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Contact structures and foliations on 3-manifolds

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Theory of foliations and contact geometry were developed practically independently, despite that the both theories exhibited a lot of striking similarities. In particular, it was understood in both cases that additional restrictions on foliations and contact structures are needed in order to make them interesting for applications. In the context of foliations this led to the theory of taut foliations ([22], [10]) and the related theory of essential laminations ([11]). In contact geometry there were studied tight contact structures ([3], [5], [12]) which exhibited a lot of similar properties. In this paper we study relations and analogies between the two structures. Proofs of the main results only sketched here. More detailed account of the subject can be found in our paper [9].

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1. Foliations, contact structures and confoliations

1.1. Definitions

Suppose ξ is a tangent plane field on a 3-manifold M. Then ξ can be defined (locally, if ξ is not co-orientable) by a Pfaffian equation $\{\alpha = 0\}$. The 1-form α is defined up to a multiplication by a non-vanishing function.

The Frobenius theorem tells that the necesary and sufficient condition for the integrability of ξ is the identity

$$\alpha \wedge d\alpha \equiv 0\,.$$

In this case the plane field ξ is tangent to a *foliation* of M by 2-surfaces. Usually we will call ξ itself a foliation, if it is integrable. If $\alpha \wedge d\alpha$ does not vanish, than ξ is nowhere integrable and is called *contact structure*. Notice that a manifold M which carries a contact structure automatically get oriented because the sign of the volume form $\alpha \wedge d\alpha$ is independent of the choice of α. On the other hand, if the manifold M is à priori oriented (and we will always assume it for the rest of the paper) than one can distinguish between *positive and negative* contact structures depending on whether $\alpha \wedge d\alpha > 0$ or $\alpha \wedge d\alpha < 0$.

The plane field ξ is called positive (resp. negative) *confoliation* if $\alpha \wedge d\alpha \geq 0$ (resp ≤ 0). Thus confoliations include contact structures and foliations on the two ends of the scale.

To understand the geometry of a 2-dimensional plane field and the geometric meaning of integrability, it is important to consider the traces which the plane field cut on 2-dimensional surfaces. Suppose F is a 2-dimensional surface in the 3-manifold M. If F is transverse to ξ then the line field $T(F) \cap \xi$ on F integrates into a 1-dimensional foliation F_ξ which is called *characteristic*. Generically, F and ξ has isolated points of tangency, and thus the characteristic foliation F_ξ still can be defined as a foliation with isolated singularities.

We will assume in this paper that ξ is co-orientable. A non-co-orientable distribution admits a co-orientable double cover, so the most of the results proven in the co-orientable case can be automatically re-proven in the non-co-orientable one. Of course, for local considerations this is irrelevant.

Suppose we are given a triangulation of M by small simplices such that each 1- and 2-dimensional simplex of the triangulation transverse to ξ. Thus the characteristic foliation on each 2-simplex is non-singular. Each 3-simplex σ of the triangulation has exactly two vertices p and q such the ξ_p and ξ_q are the supporting planes for the simplex. Let T be the edge connecting p and q. The holonomy along the leaves of the characteristic foliation $(\partial\sigma)_\xi$ defines a diffeomorphism $h_\sigma : T \to T$. If ξ is a foliation then all the holonomy maps h_σ are equal to the identity. For a general ξ this, of course, is not necessarily true, and it is important to consider two special cases:

a) $h_\sigma|_{\mathrm{int}T}$ is strictly decreasing (resp. increasing) in terms of the co-orientation of ξ;

b) h_σ is non-strictly decreasing (resp. increasing).

The case a) corresponds to contact structures, the case b) to confoliations.

It is a simple exercise to show that contact structures and foliations can be characterized by the properties of the holonomy maps h_σ, as it was explained above.

Both contact structures and foliations are locally homogeneous and admit a local normal form: foliations can be locally defined by the equation $dz = 0$, and contact structures by the equation $dz - ydx = 0$ (Darboux' theorem). In the case of contact structures it requires sometimes an orientation reversing change of coordinate to get this normal form. Confoliations, of course, need not to be locally homogeneous. It is important to observe a global distinction between contact structures and foliations. On a closed manifold there are no deformations of contact structures (see [13]), i.e. two contact structures which are homotopic as contact structures are isotopic. On the other hand, there are continuous families of pairwise non-equivalent foliations.

The following interpretation of the contact and confoliation conditions is quite useful. Suppose a 3-manifold M is split into the product $F \times \mathbb{R}$ for a 2-surface F, and the plane field ξ is tangent to the fibers of the projection $M \to \mathbb{R}$. The family of the characteristic foliations $\chi^t = (F \times t)_\xi$ on the surfaces F_t, $t \in \mathbb{R}$,completely defines ξ. Thus ξ can be thought of as a line field on a surface F which varies in time. Each foliation χ^t, $t \in \mathbb{R}$, can be defined by a non-vanishing 1-form α_t on F. The condition that ξ is a positive contact structure then reads as

$$\alpha_t \wedge \frac{d\alpha_t}{dt} > 0, \tag{1}$$

which means that for any point $p \in F$ the line χ_p^t through this point keeps rotating in the positive direction. Similarly, for a confoliation, the lines should keep rotating at the same direction; they can stop somewhere and so metimes, but never can turn backward.

