also for higher-dimensional knots of S^{2k-1} in $\mathbb{R}^{2k+1}$, and the first of these will be presented in section 3. However, in this account I will mainly concentrate on the classical case and try and fit our constructions into the picture as it emerges from the work of Drinfeld, Bar Natan, Birman-Lin, Kontsevich and others.

The first conceptual step to note in the discussion of imbeddings – versus maps – is that for an imbedding:

$$f : X \hookrightarrow Y, \tag{1}$$

the Cartesian powers of f:

$$f^n : X^n \hookrightarrow Y^n \tag{2}$$

are also imbeddings and so map the subset of *distinct* n-tuples $(x_1, \cdots, x_k)$ $\quad x_i \neq x_j$ in X^n to the corresponding subset of *distinct* n-tuples in Y^n. One calls the subset of distinct n-tuples the *n-fold configuration space of* X, denotes it by $C_n^0(X)$:

$$C_n^0(X) = \{(x_1, \cdots, x_n) \in X^n | x_i \neq x_j\}, \tag{3}$$

and the overall aim of one program is to build imbedding invariants simply out of the auxiliary sequence of arrows

$$C_n^0(f) : C_n^0(X) \longrightarrow C_n^0(Y), \qquad n \geq 1, \tag{4}$$

induced by a given imbedding f. Note that the $C_n^0(f)$ are obviously equivariant under the symmetric group action on both sides.

This principle works remarkably well for the very classical problem of imbedding a graph G into the plane $\mathbb{R}^2$.

Observe first that for every space X, the double cover

$$C_2^0(X) \to C_2(X)/\mathbb{Z}_2 \tag{5}$$

determined by the $\mathbb{Z}_2$ action on $C_2^0(X)$, defines a characteristic class $\eta_X \in H^1(C_2^0(X)/\mathbb{Z}_2; \mathbb{Z}_2)$ – the first Whitney class of the line bundle associated to the double cover.

An immediate *necessary condition for* G *to imbed in* $\mathbb{R}^2$ *is now that* $\eta_G^2 = 0$.

Indeed an imbedding

$$f : G \to \mathbb{R}^2, \tag{6}$$

gives rise to an equivariant map

$$C_2^0(G) \longrightarrow S^1 \tag{7}$$

with $\mathbb{Z}_2$ acting as the antipode on S^1, by sending $x_1, x_2 \in C_2^0(G)$ to

$$\frac{f(x_2) - f(x_1)}{|f(x_2) - f(x)|}. \tag{8}$$

It follows that η_G is a *pull-back* of η_{S^1}. On the other hand, $\eta_{S^1}^2 = 0$ by dimensional reasons. QED

Remarkably enough this condition turns out to be sufficient also! This is a result of Wu [W] who derives it from the famous Kuratovski criterion, by showing that for the two "excluded figures" $\eta_G^2 \neq 0$. I learned all this only a week or two ago from a minor thesis of Mark Wunderlich – a bright senior at Harvard this year.

The problems of knot theory are of course much more difficult because they deal with the classification of imbeddings of G in $\mathbb{R}^3$ up to isotopy, rather than an existence problem. Nevertheless, the "configuration space method" here again yields a first invariant. Namely, if

$$f : G \hookrightarrow \mathbb{R}^3 \tag{9}$$

is an imbedding, again consider

$$C_2(f) : C_2^0(G) \to C_2(\mathbb{R}^3). \tag{10}$$

On the right, $C_2(\mathbb{R}^3)$ is now seen to be equivariantly homotopy equivalent to S^2. Hence if we can find two *disjoint* 1-cycles, say S_1 and S_2 in G, then $S_1 \times S_2 \subset C_2^0(G)$ and so we have the arrows:

$$S_1 \times S_2 \subset C_2^0(C) \xrightarrow{C_2(f)} C_2^0(\mathbb{R}^3) \xrightarrow{\varphi_{12}} S^2 \tag{11}$$

whose composition defines a degree.

Put differently, if $X = S_1 \amalg S_2$ then every imbedding of X in $\mathbb{R}^3$ has an integer assigned to it by the above procedure, and that integer is of course one of the definitions of the *linking* number of the two circles. In fact, if we set

$$\omega = \frac{1}{4\pi} \frac{xdydz - ydxdz + zdxdy}{(x^2 + y^2 + z^2)^{3/2}} \tag{12}$$

so that ω serves to define the unit volume on $S^2 \subset \mathbb{R}^3$, then the pull back of ω under the arrows of (11) precisely reproduces the Gauss Integral for the linking of two circles in $\mathbb{R}^3$.

2. Knot invariants

When one tries to apply this principle to a knot, i.e., take $X = S_1$, one soon discovers that for any imbedding $f : S^1 \hookrightarrow \mathbb{R}^3$ the "Gauss Integral"

$$\int_{C_2^0(S^1)} \{C_2(f) \circ \varphi_{12}\}^* \omega \tag{13}$$

converges – but is *neither* an integer *nor* an isotopy invariant. Rather, this "*self-linking*" number is a nontrivial smooth function on the space $\mathcal{K}$ of *smooth parametrized imbeddings of* $S^1 \subset \mathbb{R}^3$.

