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STAR TOPOLOGICAL GROUPOIDS

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Abstract

In [4] a construction on topological groups was given. In this paper we generalize this costruction to more general topological groupoids and have a similar structure on the topological groupodis.

Introduction

Let G be a topological group and W an open neighbourhood of the identity e in G. Let $F(W)$ be the free group on W and N the normal subgroup of $F(W)$ generated by the elements of $F(W)$ in the form $[vu]^{-1}[v][u]$ such that u, v and vu belong to W, where for example $[u]$ is the equivalence class of u in $F(W)$. Note that where vu is the group multiplication in G while $[v][u]$ is the free group multiplication in $F(W)$. Suppose that $\tilde{G}$ is the quotient group of $F(W)$ by N. Thus we have an inclusion map $\tilde{i} : W \to \tilde{G}$ and a projection map $p : \tilde{G} \to G$.

In [4] it was proved that the group $\tilde{G}$ can be given a topology such that right translations are homeomorphisms. In general $\tilde{G}$ is not a topological group. However they prove that the projection map $p : \tilde{G} \to G$ is a covering map.

In this paper we generalize this construction to the topological groupoid case and from a topological groupoid G and an open subset W of G have a groupoid $M(G, W)$ which is not a topological groupoid but each fibre $(MG)_x$ of the initial point map has a topology such that right translations are homeomorphisms. This construction on $M(G, W)$ is called star topological groupoid (Definition 2). For groupoids and topological groupoids we refer to [6].

MONODROMY GROUPOID

A *groupoid* is a small category in which each morphism has a both sided inverse. In a groupoid G we write α and β for the initial and final maps respectively, write G_x for $\alpha^{-1}(x)$ and write $G(x, y)$ for the set of all morphisms from x to y. For a groupoid G let $I(G)$ be the set of all identities. We identify the set of object O_G with $I(G)$ and so sometimes write O_G for $I(G)$.

A *topological groupoid* is a groupoid G in which the set of morphisms G and the set of objects O_G are both topological spaces and all groupoid structure maps are continuous.

Definition 1. *Let G and H be groupoids. A local morphism of groupoids is a map $f : W \to H$ from a subset $W \subseteq O_G$ of G such that for $u \in W$, $\alpha_H(fu) = f(\alpha_G u), \beta_H(fu) = f(\beta_G u)$ and $f(vu) = f(v)f(u)$ whenever $u, v \in W$ and vu is defined and belongs to W. Suppose G and H are both topological groupoids and W is an open neighborhood of O_G. We call a continuous map $f : W \to H$ local morphism of topological groupoids if it is a local morphism on underlying groupoids.*

The following construction is similar to that of [4] stated in the introduction (see also [7] for more details and background of this area).

Let G be a topological groupoid and let W be an open subset of G such that $O_G \subseteq W$. The graph structure on W inherited from the groupoid structure gives a free groupoid $F(W)$. Let N be the normal subgroupoid of $F(W)$ generated by the elements of the form $[vu]^{-1}[v][u]$ for $(v,u) \in W \times W$ such that vu is defined and in W. The quotient groupoid $F(W)/N$ is denoted by $M(G,W)$ and called *monodromy groupoid* of G for W. The elements of $F(W)$ are written as $[u]$ and those of $M(G,W)$ are written as $< u >$ when $u \in W$ (See [3] and [5] for the details of free groupoids and quotient groupoids).

Thus by the construction of the monodromy groupoid the inclusion map $i : W \to G$ determines an injection $\tilde{i} : W \to M(G,W)$ and hence a morphism of groupoids $p : M(G,W) \to G$ such that $p\tilde{i} = i$. Write $\tilde{W}$ for $\tilde{i}(W)$. Further if $f : W \to H$ is a local morphism of topological groupoids then there exists a unique morphism $\phi : M(G,W) \to H$ of groupoids such that $\phi\tilde{i} = i$.

STAR TOPOLOGICAL GROUPOIDS

The following definition was first given in [8] under the name *"un morceau α-structuré de groupoide"* and then restated in [1] under the name "α-topological groupoid".

Definition 2. *A locally star topological groupoid is a pair (G,W) of a groupoid G and a topological space W such that:*

i) $O_G \subseteq W \subseteq G$, that is W contains all identities;

ii) W is the topological sum of the subspaces $W_x = W \cap G_x, x \in O_G$;

iii) if $g \in G(x,y)$, then for the right translation

$$R_g : W_y \to W_y g, h \mapsto hg$$

the sets $R_g^{-1}(W)$ and $W \cap W_y g$ are open in W and the map

$$R_g : R_g^{-1}(W) \to W \cap W_y g, h \mapsto hg,$$

the restriction of the right translation R_g, is a homeomorphism.

A locally star topological groupoid (G, W) is said to be a *star topological groupoid* if G and W have the same underlying set. Thus a star topological groupoid is a groupoid in which each fibre G_x has a topology such that right translations are homemorphisms.

Let G be a star topological groupoid. A subset W of G is called $star - open$ if for each $x \in O_G$, the fibre $W_x = W \cap G_x$ is open in G_x.