Notice that locally any plane field is diffeomorphic to a field which is tangent to a parallel line field, and thus has a form considered above. Moreover, one can chose a local coordinate system (x, y, z) so that z-curves are transversal to ξ while y-curves are tangent to ξ. Then the distribution ξ at these coordinates can be defined by a 1-form

$$\alpha = dz - a(x, y, z)dx\,, \tag{2}$$

and

$$\alpha \wedge d\alpha = \frac{\partial a}{\partial y}(x, y, z)dx \wedge dy \wedge dz\,.$$

Thus we have

Proposition 1.1. *Let a distribution ξ given by the differential form (2). Then*

- *ξ is a foliation if and only if the function a is independent of y;*
- *ξ is a positive (resp. negative) contact structure if and only if*

$$\frac{\partial a}{\partial y} > 0 \quad (\textit{resp. } \frac{\partial a}{\partial y} < 0);$$

- *ξ is a positive (or negative) confoliation if the function a is non-strictly increasing (or decreasing).*

Another interpretation of integrability comes from the complex world. Let us recall that a hypersurface M in a complex manifold carries a canonical real 2-dimensional plane field ξ which is formed by complex tangencies to the boundary. If M is oriented the *strict pseudo-convexity* of M is equivalent to the fact that ξ is a positive contact structure. If ξ is non-strictly pseudo-convex then ξ is a positive confoliation.

2. Perturbation of foliations into contact structures

We will study in this section the possibility of perturbing a foliation $\xi = \{\alpha = 0\}$ into a contact structure. One can try to make the perturbation in three different senses.

We say that ξ can be *linearly deformed* into a positive contact structure if there exists a deformation $\xi_t = \{\alpha_t = 0\}$, $t \in \mathbb{R}_+$ such that $\alpha_0 = \alpha$ and $\frac{d(\alpha_t)}{dt}|_{t=0} > 0$, or in other words when

$$\langle \alpha, \beta \rangle \underset{\text{def}}{=} \alpha \wedge d\beta + \beta \wedge d\alpha > 0, \tag{3}$$

where $\beta = \frac{d\alpha}{dt}|_{t=0}$. It is important to observe that this condition depends on the foliation ξ only and not on the choice of the defining form α. Indeed, we have

$$\langle f\alpha, f\beta\rangle = f^2\langle\alpha,\beta\rangle$$

for any function f.

Conversely, if there exists a form β which satisfies the inequality (3) then the deformation $\alpha_t = \alpha + t\beta$ is the required linear deformation, which defines contact structures $\xi_t = \{\alpha_t = 0\}$ for small $t \neq 0$.

We say that ξ can be *(C^k-)deformed* into a contact structure if there exists a C^k-deformation ξ_t beginning at $\xi_0 = \xi$ such that ξ_t is contact for $t > 0$.

Finally we consider also *C^k-approximations* of ξ by contact structures when it will not be clear that this could be done via a deformation.

2.1. Linear deformations

Given a foliation $\xi = \{\alpha = 0\}$ let us try to find a 1-form β such that $\langle\alpha,\beta\rangle > 0$
Let us define a real-valued symmetric form $\langle\langle\alpha,\beta\rangle\rangle$ by integrating the 3-form $\langle\alpha,\beta\rangle$:

$$\langle\langle\alpha,\beta\rangle\rangle = \int_M \langle\alpha,\beta\rangle\,. \tag{4}$$

Notice that Stokes' theorem shows that

$$\langle\langle\alpha,\beta\rangle\rangle = 2\int_M \alpha\wedge d\beta = -2\int \beta\wedge d\alpha\,.$$

Proposition 2.1. *If a foliation ξ can be defined by a closed form α, or if it admits a closed leaf with trivial linear holonomy than ξ cannot be linearly perturbed into a contact structure.*

Proof: If α is closed then $\langle\langle\alpha,\beta\rangle\rangle = 0$ for any 1-form β. This proves the first part of the proposition. Suppose ξ has a closed leaf F with trivial linear holonomy. Then there exists a trivialization $U = F\times(-\varepsilon,\varepsilon)$ of a tubular neighborhood U of F, such that ξ transverse to the fibers. Let α be a defining form for ξ normalized on U by the condition $\alpha(\frac{\partial}{\partial t}) = 1$, where t is the coordinate along the fiber of the tubular neighborhood. Then we have $d\alpha = 0$ at the points of F. Hence $\langle\alpha,\beta\rangle = \alpha\wedge d\beta$ along F, and the condition $\langle\alpha,\beta\rangle > 0$ implies that the form $d\beta$ does not vanish on F which is impossible. □

Despite this negative result, linear perturbations often exist. Moreover, they exist in most interesting cases. The following theorem is proven in our paper [9].

Theorem 2.2. *Suppose that ξ is a foliation with holonomy, and each of its closed leaves has a curve with non-trivial linear holonomy. Then ξ can be linearly deformed into a contact structure.*

2.2. Non-linear deformations

We give here an example of a non-linear deformation of a foliation into a contact structure in the situation when the linear deformation does not exist.

Let ξ_0 be the foliation of the torus T^3 by the 2-dimensional tori $T^2 \times p$, $p \in S^1$. If $x, y, \theta \in [0, 2\pi)$ are coordinates on T^3, then ξ_0 is given by the Pfaffian equation $d\theta = 0$. It is straightforward to check that

Proposition 2.3. *For any integer $n > 0$, and any $t > 0$ the form*

$$\alpha_n^t = d\theta + t(\cos n\theta dx + \sin n\theta dy)$$

defines a contact structures on T^3.