To discuss this function in detail, note that the map

$$\varphi_{12} : C_2^0(\mathbb{R}^3) \longrightarrow S^2 \tag{14}$$

defined by

$$\varphi_{12}(x_1, x_2) = \frac{x_2 - x_1}{|x_2 - x_1|} \tag{15}$$

composed with $C_2(f)$, takes two distinct points (x_1, x_2) on the knot, $k = f(S^1)$, to the unit direction of the chord joining the distinct points $f(x_1), f(x_2)$.

Now $C_2^0(S^1) = S^1 \times S^1 - \Delta$, and is therefore diffeomorphic to $S^1 \times (0, 1)$, as is apparent from Fig. 1 below:

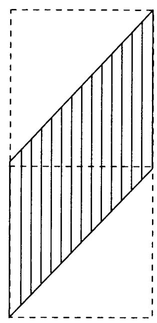


Fig.1

Due to the singularity of φ_{12} near $x_1 = x_2$, the map $C_2(f) \circ \varphi_{12}$ does *not* extend to $S^1 \times S^1$, but it does extend to the compactification $S^1 \times [0, 1]$ of $S^1 \times (0, 1)$, and in fact $C_2(f) \circ \varphi_{12}$ then clearly restricts to the tangential map

$$\psi(x) = \frac{\dot{f}(x)}{|f(x)|}, \tag{16}$$

on one boundary circle of $S^1 \times [0,1]$ and to $-\psi(x)$ on the other.

To investigate the self-linking function given by the integral (13) further, it is expedient to think of it as a pull-back push-forward – or transfer – in the following diagram:

$$\begin{array}{ccccc} \mathcal{K} \times C_2(S^1) & \overset{C_2(f)}{\longrightarrow} & C_2(\mathbb{R}^3) & \overset{\varphi_{12}}{\longrightarrow} & S^2 \\ \pi \downarrow & & & & \\ \mathcal{K} & & & & \end{array} \tag{17}$$

where now $C_2(S^1)$ denotes the compactification of $C_2^0(S^1)$ to $S^1 \times [0,1]$, and $C_2(\mathbb{R}^3)$ stands for the "compactification along the diagonal" of $\mathbb{R}^3 \times \mathbb{R}^3 - \Delta$, given by blowing up the diagonal in $\mathbb{R}^3 \times \mathbb{R}^3$ in the C^∞ sense, i.e., replacing Δ, by the unit normal bundle of Δ in $\mathbb{R}^3 \times \mathbb{R}^3$.

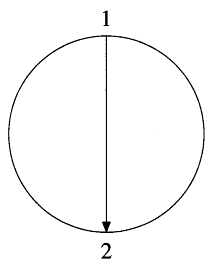
It is not difficult to check that the map φ_{12} extends to $C_2(\mathbb{R}^3)$ and in fact then restricts essentials as the *identity* on the fibers over Δ. In this picture the self-linking integral now takes the form

$$\pi_* \circ \{C_2(f) \circ \varphi_{12}\}^* \cdot \omega, \tag{18}$$

where π_* denotes integration over the fiber in the product fibration on the left. More informally, we also write

$$\int_{C_2(S^1)} \theta_{12} \tag{19}$$

for this integral, and graphically refer to it by the diagram:



(20)

To study the constancy of this function on $\mathcal{K}$ we differentiate it by applying the functorial form of Stokes' theorem for fiberings with "manifolds with boundary" as fibers. In this context, Stokes' Theorem takes the form:

$$d\pi_*^M = \pi_* d + \pi_*^{\partial M}. \tag{21}$$

Here the last symbol denotes *restriction* to the boundary followed by *integration* over that boundary. In the case at hand it therefore follows from our earlier remark concerning the value of $C_2(f)$ that

$$d\int_{C_2(S^1)} \theta_{12} = 2\int_{S^1} \psi^* \cdot \omega, \tag{22}$$

where, as before,

$$\psi : \mathcal{K} \times S^1 \longrightarrow S^2 \tag{23}$$

denotes the *tangential* map

$$(f, x) \longrightarrow \frac{\dot{f}(x)}{|\dot{f}(x)|}. \tag{24}$$

This formula (22) already exhibits an interesting topological fact about the space $\mathcal{K}$ *vis-a-vis* the corresponding space of immersions of S^1 in $\mathbb{R}^3$. If we denote the space of these immersions with $\mathcal{T}$, we clearly have $\mathcal{K} \subset \mathcal{T}$ and equally clearly, the old formula (24) extends ψ to a map:

$$\widehat{\psi} : \mathcal{T} \times S^1 \longrightarrow S^2. \tag{25}$$

Now by theorems of Whitney and Smale, the structure of $\mathcal{T}$ is well known. Namely, the transpose of $\widehat{\psi}$

$$\widehat{\psi}^t : \mathcal{T} \longrightarrow \mathcal{L}S^2, \tag{26}$$

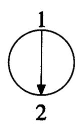
mapping $\mathcal{T}$ to the loop space of S^2 in a *homotopy equivalence.* Furthermore, by a construction of Chen the 1-form

$$\int_{S^1} \widehat{\psi}^* \omega$$

is also well known to generate the one-dimensional vector space $H^1(\mathcal{L}S^2; \mathbb{R})$.

