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LINEAR TOPOLOGIC INVARIANTS AND THEIR APPLICATIONS TO ISOMORPHIC CLASSIFICATION OF GENERALIZED POWER SPACES

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Abstract

In the present survey generalized linear topological invariants are considered as a development of classical invariants of Kolmogorov and Pelczynski (approximative and diametral dimensions). It is realized a geometric idea to construct some new invariant characteristics by applying of classical characteristics (diameter or entropy-like characteristics) to some symplest interpolational constructions under neighborhoods, taken from a given basis of neighborhoods of zero. It is considered various applications to isomorphic classification of generalized power Köthe spaces (in particular, tensor products of finite and infinite type power series spaces, spaces of analytic and infinitely differentiable vector-valued functions, spaces of analytic functions of several variables.

§0. Introduction

0.0.

The present paper is devoted to the problem of isomorphic classification of locally convex spaces (LCS). It is based mainly on the authors results [47], [49]-[54] and the lecture courses which have been given in Rostov-on-Don University (see, [55]). Most of this papers are in Russian and because of that their contents are not enough known until now.

Facts, which are discussed here, represent some certain direction of development of the classical linear topological invariants – functional and approximative dimensions (Gelfand, Kolmogorov, Pelczynski) and further version of the last one – diametral dimension (Pelczynski, Bessaga, Rolewicz, Mitiagin and so on). Moreover, the new point of view have been suggested by the author in [54], [56] (see below item 4.3), namely that the classical invariant characteristics (ϵ-entropy or n-diameters) contain essentially more

* permanent position

information about spaces than constructed with their help approximative and diametral dimensions but to recognize this information we need to utilize such characteristics to some special many-parameter constructions of neighborhoods, taken from a given basis of neighborhoods of the origin.

In this paper we restrict ourself by some special circle of questions. We do not almost concern here other results of many authors (Ahonen, Apiola, Aytuna, Djakov, Dragilev, Dubinsky, Goncharov, Kashirin, Kocatepe, Kondakov, Krone, Nurlu, Nüberg, Prada, Terzioğlu, Tidten, Vogt, Zahariuta and others) connected in one way or another with consideration of linear topological invariants.

0.1.

To show that two linear topological spaces X and Y are isomorphic ($X \simeq Y$) is enough to construct some concrete isomorphism $T : X \to Y$. But to prove a non-isomorphism of the spaces ($X \not\simeq Y$) we need to use a distinction of some properties of these spaces, which coincide if spaces are isomorphic. This is, in short, the idea of linear topological invariants.

Let $\mathcal{E}$ be a class of LCS and $\mathcal{T}$ is a set. We say that on $\mathcal{E}$ it is defined a linear topological invariant (LTI; in the following we often say only invariant) with values in $\mathcal{T}$, if a mapping $\tau : \mathcal{E} \to \mathcal{T}$ is defined, such that if $X \simeq Y$, $X, Y \in \mathcal{E}$, then $\tau(X) = \tau(Y)$. It is naturally to look for the following (unfortunately, often contradictory one to the other) properties of LTI: on the one hand, it is desirable to have LTI defined on a wider class $\mathcal{E}$, on the other hand to have stronger invariants (τ_1 is stronger then τ, if $\tau_1(X) = \tau_1(Y)$ implies $\tau(X) = \tau(Y)$); lastly, it is very important to be able to evaluate LTI (or somehow to estimate).

On occasion it may be more convenient to consider $\mathcal{T}$ as a set with some relation of equivalency and to take $\tau(X) \sim \tau(Y)$ instead of $\tau(X) = \tau(Y)$ in the definition of LTI: just suchlike invariants are mainly considered in present paper.

As an example of an universal and rather strong invariant one can consider so called linear dimension $d_\ell(X)$ of the space X, which was introduced by S. Banach [2] and has great theoretical importance.

After the classical results of S. Banach and S. Masur on topological distinction of spaces ℓ^p, L^p (see [2], Ch. XII and remarks to it) appeared many papers in which the isomorphisms of Banach spaces was studied with the help of some LTI.

In the following we consider only questions connected with LTI and isomorphisms of non-normed LCS. The basic results here are due to Kolmogorov [23], [24] and Pelczynski [35], who introduced important LTI - so called approximative dimensions. With the help of these invariants and their modifications were obtained results on isomorphic classification for many concrete LCS (see [5]-[7], [12], [16]-[21], [24]-[27], [29]-[30], [36]-[39], [42]-[43]). We give here two at the most significant examples:

Proposition 0.1. *(Kolmogorov [23], [24]). The space $A(D)$ (of all analytic function* *in D) is not isomorphic to the space $A(G)$, if D is domain in $\mathbb{C}^n, G$ is domain in $\mathbb{C}^m$, and $n \neq m$.*

Proposition 0.2. *(see [30], [36], [39]). The space $A(U^n)$, where U^n is the unit polydisc in $\mathbb{C}^n$, is not isomorphic to the space $A(\mathbb{C}^n)$.*

A little later were introduced so called diametral dimensions* $\Gamma(X), \Gamma'(X)$ (Pelczynski, Bessaga, Rolewicz [6], Mitiagin [29]). These invariants turn to be more convenient (especially for investigation of Köthe spaces) and stronger then approximative dimensions (see [29]). In §3 we consider diametral dimensions more in detail.