According to a theorem of Giroux (see [12]), also independently proved by Kanda (see [17]), the contact structures $\xi_n = \{\alpha_n^1 = 0\}$ are pairwise non-diffeomorphic. On the other hand, according to Gray's theorem [13] the contact structures $\{\alpha_n^t = 0\}$ are isotopic, when n is fixed and t varies in $(0, \infty)$. Thus

Proposition 2.4. *The foliation ξ_0 can be C^∞-deformed into an infinite number of non-equivalent contact structures.*

Remark 2.5. It is likely that any foliation on an orientable manifold can be deformed into a contact structure. However, we are able to prove only a weaker approximation result (see 2.18 which we consider in the Section 2.4 below).

2.3. Dynamics of codimension one foliations

2.3.1. Basic notions of the theory of codimension one foliations

Although we are concerned in this paper only with the 3-dimensional manifolds, all the results which we discuss in this section holds for codimension 1 foliations on manifolds of arbitrary dimensions.

In view of the Reeb stability theorem a foliation on a closed manifold which contains a closed simply connected leaf is a fibration over a circle, and thus in the three-dimensional (orientable) case it has to be diffeomorphic to the foliation ζ on $S^2 \times S^1$ which we excluded from the considerations. Thus we will assume without further notice, that the foliations which we consider has no simply-connected closed leaves.

Let ξ be a co-oriented codimension 1 foliation on a closed manifold M. Given an oriented embedded curve Γ which is contained in a leaf S of the foliation, we define the *holonomy* along Γ as a germ at 0 of a diffeomorphism $\mathbb{R} \to \mathbb{R}$ as follows. Take an annulus $A = \Gamma \times [-1, 1]$ with $\Gamma = \Gamma \times 0$ which is transversal to ξ. Then ξ defines on A the characteristic foliation A_ξ which has Γ as its closed leaf. Then the holonomy is the germ of the corresponding Poincaré return map which is well defined and depends only on the homotopy class of the loop Γ in S.

Foliation without holonomy is a foliation for which the holonomy along any curve is trivial. We say that the holonomy along Γ is *non-trivial* if it is different from the identity.

For a co-oriented foliation we say that the holonomy is *non-trivial on the positive side* if it is not the identity being restricted to $\mathbb{R}_+$. We assume here that the splitting of A respects the given co-orientation of ξ. We say that the holonomy is *contracting on positive side* if it is contracting in a neighborhood of 0 in $\mathbb{R}_+$. We say that Γ has a non-trivial *linear holonomy* if the differential of the holonomy map at 0 is different from 1. Finally we say that the holonomy is *weakly contracting* (resp. weakly contracting on the positive side) if the holonomy diffeomorphism has arbitrarily close to 0 intervals (resp. intervals on the positive side of 0) where it is contracting.

Remark 2.6. If a holonomy curve Γ has non-trivial holonomy then arbitrary close to Γ one can find another curve (in a different leaf) which has contracting holonomy on its positive side.

2.3.2. Structure of foliations without holonomy

Let Y be a 1-dimensional manifold. A *foliated Y-bundle* over a manifold N is a pair (W, ξ) where W is a fibration over N with a fiber Y and ξ is a codimension 1 foliation on W which is transversal to the fibers of the fibration.

Theorem 2.7. (See [15] and [20]) *Let (M, ξ) be a foliation without holonomy. Then there exists a foliated trivial bundle $(T^n \times S^1, \eta)$ without holonomy and a map $h : M \to T^n \times S^1$ transversal to η which induces ξ from η.*

Corollary 2.8. *If (M, ξ) is as in Theorem 2.7 and C^2-smooth then it can be C^0-approximated by a fibration over a circle.*

Proof: According to [2] there exists a homeomorphism $g : S^1 \to S^1$ such that the diffeomorphism $\mathrm{id} \times g$ sends η to a linear foliation η_0 on the torus $T^{n+1} = T^n \times S^1$. We can C^0-approximate g by a diffeomorphism $\tilde{g}$, and approximate η_0 by a linear foliation $\tilde{\eta}_0$ with a rational slope. Then $\tilde{\eta}_0$ is a fibration $p : T^{n+1} \to S^1$. If the chosen approximations are sufficiently good then the map h constructed in 2.7 is still transversal to the foliation $\tilde{\eta}_0 = \mathrm{id} \times \tilde{g}^*(\tilde{\eta}_0)$, and therefore the foliation $\tilde{\xi} = h^*(\tilde{\eta}_0)$ which is C^0-close to ξ is the foliation by the fibers of the fibration $p \circ \tilde{h} : M \to S^1$. □

2.3.3. Foliations with holonomy

A minimal set in a foliated manifold (M, ξ) is a closed subset which consists of whole leaves and which does not contain non-empty smaller subsets with these properties. A minimal leaf is called *exceptional* if the intersection of A with any transversal is a nowhere dense Cantor set.

The center $Z(\xi)$ of ξ is the union of all its minimal sets. The union of all closed leaves of ξ will be denoted by $C(\xi)$.