In short, the relation (22) asserts that this generator of $H^1(\mathcal{T})$ *restricts to 0 in* $H^1(\mathcal{K})$ and in fact is there the boundary of one-half the self-linking function :

$$\iota_{\mathcal{K}}^* \cdot \int_{S^1} \widehat{\psi}^* \cdot \omega = 2d \tag{27}$$



All these remarks are essentially old hat, even if expressed in a new form. Here they mainly serve to prepare us for the next conceptual step in the study of imbeddings. Namely, we need to understand *compactifications* of the spaces $C_n^0(X)$ for all $n \geq 2$, which are *functorial* for *imbeddings*, and to which our maps φ_{ij}:

$$\varphi_{ij} : C_k^0(\mathbb{R}^3) \longrightarrow S^2, \tag{28}$$

given by $\varphi_{ij}(x_1 \cdots x_k) = \dfrac{x_j - x_i}{|x_j - x_i|}$, $i, j \leq n$, extend smoothly. In short, we need compactifications where two distinct points never quite lose their identity. Such a compactification was carefully described by Fulton and MacPherson in the algebraic context a year ago [F-M], and it is not difficut to adapt their work to the C^∞ case – as was done very explicitly by Axelrod-Singer in [A-S]. Presumably this is also the compactification Kontsevich had in mind in [K].

Actually, the *definition* is very simple and it is only the careful description of the *strata* that is difficult, confusing, and at times counter-intuitive.

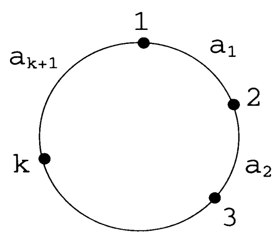
The *definition* of [F-M] is the following one. Given a finite set S, M^S denotes the space of maps of S to M, and $B\ell(M^S, \Delta_S)$ denotes the C^∞ blow up of the "diagonal" $\Delta_S \subset M^S$, consisting of the *constant* maps of S to M. If we write $C_S^0(M)$ for the space of *imbeddings* of S to M, then these spaces are clearly the functorial versions of our earlier configuration spaces. It is also clear that for every inclusion $K \subset S$ there are natural projections $M^S \to M^K$ and corresponding arrows $C_S^0(M) \to C_K^0(M)$ induced by restricting maps from S to K. Furthermore the natural inclusions $C_S^0(M) \subset M^S$ *lift* to *inclusions* $C_S^0 M \subset B\ell(M^S, \Delta_S)$, as these sets avoid all diagonals. In short, given a finite set N, there is a canonical inclusion

$$C_N^0(M) \hookrightarrow \bigotimes_{\substack{S \subset N \\ |S| \geq 2}} B\ell(M^S, \Delta_S) \times M^S. \tag{29}$$

The Fulton-MacPherson compactification, $C_N(M)$ is now defined to be the closure of $C_N^0(M)$ in this imbedding.

This compactification has all the desired properties: it turns out to have a natural structure of a *manifold with corners*; it clearly has the desired *equivariant functorial properties* under imbeddings, and the *maps* φ_{ij} on $C_n^0(\mathbb{R}^3)$ *do extend to* $C_n(\mathbb{R}^3)$. In short, it meets all the criteria we had laid out for it.

I do not have time to elaborate on this construction, but let me give you a feel for it by drawing the first few $C_k(S^1)$. The space $C_1(S^1)$ is of course S^1 while $C_2(S^1) = S^1 \times [0, 1]$ as we already saw. For $k \geq 3$, $C_k^0(S^1)$ breaks into $(k-1)!$ components, and once we choose an orientation for S^1 the components are characterized by the cyclic order of the points as they appear on S^1. The "identity component" is taken to be the one in which they appear in their natural order, i.e.,



$$a_1 + a_2 + a \cdots a_{k+1} = 1$$

Fig. 2

Using the parameters a_i of Figure 2, this component is now seen to be

$$\text{id. comp. } \mathrm{C}^0_\mathrm{k}(\mathrm{S}^1) = \mathrm{S}^1 \times \mathring{\Delta}_{\mathrm{k}-1}, \tag{30}$$

where $\mathring{\Delta}_k$ denotes the *open* k simplex.

Thus the *naive* compactification of the identity components of $C^0_k(S^1)$ would simply be $S^1 \times \Delta_{k-1}$: the circle times the closed $(k-1)$ simplex.

Pictorially the naive compactifications would therefore give rise to S^1 crossed with the sequence:

$$\mathrm{pt} \rightrightarrows \quad \Delta_1 \quad \begin{matrix} \to \\ \to \\ \to \end{matrix} \quad \Delta_2 \quad \ldots \tag{31}$$

Fig. 3

of closed simplexes, so well known from semisimplicial theory.