We now give a theorem which was proved in [7] in a different way using a result given in [1] on the extendibility of locally star topological groupoids to star topological groupoids. The following proof is completely different.

Earlier we have written $\tilde{W}$ for $\tilde{i}(W)$. Now impose on $\tilde{W}$ the topology such that the bijection $\tilde{i} : W \to \tilde{W}$ is a homemorphism. We then give the following main theorem.

Theorem 1. *The groupoid $M(G,W)$ is a star topological groupoid such that $\tilde{W}$ is star open in $M(G,W)$. Further the projection map $p : M(G,W) \to G$ is a covering map on each fibre $M(G,W)_x$.*

Proof. Let $\sigma_g : W_{\beta g} \to \tilde{W}g$ be defined by $u \to < u > g$. Then σ_g is a bijection. For each $g \in M(G,W)_x$, impose on $\tilde{W}g$ the topology induced from that of W by the bijection $\sigma_g : W_{\beta g} \to \tilde{W}g$ and then impose on $M(G,W)_x$ the final topology with respect to the inclusions $i : \tilde{W}g \to M(G,W)_x$, for all $g \in M(G,W)_x$.

We now show that for $h \in M(G,W)(x,y)$, the right translation

$$R_h : M(G,W)_y \to M(G,W)_x, g \mapsto gh$$

is a homeomorphism. It is obvious that R_h is bijective and the right translation $R_h - 1$ is the inverse of R_h. Hence it is enough to prove that R_h is continuous. Since the topology on $M(G,W)_y$ is the final topology with respect to the inclusions $i : \tilde{W}g \to M(G,W)_y$, R_h is continuous if and only if the composition map $R_h i : \tilde{W}g \to M(G,W)_x$ is continuous. But $R_h i$ is continuous, since $R_h i|\tilde{W}g, (\tilde{W}g)h$, the restriction of $R_h i$, is a homeomorphism by the homeomorphisms $\sigma_g : W_{\beta g} \to \tilde{W}g$ and $\sigma_{gh} : W_{\beta g} \to \tilde{W}gh$. So that $M(G,W)$ becomes a star topological groupoid.

Further by the construction of the topology on $M(G,W)_x$, obviously $\tilde{W}_x$ is open in $M(G,W)_x$. Hence $\tilde{W}$ is star open in $M(G,W)$.

Next we prove the following

I) for $g, h \in M(G,W)_x$, the set $\tilde{U}_{g,h} = \tilde{W}g \cap \tilde{W}h$ is open in both $\tilde{W}g$ and $\tilde{W}h$;

II) the topologies on $\tilde{U}_{g,h}$ induced by $\tilde{W}g$ and $\tilde{W}h$ as subspaces coincide.

In order to prove these we need the following lemma which is given in group case in [4]. □

Lemma 1. *Let $g \in M(G,W)_x$ and U_g be the set of elements $u \in W$ such that $< u > \in \tilde{W}g$, that is, $U_g = (\tilde{i})^{-1}(\tilde{W}_g)$. Then U_g is open in W_x and the map $\psi_g : U_g \to W$,*

$u \mapsto u'$ *with* $< u >=< u' > g$, *is continuous.*

Proof. Let $u_0 \in U_g$ and let $u'_0 \in W$ such that $\psi_g(u_0) = u'_0$. So that $< u_0 >=< u'_0 > g$. We choose a subset V of W as $(Wu_0^{-1}) \cap (W{u'_0}^{-1}) \cap W$. Then V is open in G_{u_0} using the right translations $R_{u_0^{-1}}$ and $R_{{u'}_0^{-1}}$. Hence $1_{\beta u_0} \in V$, and for $v \in V$ we have $(u_0, v) \in D$ and $(u'_0, v) \in D$. Where D is the set $\gamma^{-1}(W) \cap (W_\alpha X_\beta W)$ and

$$W_\alpha X_\beta W = \{(v, u) \in W \times W : \alpha v = \beta u\}.$$

For $u \in Vu_0$, we have $u = vu_0$ with a $v \in V$ so that

$$< u >=< vu_0 >=< v >< u_0 >=< v >< u'_0 > g =< vu'_0 > g.$$

Hence $u \in U_g$ so that $u_0 \in Vu_0 \subset U_g$ (We notice that $U_g \subseteq W_x$). But Vu_0 is open in G_x by the right translation

$$R_{u_0} : G_{u_0} \to G_x.$$

Hence Vu_0 is open in W_x, so U_g is open in W_x. Further $< u >=< vu'_0 > g$ and $< u >=< u' > g$ so that

$$< u' >=< vu'_0 >, u' = vu'_0 = uu_0^{-1}u'_0.$$

Hence ψ_g is continuous at $u \in Vu_0$. But since Vu_0 is open in U_g, so ψ_g is continuous on all U_g which completes the proof of Lemma 1. □

Proof. of (I) and (II)

$$\begin{aligned}
\sigma_g^{-1}(\tilde{U}_{g,h}) &= \{u \in W :< u > g \in \tilde{U}_{g,h}\} \\
&= \{u \in W :< u > g \in \tilde{W}g \cap \tilde{W}h\} \\
&= \{u \in W :< u > g \in \tilde{W}h\} \\
&= \{u \in W :< u > \in \tilde{W}hg^{-1}\} \\
&= U_{hg^{-1}}
\end{aligned}$$

and similarly we can see that $\sigma_h^{-1}(\tilde{U}_{gh}) = U_{gh^{-1}}$. By Lemma 1 $U_{hg^{-1}}$ (resp. $U_{gh^{-1}}$) is open in $W_{\beta g}$ (resp. $W_{\beta h}$). So $\tilde{U}_{g,h}$ is open in both $\tilde{W}g$ and $\tilde{W}h$ which proves (I).