Theorem 2.9. (see [15]) *Let A be a minimal set. Then A is either the whole manifold M, or a closed leaf, or an exceptional minimal set. The sets $C(\xi)$ and $Z(\xi)$ are closed. Each exceptional minimal set is isolated in $Z(\xi)$, i.e. has a neighborhood which contains no other minimal sets. In particular, there are only finitely many exceptional minimal sets.*

The previous theorem holds without any assumptions about the smoothness of the foliation. The next theorem which belongs to R. Sacksteder, requires that the foliation is at least C^2-smooth.

Theorem 2.10. (see [21]) *Any exceptional minimal set of a C^2-foliation contains a curve with non-trivial linear holonomy.*

We finish this section by the folowing simple property of the set of closed leaves.

Proposition 2.11. *Any C^k-foliation can be C^k-approximated by a foliation with only finite number of closed leaves.*

Proof: Take a closed leaf S. Then it is either isolated in $C(\xi)$, or at least on one of its sides there are other closed leaves, arbitarily close to S. Take one of these leaves, say S'. If S' is chosen sufficiently close to S then S and S' bound a foliated I-bundle U over S which is C^k-close to the trivial horizontal foliation. Let us delete this foliation and insert instead a foliated I-bundle, C^k-close to (U, ξ), which has no interior closed leaves. Repeat this procedure, if necessary, on the other side of S. This operation destroy all closed leaves in a neighborhood of S and do not change the structure of the set $C(\xi)$ outside of this neihborhood. Thus compactness of $C(\xi)$ garantees us that in a finite number of steps we get a foliation with isolated closed leaves. □

2.3.4. Laminations

Let A be a closed subset of a 3-manifold M. A tangent to M plane field ξ given at the points of A is called a *lamination* if for each point $p \in C$ there exists a surface F and an almost proper embedding $F \to M$ which is tangent to ξ and contains the point p. Let us recall that a map is called almost proper if all connected components of the preimage of a compact set are compact.

The following proposition shows how laminations interfere into the study of plane fields on 3-manifolds.

For any plane field ξ on a closed manifold M we denote by $\mathrm{Fol}(\xi)$ its *fully foliated part*, i.e. the set of points $q \in M$ such that there exists an almost proper embedding of a surface F into M which contains the point q.

Proposition 2.12. *The set* $\mathrm{Fol}(\xi)$ *is closed, and the restriction of* ξ *to* $\mathrm{Fol}(\xi)$ *is a lamination.*

Proof: Fix a Riemannian metric in M. It is sufficient to prove that there exists an $\varepsilon > 0$ that each point $p \in \overline{\mathrm{Fol}(\xi)}$ is contained in an integral surface S_p such that $S_p \cap B_\varepsilon(p)$ is a closed submanifold of the ball $B_\varepsilon(p)$ of radius ε, centered at the point p. This is true, by the definition, for $p \in \mathrm{Fol}(\xi)$. Take a boundary point $p \in \partial\mathrm{Fol}(\xi)$, and let $p_k, k = 1, \dots,$ be a sequence of points of $\mathrm{Fol}(\xi)$ which converges to p. Notice that in view of compactness of M, the normal curvatures of any integral surface of the plane field ξ are à priori bounded by a constant, which depends only on ξ. Let D_k, $k = 1, \dots,$ be the intersection of the leaf of ξ through the point p_k with the ball $B_\varepsilon(p)$. If ε is sufficiently small and k is large then D_k is a disc. The sequence of these discs converges to a C^1-smooth disc D through the point p which is automatically integral for ξ. Notice that à posteriori we can conclude that the disc is as smooth as the plane field ξ. □

The results from the previous section about the foliations with-holonomy hold with practically no changes for laminations. Notice that the notion of non-trivial linear holonomy has sense even for the laminations. We say that a lamination (F, ξ) has non-trivial linear, say contracting holonomy along an integral curve Γ if, given a transversal curve T through a point $p \in \Gamma$, the point p is an accumulation point of $F \cap T$ from both sides, and the germ h at p of the holonomy map $F \cap T \to F \cap T$ satisfies the inequality $h(x) < cx$ for all $x \in F \cap T$ for a positive constant $c < 1$.

The following lemma contains a summary of the results about the dynamics of laminations, which we need for our purposes.

Lemma 2.13. *Let (A, ξ) be a C^k-lamination, $k > 1$. All minimal sets of the lamination ξ are either closed leaves or exceptional sets. The union $Z(\xi)$ of all minimal sets, as well as the union $C(\xi)$ of all closed leaves are compact. All exceptional minimal sets are isolated and each of them contains a curve, along which ξ has non-trivial linear holonomy.*

As in the case of foliations, only the last statement about linear holonomy requires an assumption about the smoothness of ξ.

2.4. Approximations of foliations by contact structures

2.4.1. Reeb stability theorem for confoliations

Let us denote by ζ the product-foliation (fibration) of the manifold $S^2 \times S^1$ by the spheres $S^2 \times z$, $z \in S^1$.