On the other hand, the identity components of the functorial compactification $C_k(S^1)$ give rise to S^1 times a sequence of polyhedra obtained from the above sequence by a series of "blow ups". The polyhedra W_i in question takes the following from:

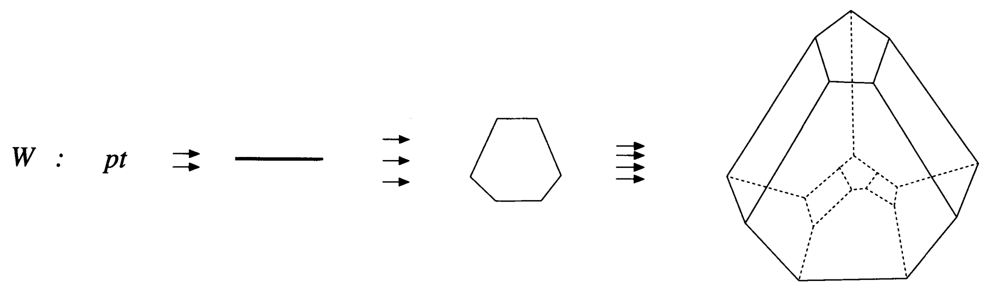
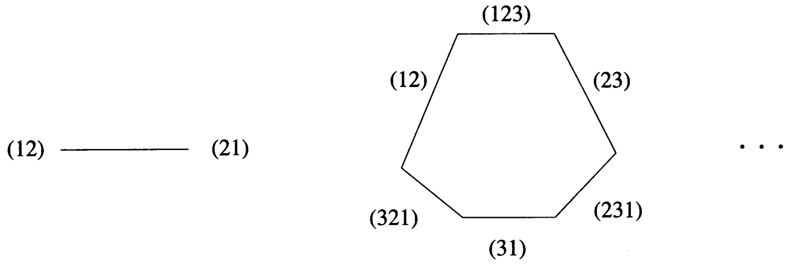


Fig. 4

Thus the identity component of $C_3(S^1) = S^1 \times W_3$ with W_3 a hexagon, while that of $C_4(S^1)$ is given by $S^1 \times W_4$ with W_4 manifestly already a more formidable figure. The interesting, and possibly at first counter-intuitive, state of affairs is that the open strata of codimension 1 in $C_n(M)$ are indexed by *all* the subsets, S, with cardinality $|S| \geq 2$, of the set $N = (1, 2, \cdots n)$, an open stratum associated to S describing the coming together of the points x_i, $i \in S$, at *distinct but commensurate speeds.*

In the one-dimensional case these strata need not be connected, but in the identity component of $C_n(S^1)$ one can keep track of the connected components of a stratum by indicating in what cyclic order the points come together.

Thus one labels the W's by:



(32)

Fig. 5

The higher "faces" of W_k are indexed by nested subsets of the set $N = (1, 2, \cdots n)$, that is, by collections $\Lambda = (S_1, \cdots S_r)$ of subsets of N, each of cardinality ≥ 2, such that for any two S_i and S_j:

$$S_i \subset S_j \qquad \text{or} \qquad \mathrm{S_j} \subset \mathrm{S_i} \qquad \text{or} \qquad \mathrm{S_i} \cap \mathrm{S_j} = \emptyset. \tag{33}$$

A glance at the polyhedra W_i shows that certain faces of W_{i+1} are isomorphic to W_i. These are the faces indexed by subsets of cardinality 2 and correspond to the collision of *precisely two points*. Correspondingly, we call these the "principal" faces of W_{i+1}, and the others the "hidden" faces. Thus the pentagons and quadrilaterals in W_4 are *hidden* while the 4 *principal* faces are the hexagons.

From this point of view the great virtue of the sequence of simplexes $\{\Delta_k\}$ is of course that *all* the faces of Δ_k are isomorphic to Δ_s with $s < k$ so that, for instance, integrals over the boundary of Δ_k can always be collated to a sum of integrals over Δ_{k-1}.

Unfortunately this is not quite true for the sequence W_i and I therefore like to think of the W-sequence as a "leaking" semisimplicial space.

With these basics understood it is time to continue our search for new knot invariants, and armed with the compactifications $C_k(S^1)$ and $C_k(\mathbb{R}^3)$ we see that one again has natural arrows:

$$\begin{array}{ccc} C_k(S^1) \times \mathcal{K} & \xrightarrow{C_k} & C_k(\mathbb{R}^3) \\ \pi \big\downarrow & & \\ \mathcal{K} & & \end{array} \tag{34}$$

so that we can again construct forms on $\mathcal{K}$ from forms on $C_k(\mathbb{R}^3)$ by the pull-back push-forward procedure.

Furthermore, there is a natural candidate for the forms on $C_n(\mathbb{R}^3)$ to which we can apply this transfer procedure, namely the ring generated by the pull backs of the volume form $\omega \in \Omega^2(S^2)$ under the projections

$$\varphi_{ij} : C_n(\mathbb{R}^3) \longrightarrow S^2 \tag{35}$$

sending $(x_1 \cdots x_n)$ to $(x_j - x_i)/|x_j - x_i|$.

We correspondingly write

$$\theta_{ij} = \varphi_{ij}^* \cdot \omega, \qquad i, j \leq n \tag{36}$$

and refer to the ring generated by the θ_{ij} as the ring of "*tautologous forms*" on $C_n(\mathbb{R}^3)$.

It is well known and easy to prove by induction that the θ_{ij} generate $H^*C_n(\mathbb{R}^3)$. They clearly also satisfy the relations

$$\theta_{ij}^2 = 0 \qquad \theta_{ij} = -\theta_{ji} \tag{37}$$

so that this ring is spanned by *monomial expressions* in the θ_{ij}, $i < j$. In short, we seek candidates for new invariants among the transfers, $\pi_* \circ C_k^* \cdot w$, of monomials w in the θ's.