For (II) let $\tilde{U}$ be a subset of $\tilde{U}_{g,h}$ and let $w \in \tilde{U}$. Then there are $u, u' \in W$ such that $w =< u > g$ and $w =< u' > h$. So

$$< u > g =< u' > h, < u >=< u' > hg^{-1} \text{ or } < u' >=< u > gh^{-1}.$$

So that

$$\begin{aligned} \tilde{U} &= \{< u > g : \text{ for some } u \in U_{hg^{-1}}\} \\ &= \{< u' > h : \text{ for some } u' \in U_{gh^{-1}}\} \end{aligned}$$

and

$$\begin{aligned} \sigma_g^{-1}(\tilde{U}) &= \{u \in W :< u > g \in \tilde{U}\} \\ &= \{u \in W :< u > g \in \tilde{U} \text{ and } < u > g =< u' > h \text{ for a } u' \in W\} \\ &= \{u \in W :< u > g \in \tilde{U} \text{ and } < u >=< u' > hg^{-1} \text{ for a } u' \in W\} \\ &= \{u \in U_{hg^{-1}} :< u > g \in \tilde{U}\}; \end{aligned}$$

and similarly

$$\sigma_h^{-1}(\tilde{U}) = \{u \in U_{gh^{-1}} :< u > h \in \tilde{U}\}$$

Hence,

$$\psi_{hg^{-1}}^{-1}(\sigma_h^{-1}(\tilde{U})) = \sigma_g^{-1}(\tilde{U}), \text{ and } \psi_{gh^{-1}}^{-1}(\sigma_g^{-1}(\tilde{U})) = \sigma_h^{-1}(\tilde{U}).$$

So if $\tilde{U}$ is open in $\tilde{W}g$, then $\sigma_g^{-1}(\tilde{U})$ open in $W_{\beta g}$. Since $\psi_{gh^{-1}} : U_{gh^{-1}} \to W_{\beta g}$ is continuous, $\psi_{gh^{-1}}^{-1}(\sigma_g^{-1}(\tilde{U})) = \sigma_h^{-1}(\tilde{U})$ is open in $U_{gh^{-1}}$. But by Lemma 1 $U_{gh^{-1}}$ is open in $W_{\beta h}$. Hence $\tilde{U}$ is open in $\tilde{W}h$. Conversely (with the same idea) we can show that if $\tilde{U}$ is open in $\tilde{W}h$ then it is open in $\tilde{W}g$. This shows that the topologies on $\tilde{U}_{g,h} = \tilde{W}g \cap \tilde{W}h$ induced by $\tilde{W}g$ and $\tilde{W}h$ coincide, which is required in (II).

So by the Proposition 8, page 33 in [2] the star $M(G,W)_x$ together with the topology imposed on it as above satisfies the following conditions.

A) for each $g \in M(G,W)_x$, $\tilde{W}g$ is open in $M(G,W)_x$;

B) for $g \in M(G,W)_x$, the topologies on $\tilde{W}g$ induced by the bijection $\sigma_g : W_{\beta g} \to \tilde{W}g$ and by $M(G,W)_x$ as a subspace coincide.

Finally we prove that $p_x : M(G,W)_x \to G_x$, the restriction of the projection map $p : M(G,W) \to G$, is a covering map. Let $g \in G(x,y)$. Since W_y is open in G_y then by the right translation $R_g : G_y \to G_x$, $(W_y)g$ is a open neighbourhood of g in G_x, and

$$\begin{aligned} p^{-1}(W_y g) &= p^{-1}(W_y)p^{-1}(g), \\ &= U\{(\tilde{W}_y)h : h \in p^{-1}(g)\}. \end{aligned}$$

We notice that since O_p is identity, $\beta h = y$. But for each $h \in p^{-1}(g)$, the restriction $p : (\tilde{W}_y)h \to (W)_y)g$ is a homeomorphism by the homeomorphisms $R_g : W_y \to (W_y)g$ and $R_h\tilde{i} : W_y \to (\tilde{W}_y)h$. Hence $p_x : M(G,W)_x \to G_x$ is a covering morphism. □

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YILDIZ TOPOLOJİK GROUPOİDLER

Özet

Referanslardan [4]'de topolojik gruplar hakkında bir kavram geliştirilmiştir. Bu makalede bu kavramı topolojik gruplardan daha geniş olan topolojik groupoidlere genelleştiriyor ve topolojik grupoidlerde bir kavram elde ediyoruz.

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