Proposition 2.14. a) *Let ξ be a confoliation on $M = S^2 \times S^1$ which transverse to the circles $u \times S^1, u \in S^2$. Then ξ is a foliation diffeomorphic to the fibration ζ.*

b) *Let ξ be a confoliation on $M = S^2 \times \mathbb{R}$ which transverse to the lines $u \times \mathbb{R}, u \in S^2$, and is tangent to the sphere $S = S^2 \times 0$. Then there exists a neighborhood U of the the sphere S such that $\xi|_U$ is a foliation diffeomorphic to the fibration of $S^2 \times I$ by the spheres $S^2 \times v$, $v \in I$.*

Proof: We prove here the case b). The case a) is similar. Present S^2 as the union of two hemispheres $S^2 = D_+ \cup D_-$. Set $M_\pm = D_\pm \times \mathbb{R}$. Changing, if necessary, the splitting $M_+ = D_+ \times \mathbb{R}$ we can arrange that in the cylindrical coordinates $z \in \mathbb{R}$, $\rho \in [0,1], \theta \in [0, 2\pi)$, corresponding to the splitting the confoliation $\xi|_{M_+}$ is given by the form $\alpha_+ = dz + f(z, \rho, \theta)d\theta)$, where the function f vanishes on the axis $\{\rho = 0\}$. According to Proposition 1.1 the confoliation condition amounts to the inequality $\frac{\partial f}{\partial \rho} \geq 0$. Hence, $f(z, 1, \theta) \equiv 0$. Let φ be the germ at 0 of the holonomy diffeomorphism $\mathbb{R} \to \mathbb{R}$ along the equator $E = \partial D_- = \partial D_+$ oriented as the boundary of D_+. Notice that φ is the time 2π map of the flow Φ_t which starts at $\Phi_0 = \text{id}$ and defined by the differential equation

$$\frac{d\Phi_t(\Phi_t(z))}{dt} = -f(z, 1, t)\,.$$

Thus we have $\varphi \leq \text{id}$. On the other hand the similar argument applied to $\xi|_{M_-}$ gives the opposite inequality $\varphi \geq \text{id}$. Hence, in a neighborhood of S the holonomy defined by the characteristic foliation on any cylinder $\Gamma \times \mathbb{R}$ is trivial for any circle $\Gamma \subset S^2$. This alllows to construct diffeomorphism of the restriction of the foliation ξ to a neighborhood of S onto the restriction of the fibration ζ to a neighborhood of one of its fibers. □

Corollary 2.15. *Any confoliation on $S^2 \times S^1$ which is C^0-close to the foliation ζ is a foliation, diffeomorphic to ζ.*

In particular, ζ cannot be approximated by contact structures.

Proposition 2.16. ("Reeb stability theorem" for confoliations). *Suppose that a confoliation ξ on a closed oriented manifold M has an integral embedded 2-sphere S. Then ξ is a foliation and (M, ξ) is diffeomorphic to $(S^2 \times S^1, \zeta)$.*

Proof: According to Proposition 2.14b) the restriction of ξ to a neighborhood of the leaf S is a foliation by spheres. Moreover, it follows that the subset F of M, where ξ is a foliation by 2-spheres, is open. On the other hand, according to 2.12 and 2.9 the set F is closed. Hence $M = F$ and ξ is a fibration by 2-spheres, which implies that (M, ξ) is diffeomorphic to $(S^2 \times S^1, \zeta)$. □

Corollary 2.17. *Let ξ be a confoliation on the 3-ball B which is standard (i. e. coincides with the foliation of the round ball by planes) near the boundary. Then ξ is a foliation and it is diffeomorphic to the standard one.*

In other words, one cannot perturb a foliation into a confoliation by a local perturbation near a point.

Proof: Take the foliation ζ on $S^2 \times S^1$ and a small round ball B which does not intersect one of the spheres $S^2 \times z_0$, $z_0 \in S^1$, and implant instead of B with the standard foliation $\zeta|_B$ a confoliated ball (B, ξ). The new confoliation is still tangent to the sphere $S^2 \times z_0$, and thus, according to 2.16, it is a foliation diffeomorphic to ζ. It follows then that the confoliation (B, ξ) is a foliation diffeomorphic to the standard one. □

2.5. Existence of C^0-perturbations of foliations into contact structures

According to Corollary 2.15 the foliation ζ on $S^2 \times S^1$ does not admit a C^0-approximation by contact structures (and even by somewhere positive confoliations). Moreover, Proposition 2.1 shows that in certain cases a foliation cannot be *linearly deformed* into a contact structure. Despite these negative facts we have

Theorem 2.18. *Any C^2-confoliation ξ on an oriented 3-manifold, different from the foliation ζ on $S^2 \times S^1$, can be C^0-approximated by contact structures. When ξ is a foliation then it can be approximated both, by positive and negative contact structures.*

We will give in this paper only sketch of the proof for the case when ξ is a foliation. The details can be found in [9].

The perturbation procedure consists of few steps.

Let M be a manifold with a positive confoliation $\xi = \{\alpha = 0\}$. Let

$$H(\xi) = \{x \in M \mid \alpha \wedge d\alpha(x) > 0\}$$

be the contact part of M. For a contact structure ξ we have $H(\xi) = M$.

Let us denote by $\hat{H}(\xi)$, $\hat{H}(\xi) \subset M$, the set of points $p \in M$ which satisfy the following condition

- there exists an embedded curve Γ_p, tangent to ξ, which begins at p and ends at a point $p' \in H(\xi)$.

A confoliation ξ on a compact 3-manifold M is called a *contact-to-be* confoliation if $\hat{H}(\xi) = M$.