For instance, if we take the monomial $\theta_{13}\theta_{24}$ in $\Omega^4 C_4(\mathbb{R}^3)$ and consider its transfer:

$$\alpha = \pi_* C_4^* \theta_{13}\theta_{24} = \int_{C_4(S^1)} \theta_{13}\theta_{24} \tag{38}$$

we get a new *function* on $\mathcal{K}$. However, as before, this function will *not* be locally constant, i.e., a knot invariant, unless:

$$d\alpha = \int_{\partial C_4(S^1)} \theta_{13}\theta_{24} \tag{39}$$

vanishes.

Now the boundary of $\partial C_4(S^1)$ consists of the "hidden faces" and the "principal faces" – which are all isomorphic to $C_3(S^1)$.

A local computation now shows that although $C_4^*\theta_{13}\theta_{24}$ does *not* vanish on a hidden face, its push-forward *does*! Hence this part of $\partial C_4(S^1)$ can be forgotten in the computation of $d\alpha$.

On the other hand, the contribution of the principal faces can be collated to yield an integral over $C_3(S^1)$. In this way – and using the invariance of the push forward under the orientation preserving diffeomorphism given by the $\mathbb{Z}_3$ action on $C_3(S^1)$ – one finds the new representation:

$$d\alpha = 4 \int_{C_3(S^1)} \theta_{12}\theta_{23}. \tag{40}$$

In the attempt to cancel this term we come now to the *third* essential principle needed for making imbedding invariants in the context of configuration spaces. I will call it the *exchange principle*, and for those in the know, it clearly has its antecedents in the Feynmann integral.

The idea is simply to extend our notion of "configuration spaces" in a functorial manner.

Namely, given an imbedding

$$f : X \subset Y \tag{41}$$

we define $C_{r,s}(X, Y)$ to be the compactification of configurations of *r-distinct points in* X and *s-distinct points in* Y, which are also distinct from the $f(x_j)$, $1 \leq j \leq r$.

Technically $C_{r,s}(X,Y)$ fits into the Cartesian Square

$$\begin{array}{ccc} \widetilde{C}_{r,s}(X,Y) & \xrightarrow{C_{r,s}} & C_{r+s}(Y) \\ \pi' \big\downarrow & & \big\downarrow \\ \mathrm{Im\ b(X,Y) \times C_r(X)} & \longrightarrow & C_r(Y) \end{array} \tag{42}$$

with π' a bundle projection. *In particular,* $\widetilde{C}_{r,s}(X,Y)$ *is a space over* $Imb(X,Y)$ *with fiber* $C_{r,s}(X,Y)$ so that one again has the transfer diagram:

$$\begin{array}{ccc} \widetilde{C}_{r,s}(X,Y) & \xrightarrow{C_{r,s}} & C_{r+s}(Y) \\ \pi \big\downarrow & & \\ \mathrm{Imb(X,Y).} & & \end{array} \tag{43}$$

In the case at hand one will see that the transfer of $\theta_{14}\theta_{24}\theta_{34}$ in $\Omega^6 C_4(\mathbb{R}^4)$ is the integral[1]

$$\beta = \int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34} \tag{44}$$

can be used to cancel the integral for $d\alpha$.

Precisely one has:

Theorem 1. (Bar-Natan, Guadagnini, Martellini, Mintchev) *The integral expression*

$$V_1 = \frac{1}{4}\int_{C_4(S^1)} \theta_{13}\theta_{24} - \frac{1}{3}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34} \tag{45}$$

represents a locally constant function on $\mathcal{K}$ *and is thus a knot invariant.*

The reason for this cancellation is maybe best understood by the following considerations.

We have four natural boundary maps

$$C_3(S^1) \quad \begin{smallmatrix} \rightrightarrows \\ \rightrightarrows \end{smallmatrix} \quad C_4(S^1) \tag{46}$$

of which the first is given by

$$\partial_1(x_1, x_2, x_3) = \lim_{\epsilon > 0 \to 0} (x_1, x_1 + \epsilon, x_2; x_3) \tag{47}$$

and the rest can be taken to be the transforms of ∂_1 by the $\mathbb{Z}_4$ action on $C_4(S^1)$. These maps induce corresponding maps

$$\widetilde{C}_{3,0} \quad \begin{smallmatrix} \rightrightarrows \\ \rightrightarrows \end{smallmatrix} \quad \widetilde{C}_{4,0} \tag{48}$$

[1]We have here abbreviated the identity component of $C_{3,1}(S^1, \mathbb{R}^3)$ to $C_{3,1}$ and will adhere to this notation hereafter.

where we have now abbreviated the notation of $\widetilde{C}_{r,s}(X,Y)$ to $\widetilde{C}_{r,s}$.

But we also have three natural arrows:

$$\begin{array}{c} \widetilde{C}_{3,1} \\ \uparrow\uparrow\uparrow \\ \widetilde{C}_{3,0} \times S^2 \end{array} \tag{49}$$

of which the first is defined by

$$\partial_1\{f_1, x_1, x_2, x_3, \xi\} \longrightarrow \lim_{\epsilon \geq 0}(f, x_1, x_2, x_3, x_4 + \epsilon \cdot \xi), \tag{50}$$

so that the S^2-factor records the direction in which the two points x_1 and x_4 collide.