The following Proposition 2.19 belongs to Steve Altschuler (see [1]). His proof uses existence results for a linearized heat type equation associated to the problem. In [9] we give a more direct geometric proof of this result.

Proposition 2.19. *Any C^k-smooth contact-to-be confoliation, $k \geq 1$, can be C^k-approximated by by a contact structure.*

Theorem 2.18 follows from 2.19 and the following

Proposition 2.20. *Any C^2-confoliation can be C^0-approximated by a contact-to-be confoliation.*

The following Proposition 2.21 plays the central role in the proof of 2.20.

Let Γ be an embedded closed curve in a leaf S of a co-oriented foliation ξ. By a *positive (resp. negative) semi-neighborhood* of Γ we mean the positive (resp. negative) component of $U \setminus S$ for a neighborhood U of Γ.

Proposition 2.21. *Let (M, ξ) be a C^k-foliation, $k \geq 0$.*

a) *Suppose Γ is a curve with non-trivial linear holonomy. Then ξ can be C^k-perturbed into a positive or negative (at our choice) confoliation which is contact in a neighborhood U of Γ, coincides with ξ outside of a bigger neighborhood, and diffeomorphic to ξ outside of U.*

b) *Suppose the curve Γ has contracting holonomy on its positive side. Then there exists a positive (or negative) confoliation $\tilde{\xi}$ which is contact in a positive semi-neighborhood U_+ of Γ, C^0-close to ξ, diffeomorphic to ξ outside U_+, and coincides with ξ outside a bigger positive semi-neighborhood.*

c) *Suppose Γ has weakly contracting (two-sided) holonomy. Then one can C^0-perturb ξ into a positive (or negative) confoliation which is contact in a neighborhood U of Γ, diffeomorphic to ξ outside of U, and coincides with ξ outside a bigger neighborhood.*

We prove here only the case a). Proofs of b) and c) can be found in [9].

Proof: We will prove here even a stronger version of 2.21a). Namely we will show that the required approximation can be done via a linear deformation. According to 2.1 this amounts to finding a form β on a neighborhood $U \supset \Gamma$ which vanishes outside U and satisfies the inequality

$$\langle \alpha, \beta \rangle > 0$$

inside U. Let us split a neighborhood U into the product $\Gamma \times [-1,1] \times [-1,1]$, so that the foliation $\xi|_U$ can be defined in a neighborhood by the form

$$\alpha = dz + v(x,z)dx,$$

where $\frac{\partial v}{\partial z} \geq C$ for a positive constant $C > 0$. Take a monotone smooth function $h : \mathbb{R} \to [0,1]$ which is equal to 1 near 0, positive on $[0,1)$, and vanishes on $[1,\infty)$. It is straightforward to check that the form $\beta = h(x^2 + z^2)dy$ satisfies the inequality

$$\langle \alpha, \beta \rangle > 0$$

everywhere inside U. □

Proof of Proposition 2.20:

The foliation ξ can be either with or without holonomy. Consider first the case of a foliation with holonomy. If ξ has everywhere dense leaves then by perturbing ξ into a contact structure in any neighborhood and keeping it diffeomorphic to ξ otside the neighborhood, we transform ξ into a contact-to-be confoliation. Thus take any curve Γ with non-trivial holonomy, say on positive side. According to Remark 2.6 we can choose Γ in such a way that the holonomy is contracting on the positive side. Then apply Proposition 2.21 to perturb ξ into a confoliation $\tilde{\xi}$ which is contact in a positive semi-neighborhood of Γ and diffeomorphic to ξ outside this neighborhood. Then $\tilde{\xi}$ is a contact-to-be confoliation.

Suppose now that ξ has minimal sets different from the whole M. According to Theorem 2.9 ξ has finitely many exceptional minimal sets and according to Proposition 2.11 one can perturb ξ into a foliation with only finite number of compact leaves as well. Sacksteder's theorem 2.10 asserts that each exceptional minimal set has a curve with non-trivial holonomy. Each isolated closed leaf S has a curve Γ_+ with weakly contracting holonomy on the positive side, and possibly a different curve Γ_- with weakly contracting holonomy on the other side. Moreover, one can arrange that these curves form a part of the standard basis of the fundamental group of the surface S.

There exists a foliated I-bundle $(S \times I, \eta)$ which has exactly one interior closed leaf with non-trivial linear holonomy and which has contracting holonomy along the curves $\Gamma_+ \times 1$ and $\Gamma_- \times 0$. To construct such bundle take any diffeomorphism $g : I \to I$ such that $g(1/2) = 1/2$, $g'(1/2) > 1$, g is C^∞-tangent to the identiy near the boundary ∂I and has no other interior fixed points except the point $1/2$. There exists a representation of $\pi_1(S)$ into $\mathrm{Diff}(I)$ which sends the homotopy classes of the curves Γ_+ and Γ_- to the diffeomorphism g. The required foliated bundle corresponds to this representation.

We can cut now the manifold M open along S and insert the bundle $(S \times I, \eta)$. The new foliation has three closed leaves instead of one. However the central leaf has a curve with non-trivial linear holonomy, and the two other leaves have curves with weakly contracting two-sided holonomy. Notice that by choosing the interval I sufficiently small one can make the new foliation to be C^∞-close to the original one. Repeating this procedure we get a foliation for which each closed leaf and each exceptional minimal set contains a curve with either non-trivial linear holonomy or with two-sided weakly contracting holonomy.