To bring these two pictures into synchronization we multiply the first one by S^2 to obtain

$$\begin{array}{ccc} \widetilde{C}_{3,1} & & \\ \uparrow\uparrow\uparrow & & \\ \widetilde{C}_{3,0} \times S^2 & \begin{array}{c}\rightrightarrows \\ \rightrightarrows\end{array} & \widetilde{C}_{4,0} \times S^2. \end{array} \tag{51}$$

Except for numerical factors and using the invariance under cyclic permutations of the push forward, one then sees that boundary contributions of the two forms $\theta_{13}\theta_{24} \cdot \omega$ on the right, and $\theta_{14}\theta_{24}\theta_{34}$ above, cancel each other on the $\widetilde{C}_{3,0} \times S^2$ part of their common boundary! But we are not quite done, for there remain new "principal boundary" parts, of $\widetilde{C}_{3,1}$, to be considered, those corresponding to the collision of two points on the circle. But on the part where x_1 and x_2 tend to each other the form $\theta_{14}\theta_{24}\theta_{34}$ restricts to $\theta_{14}\theta_{14}\theta_{34}$ and so vanishes thanks to the relation $\theta_{ij}^2 = 0$. Similarly on the faces where x_2 approaches x_3, and x_3 approaches x_1. Unfortunately, new hidden faces of C_{31} have to be considered also – but again one finds that the *push forward from these hidden faces vanishes.* Q.E.D.

In summary, one can fit this invariant into the compactified configuration space picture and deduce its invariance from Stokes, the vanishing of push forwards on hidden faces, the equivariant behaviour of the push forward in general, and the exchange principle.

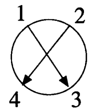
Once this is done it is more or less clear how to generalize this construction and create new integrals which give invariants of knots, and with Cliff Taubes we laid the foundation for this extension in our paper [B-T].

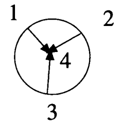
What emerges can be summarized as follows: For each integer ℓ, the transform of the tautologous forms from the diagrams

$$\begin{array}{ccc} \widetilde{C}_{r,s} \times (S^2)^{2\ell - s} & \longrightarrow & C_{r+s}(\mathbb{R}^3) \times (S^2)^{2t} \\ \downarrow & & \\ \mathcal{K} & & \end{array} \quad , \quad r + s = 2t \tag{52}$$

give rise to a sub-complex F_ℓ of $\Omega^*(\mathcal{K})$ and the relations between these forms on the "principal faces" induce a combinatorial differential operator δ on F_ℓ. Furthermore, *in dimension 0, $H^0_\delta(F_L)$ describes potential knot invariants.*

In many ways, as physicists noted years ago, this complex F_ℓ – at level ℓ, is best described graphically. Thus we denote the transfer of a form from the diagram above by an oriented circle on which r points are indicated, labelled in the direction of the orientation, together with s interior points, labelled by the integers $r+1, \cdots, r+\ell$. Now the form θ_{ij} is graphically indicated by an oriented line joining the vertex i to the vertex j.

Thus the form $\displaystyle\int_{C_{4,0}} \theta_{13}\theta_{24} \cdot \omega$ is indicated by 

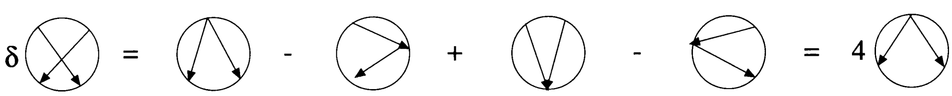
, and $\displaystyle\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}$ by 

Actually these diagrams stand for equivalence classes; changing the orientation of an arrow flips the sign and the change of labellings acts according to its effect on the orientation of $C_{r,s}$. In this notation the graphical transcription which correspond to the boundary and interchange relations:

$$
\begin{array}{ccc}
C_{r,s} & \begin{array}{c}\rightarrow\\ \cdots\\ \rightarrow\end{array} & C_{r+1,s} \\
\text{and} & & \\
C_{r,s}\times S^2 & \begin{array}{c}\rightarrow\\ \cdots\\ \rightarrow\end{array} & C_{r,s+1}\ ,
\end{array}
\qquad (53)
$$

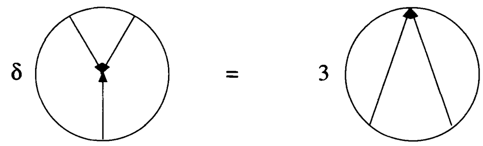
respectively, are as follows. Each boundary arrow corresponds to contracting of the corresponding arc on the circle. Each interchange arrow corresponds to a contraction of an edge joining an interior vertex with a vertex on the circle, or two interior vertexes.

In short, the combinatorial δ of F_ℓ assigns to every diagram D the linear combination "with appropriate signs" of the graphs obtained from D by these operations. For example,



(54)

and similarly



(55)

The rules of this complex are therefore very similar to the ones in Kontsevich's "Graph Cohomology" [K]. But notice that they differ from his general graph cohomology in the presence of the circle and the fact that not *all* edges are contracted when forming δ. In F_ℓ the edges joining two points of a circle – other than the "circle arcs" are *not contracted.* This is then some sort of relative version of Kontsevich's construction. His rules were of course devised for the construction of 3-manifold, rather than knot invariants.

At this stage we have arrived at a graphical complex which is also reminiscent of a complex occurring in the "weight system" approach to Vassilief invariants of Dror Bar-Natan, Birman-Lyn and others.

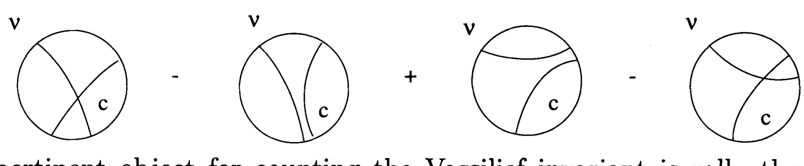
In this approach one works with the vector space of "chord diagrams" A', generated by graphs consisting of an oriented circle in which an even number of "vertexes", i.e., points, are "paired off", such a pairing being denoted by a chord joining the two vertexes. For example,

(56)

denote two generators of A' of weight $\ell = 3$. In this vector space A', the subspace A'' of "4-term relations" is singled out. Here a 4-term relation is generated in the following manner:

Starting with any chord diagram D, choose a chord c in D and introduce a new vertex v in the bounding circle. Now consider the four new chord diagrams obtained by adding to D a chord starting at v and ending "near the endpoints" of c.

There are clearly 4 of these and their "oriented" sum, as indicated below, is a generator of A'':



(57)

The pertinent object for counting the Vassilief invariant is rally the quotient space $A = A'/A''$, or in the terminology of Bar-Natan a subset of its dual space, the "weight systems" W on A.

These are linear forms on A which vanish on any diagrams having an "exposed chord", that is, a chord whose endpoints are joined by an arc of the boundary circle which contains no other vertexes of D. W is clearly a graded space

$$W = \bigoplus W_k$$

with W_k nontrivial only on diagrams with k chords.

Recall now that in the Vassilief scheme of things the space of Vassilief knot invariants V, is naturally filtered $V \supset V_1 \supset V_2, \cdots$ and the space of weight systems constitutes a natural "upper bound" for the graded group associated to this filtering, in the sense that there is a natural *inclusion*:

$$V_k/V_{k-1} \hookrightarrow W_k. \tag{58}$$

That this arrow is actually *onto* was proved by Kontsevich, using a beautiful but mysterious integral formula, see [BN], and constitutes the main existence theorem of the subject.

The nature of this integral, and its antecedents, make an interesting story, but I cannot go into here, except to say that it is arrived at from the "braid point of view" towards knot theory and thus ultimately related to configuration space $C_n(\mathbb{R}^2)$, rather than to $C_n(\mathbb{R}^3)$!

The combinatorial relation of W_k to our complex of tautologous forms F_k intrigued me all last year, and I managed to get Dylan Thurston – a senior at Harvard – interested in this question. His combinatorial gift soon resolved it by showing, among other things, that every "weight system" *induces* a cocycle of *dimension* 0 in *our complex F*.

To prove this relation, he uses an equivalent description of A in terms of the "Chinese character diagrams" introduced by Bar-Natan [BN]. These "Chinese character diagrams" actually looked just like the diagrams in our complex F, except that their orientation conventions seemed quite *different* to me. However, Dylan Thurston cleared this matter up in a most satisfactory way by asserting that if we endow the space A' with an inner product given by:

$$\langle D, D' \rangle = \#\text{ of isomorphisms of } D \text{ with } D', \tag{59}$$

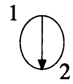
so that every weight system w has a "geometric realization" in A' given by:

$$\vec{w} = \sum w(D) \cdot \frac{D}{|D|} \qquad |D| = \#\text{ of automorphisms of } D; \tag{60}$$

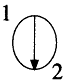
then $\vec{w}$ will be a cocycle in our complex.

After this extended excursion into combinatorial matters it is again time to return to the main question, namely whether the combinatorial cohomology groups $H^*(F_\ell)$, actually determine potential cohomology classes in $H^*(\mathcal{K})$.

Put differently, the question is whether the push forward vanishes on all hidden faces, for the complex F takes care precisely of all the *principal* ones.

Unfortunately, it does *not* seem to be true that all push forwards along all hidden faces vanish, but what Cliff Taubes and I were able to show in [B-T] is that for "connected diagrams" D in F the contributions of all hidden faces to the push forward is zero, except for the "maximally degenerate face" i.e., the one in which all the points approach each other. On the other hand, we *can show* that the contribution of this *"bad" face*, is always a multiple $\mu(D)$ times the self-linking integral 

!

From this it follows that at least in dimension 0, *every cocycle* α *in* $H^0(F_\ell)$ *can be modified by a suitable multiple* $\mu(\alpha)$ *of* 

to yield a knot invariant.

Combined with Dylan Thurston's assertion that each weight system gives rise to a cocycle in F, one therefore obtains a new proof of the basic existence theorem that $V_k/V_{k-1} \hookrightarrow W_k$ is onto, and so brings – to my mind – a much less mysterious though more pedestrian explanation for this fact than Kontsevich's integral does.