Thus we can apply the cases a) and c) of Proposition 2.21 to perturb ξ into a confoliation which is contact in neighborhoods of these curves and which is diffeomorphic to ξ outside the neighborhoods. By the definition of a minimal set any leaf of ξ come arbitrarily close to one of the chosen curves. Hence the constructed confoliation is contact-to-be.

Consider now the case when ξ is a foliation without holonomy. According to Corollary 2.8 the foliation ξ can be approximated, in this case, by a fibration. Hence, using Proposition 2.11 we can further approximate ξ by a foliation with finite number of closed leaves and no other minimal sets. Thus we returned to the case of a foliation with holonomy which we already considered above. This finishes off the proof for the case of a foliation. □

3. Taut and Tight

A contact structure ξ on a 3-manifold M is called *overtwisted* (see [4]) if there exists an embedded disc $D \in M$ such that ∂D is tangent to ξ but D itself is transversal to ξ along ∂D. A contact structure is called *tight* if it is not overtwisted.

As it was shown in [4], [5], [12] tight and overtwisted structures constitutes two completely different worlds. Overtwisted structures are very flexible and abide an h-principle: the isotopy classification of overtwisted structures coincides with their homotopy classification as tangent plane distributions (see [4]). On the other hand tight contact structures exhibit a lot of rigidity properties. For instance, on some manifolds (like $S^3, \mathbb{R}P^3, \mathbb{R}^3, S^3 \times S^1$) there exists a unique, up to isotopy, contact structure (see [5]). Another manifestation of rigidity of tight contact structures is the inequality which we formulate in Theorem 3.8 below.

Moving to the other end of the confoliation scale, i.e. to the foliations, let us recall that a foliation ξ is called *taut* (see [22] and [10]) if it is different from the foliation ζ on $S^2 \times S^1$, and satisfies any of the following equivalent properties:

O_1: each leaf is intersected by a transversal closed curve;

O_2: there exists a vector field X on M which is transversal to ξ and preserve a volume form Ω on M;

O_3: ξ has no generalized Reeb components;

O_4: M admits a Riemannian metric for which all leaves are minimal surfaces.

Generalized Reeb component is a foliation η on the manifold $F \times S^1$, where F is a compact surface with boundary, which characterized by the following properties:

- near each component of the boundary η behaves as the standard Reeb foliation on $D^2 \times S^1$ near its boundary.;
- for a slightly smaller surface $F' \subset F$ the restriction $\eta|_{F' \times S^1}$ is diffeomorphic to the fibration by surfaces $F' \times z$, $z \in S^1$.

3.1. Symplectic filling

Let (M, ξ) be a confoliation and ω a closed 2-form on M. We say that ω *dominates* ξ if $\omega|_\xi$ does not vanish.

For a foliation the existence of the dominating form ω is equivalent to the definition O_2 of a taut foliation. Indeed, if the vector field X transverse to ξ and preserves Ω then the closed 2-form $\omega = X \rfloor \Omega$ dominates ξ. Conversely, suppose ω dominates $\xi = \{\alpha = 0\}$. Then the vector field X, such that $X \rfloor \omega = 0$ and $\alpha(X) = 1$, transverse to ξ and preserves the volume form $\Omega = \omega \wedge \alpha$.

Thus,

O_2': taut foliations are exactly those which admits a dominating 2-form.

Suppose now that a 3-manifold M with a positive confoliation ξ bounds a symplectic 4-manifold (W, ω). We call (W, ω) a *symplectic filling* of the confoliated manifold (M, ξ) if $\omega|_M$ dominates ξ and M is oriented as the boundary of the canonically oriented symplectic manifold (W, ω). A confoliated manifold which admits a symplectic filling is called *symplectically fillable.* It is called *symplectically semi-fillable* if it is a connected component of a symplectically fillable confoliated manifold.

Of course, when ξ is a foliation then the orientation condition is irrelevant.

Now we observe that

Proposition 3.1. *Taut foliations are symplectically semi-fillable.*

Proof: Let ω be a dominating 2-form for the taut foliation $\xi = \{\alpha = 0\}$ on M. Set $W = M \times [0, 1]$, and define a closed 2-form $\tilde{\omega} = p^*\omega + \varepsilon d(t\alpha)$, where p is the projection $W \to M$. When $\varepsilon > 0$ is small then the form $\tilde{\omega}$ is non-degenerate and dominates ξ on $\partial W = M \times 0 \cup M \times 1$. □

Notice here that the orientation defined by $\tilde{\omega}$ on the boundary of W coincides with the orientation of M on $M \times 1$ and opposite to it on $M \times 0$.

Remark 3.2. It is possible that all taut foliations are symplectically fillable even in the strong sense. However, this is unknown.