In higher dimensions, this basic "anomaly" of the maximally degenerate face, propagates more errors into the passage from $H^*(F)$ to $H^*(\mathcal{K})$, so that the best way to describe the situation is that $H^*(F)$ plays the role of the E_2-term of a spectral sequence converging to potential cohomology classes in $\mathcal{K}$.

Let me close this report, which has become much longer than I had expected, with some of the questions which I would very much like to have answered.

1. This first problem deals with the question of the anomaly. The number $\mu(D)$ as defined in [B-T] is given by a kynematical integral and will in general not be zero on a given diagram, D, but it is conceivable that for a cocyle $w = \sum a_D D$, we also have the relation $\sum a_D \mu(D) = 0$, whence *no* anomaly would exist in $H^0(F)$.
   In short, the first question is then: Do the anomalies cancel for cycles in $H^0(F_\ell)$?
2. Granting that the answer to 1 is yes, the next question is: Do all "rational" weight systems give rise to rational knot invariants?
3. Construct nontrivial elements in $H^i(F)$ for $i \geq 1$.
   No such classes are known to me, but I would expect at least one such class in dim 1. By a little known theorem of Hatcher the knot components are all $K(\pi, 1)$'s and the possible π's include $\mathbb{Z}$. Hence some components carry a nontrivial H^1 and it would be interesting if this H^1 came from a universal element in $H^1(F)$.
4. It would be interesting if some of the famous anomalies in the physics literature could also be traced to "hidden faces" in configuration spaces.

3. The configuration space invariants for imbeddings of S^{2k-1} in $\mathbb{R}^{2k+1}$, $k > 1$

To apply our general procedures to imbeddings of $X \subset Y$, we first of all need to find some "resonance" in the cohomological behaviour of the transfer diagram:

$$\begin{array}{ccc} \mathrm{Imb}(X,Y) \times C_k(X) & \longrightarrow & C_k(Y) \\ \Big\downarrow \pi & & \\ \mathrm{Imb}(X,Y) & & \end{array} \tag{61}$$

For this purpose we need to understand $H^*C_k(Y)$ and $H^*C_k(X)$. In the case at hand, $H^*(\mathbb{R}^m)$ is well known to be generated by the classes $\theta_{ij} = \varphi_{ij}^* \cdot \omega$ with

$$\varphi_{ij}(x) = \frac{x_j - x_i}{|x_j - x_i|}, \qquad \omega = \text{vol on } \mathrm{S}^{\mathrm{m}-1}, \tag{62}$$

in short the classes we have already met for $m = 3$. The behaviour of these forms is then quite uniform in m. On the other hand, $C_n(S^m)$ is homologically different for m even and m odd.

Note that in general, if $p \in S^m$ is a base point, then $S^m \times S^m - p \times S^m \cup S^m \times p$ is diffeomorphic to $\mathbb{R}^m \times \mathbb{R}^m$. If we now remove the diagonal Δ in $\mathbb{R}^m \times \mathbb{R}^m$ to obtain $C_2^0(\mathbb{R}^m)$, then our old map $\varphi_{12}C_2^0(\mathbb{R}^m) \to S^{m-1}$, can be used to pull back the volume form yielding a closed $(m-1)$-form η_{12}. Put differently, define the relative configuration space $C_n^0(S^m; p)$ as distinct "n-tuples of points in S^m, *which are also different from p.* Then the η_{ij} are defined on $C_n^0(S^m; p)$ and extend to the natural compactification of $C_n^0(S^m; p)$ to $C_n(S^m; p)$.

The space $C_n(S^m; p)$ is a blow-up over $C_n(S^m)$ and although the individual η_{ij} do not descend to $C_n(S^m)$ the sign in the diagonal map, $\Delta^m \to S^m \times S^m$, sees to it that for *odd spheres* $m = 2k - 1$, any *cyclic sum* of the η_{ij}'s *does* descend to $C_n(S^m)$. Thus, for instance, for $n = 3$, one has the closed form:

$$\eta_{12} + \eta_{23} + \eta_{31} \in H^{2k-2}(C_3(S^{2k-1})). \tag{63}$$

With these preliminaries understood, we are ready to write down the "first" knot invariant of a $(2k-1)$ knot in $\mathbb{R}^{2k+1}$ mentioned in the introduction:

$$\begin{aligned} \nu &= \frac{1}{4}\int_{C_{4,0}} \theta_{13}\theta_{24}(\eta_{12} + \eta_{23} + \eta_{34} + \eta_{41})^2 \\ &- \frac{1}{3}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}(\eta_{12} + \eta_{23} + \eta_{34}). \end{aligned} \tag{64}$$

Let us check at least that in these integrals "resonance" does occur. The dimension of the fiber in the first integral is $4 \cdot (2k-1) = 8k - 4$. The form to be integrated has dim $2k + 2k + 2(2k-2) = 8k - 4$.

In the second integral the fiber-dimension is $3(2k-1) + 2k + 1 = 8k - 2$ while the form is of dimension $3 \cdot 2k + 2k - 2 = 8k - 2$.

The proof that ν is a knot invariant follows the same pattern as before. One checks that on the principal faces the integrals for $d\nu$ cancel, and that on the hidden ones they vanish. On the other hand, at the time of this writing we do not know whether ν distinguishes between different knots.

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