As it was proven in [14] and [6]

Theorem 3.3. *Symplectically semi-fillable contact structures are tight.*

Corollary 3.4. *Contact structures, C^0-close to a taut foliation, are symplectically fillable and, therefore, tight.*

Proof: Symplectic fillableness is a C^0-open condition. □

The following theorem of D. Gabai (see [10]) shows that there is a lot of closed 3-manifolds which admits taut foliations. Namely, he proved

Theorem 3.5. (D. Gabai, [10]) *Let M be an irreducible 3-manifold with non-trivial homology group $H_2(M)$. Let $F \subset M$ be an oriented surface which represents a non-trivial homology class and has minimal genus among surfaces representing the same class. Then there exists a taut foliation ξ on M which has F as one of its leaves.*

Thus from 3.5, 2.18 and 3.4 we can conclude

Corollary 3.6. *Any 3-manifold M with $H_2(M)/\pi_2(M) \neq 0$ admits a tight contact structure.*

Remark 3.7. In [7] there was described a class of index 1 and 2 surgeries of contact manifolds. These surgeries have the following important property: if the original contact manifold is (weakly) symplectically fillable then after the surgery it is (weakly) symplectically fillable as well. However, it is unknown whether the tihgtness condition is preserved by these surgeries. Proposition 3.1 shows that when performing contact surgeries on contact manifolds which are deformations of taut foliations, we remain in the class of symplectically fillable and, therefore, tight contact structures.

3.2. The inequality

It sounds surprising that while the definitions of a tight contact structure look completely unrelated, they both satisfy a remarkable inequality which was independently proven in the case of taut foliations by Thurston in [22] and in the case of tight contact structures by Bennequin and Eliashberg (see [3] and [5]).

Let ξ be any orientable tangent plane distribution on an oriented 3-manifold M, and $F \subset M$ an orientable 2-surface which is either closed or have a boundary Γ transversal to ξ. If F is closed then we orient F and ξ anyhow, but if $\Gamma \neq \emptyset$ then the orientations of M, F and ξ should be related in the following way:

Let us orient Γ as the boundary of the oriented surface F. Then at a point $x \in \partial F$ the orientation of the plane $\xi(x)$ together with the orientation of $T_x(\Gamma)$ should give the orientation of the tangent space $T_x(M)$.

We denote by $\chi(F)$ the Euler characteristic of F and by $e(\xi)[F]$ the value of the Euler class $e(\xi) \in H^2(M)$ evaluated on F in the case when F is closed, and the relative Euler number of $\xi|_F$ if $\Gamma \neq \emptyset$. The relative Euler number can be defined as follows. Consider a vector field X along Γ which generates the line field $\xi \cap T(F)$. Then $e(\xi)[M]$ is the obstruction for the extension of X to F as a vector field in ξ.

Notice that the number $e(\xi)[M]$ remains unchanged when the orientations of ξ and F are simultaneously reversed. In particular, in the case of the surface with boundary our orientation agreement shows that $e(\xi)[M]$ is determined by the choice of the orientation of M.

Theorem 3.8. *Let ξ be a tight positive contact structure or a taut foliation on an oriented 3-manifold M. If F is a closed embedded orientable 2-surface $F \subset M$, which is different from S^2 then the following inequality holds:*

$$|e(\xi)[F]| \leq -\chi(F)) . \tag{5}$$

If $F = S^2$ then

$$e(\xi)[F] = 0 . \tag{6}$$

If F is a surface with boundary transversal to ξ then we have the inequality

$$e(\xi)[F] \leq -\chi(F) . \tag{7}$$

Notice that in view of 3.4 and 3.12 the case of taut foliations in Theorem 3.8 follows from the case of tight contact structues.

The inequality (5) implies (see [6])

Corollary 3.9. *For any closed manifold M only finitely many cohomology classes from $H^2(M)$ can be represented as Euler classes of tight contact structures.*

It is still unknown whether only finite number of homotopy classes of plane fields can be represented by tight contact structures. However, recently Kronheimer and Mrowka [18] proved, using Seiberg-Witten equation that *weakly symplectically fillable contact structures may represent only finite number of homotopy classes of plane fields.* This result, together with 2.18 and 3.4 implies

Corollary 3.10. *Only finitely many homotopy classes of tangent plane fields are representable by taut foliations.*

The natural question which arises in connection with Theorem 3.8 is to describe the general class of *confoliations* for which the above inequalities hold. Let us denote by $\mathcal{T}$ the class of all confoliations which satisfy 5–7. The intersection of the class $\mathcal{T}$ with the class of contact structures consists exactly of tight contact structures. However, in the case of foliations the class $\mathcal{T}$ contains more than just taut foliations. In fact, Thurston proved in [22] the inequalities (5)–(7) for a larger class of foliations, namely for foliations without Reeb components. However, the following example suggests that even when certain Reeb components are allowed the foliation still can belong to $\mathcal{T}$.

Example 3.11. Let $M = S^3$, and ρ be the Reeb foliation. Actually, up to an orientation preserving homeomorphism, there are two different Reeb foliations (notice that there is a continuum of *non-diffeomorphic* Reeb foliations). To distinguish between them let us choose a co-orientation of ρ and pick positively oriented closed transversals T_1 and T_2 in each of the Reeb components of the foliation. We call the Reeb foliation positive or negative depending on the sign of the linking number $l(T_1, T_2)$. These two Reeb foliations will be denoted by ρ_+ and ρ_-. We leave it as an exercise to the reader to check that

the positive Reeb foliation ρ_+ belongs to the class $\mathcal{T}$ while the negative one does not.

It is interesting to observe that a perturbation of ρ_+ into a positive contact structure (which exists according to Theorem 2.18) is tight while its perturbation into a negative contact structure is overtwisted.

Our last (and obvious) observation is

Proposition 3.12. *Class $\mathcal{T}$ is closed in C^0-topology.*

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