Later on Dragilev [16] makes clear that the invariants $\Gamma(X), \Gamma'(X)$ are extremely convenient for investigations of a special class of spaces with regular absolute basis, i.e. (up to isomorphism) Köthe spaces $K(a_{np})$ such that

$$\frac{a_{np}}{a_{nq}} \downarrow 0, \quad n \to \infty, \quad p < q.$$

Dragilev [17], [18] studied the invariant $\Gamma(X)$ on the class of nuclear spaces with regular basis. He found some smaller classes, on which the invariant $\Gamma(X)$ is complete (i.e. $\Gamma(X) = \Gamma(Y)$ implies $X \simeq Y$) and consider very interesting examples of nonisomorphic spaces with regular basis, which are not distinguished by $\Gamma(X)$.

Krone and Robinson [13], Kondakov [26] proved, that the invariant $\Gamma'(X)$ (which was not studied actively till that moment) is complete on the class of all nuclear spaces with a regular basis.

The next example shows that the invariants $\Gamma(X), \Gamma'(X)$ are too rough for investigation of isomorphic classifications of spaces without regular bases:

Proposition 0.3. *([17], [30], [39]). The spaces $A(U)$ and $A(U) \times A(\mathbb{C})$ are not isomorphic, although $\Gamma'(A(U)) = \Gamma'(A(U) \times A(\mathbb{C}))$. There is no regular basis in the space $A(U) \times A(\mathbb{C})$.*

Obviously it is necessary to consider more general invariants which take into account the "irregularity" of bases. First invariant of this type was introduced by Mitiagin [30], [31] in the special case: on the class of all centers of Hilbert scales.

Almost in the same time the author suggested another approach to investigations of some class of spaces without regular bases (including, in particular, the space $A(U) \times A(\mathbb{C})$ from Proposition 0.3.). This approach is based on Riesz theory of compact operators; see more detailly about this method [46], [48], [49], [53] (see also [3], [4], [20]).

* We follow the notations of [16], [29].

0.3.

In the present paper we study general invariant characteristics for Köthe spaces (see Zahariuta [47], [50]-[52]), which generalize the invariant, introduced in [32], [33].

The main point of the paper is §4, where general invariant characteristics are defined and studied on the class of all Köthe spaces. There the invariantness of these characteristics is proved and some new geometric approach to the definition of invariant characteristics on LCS is considered.

We consider applications of general invariant characteristics to problem of isomorphic classification of concrete classes of spaces (in particular, classes of spaces of analytic functions) and to the problem of quasiequivalence of bases. Significiant attention is paid to applications for isomorphic classification of some special, in particular spaces of C^∞-functions (or analytic functions) with values in Fréchet spaces (§5).

In §6 we build rather special invariants on the base of invariant characteristics of §4. These invariants are specially adapted to the problem of isomorphic classification of spaces of analytic functions in unbounded multicircular domains. We prove here the existence of a continuum of pairwise nonisomorphic spaces of that kind.

We also use general invariant characteristics to get new results connected with the problem on quasiequivalence of bases.

Let us remind that two bases $\{x_i\}, \{y_i\}$ in LCS X are said to be quasiequivalent, if there exist a sequence of numbers (λ_i), a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and an isomorphism $T : X \to X$, such that $Tx_i = \lambda_i \, y_{\sigma(i)}$.

After Dragilev [15] proved that any two basis in the space $A(U)$ are quasiequivalent, it arised the quasiequivalence problem for nuclear Fréchet spaces (see [29]) or more generally, for unconditional bases in Montel Fréchet spaces.

The quasiequivalence problem is closely connected with the problem of isomorphic classification. So, with the help of diametral dimensions it was proved the quasiequivalence of bases in nuclear spaces with regular basis (see [7], [12], [14], [26], [29], [30]). Let us note, that in [45] the quasiequivalence of all unconditional bases in Montel centers of scales had been proved - that is the first result on quasiequivalence problem for nonnuclear F-spaces. The quasiequivalence of unconditional bases was proved in [32], [33] for arbitrary (non montel!) centers of Hilbert scales with the help of a new invariant characteristics, introduced there.

The quasiequivalence problem is rather more difficult for spaces without regular basis. Using invariants, based on the Riesz theory (see [20], [46], [48]), Dragilev and Zahariuta proved that all bases are quasiequivalent in spaces of the kind $X \times Y$, where the spaces X, Y are "with very different properties" (for example, $X = A(U), Y = A(\mathbb{C})$). The general invariant characteristics of Köthe spaces give us another approach to the quasiequivalence problem for spaces without regular basis (see [47], [50]-[51]). We consider this matter in §5 and use it to prove that all bases are quasiequivalent for wide classes of Köthe spaces without regular basis.

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§1. Köthe Spaces

1.0.

Let $A = (a_{ip})_{i \in I, p \in \mathbb{N}}$ be a matrix of real numbers such that $0 \le a_{ip} \le a_{ip+1}$. Köthe space, defined by the matrix A, is said to be the locally convex space $K(A)$ of all sequences $\xi = (\xi_i)$ such that

$$|\xi|_p := \sum_{i \in I} a_{ip} |\xi_i| < \infty \qquad \forall\, p \in \mathbb{N},$$

with the topology, generated by the system of seminorms $\{|\xi|_p, p \in \mathbb{N}\}$. The set of indices I is supposed to be countable, but in general $I \ne \mathbb{N}$. This is convenient for applications, especially when multiple series are considered.

It is well known (e.g. [29], [38]) that every Fréchet space with absolute basis is isomorphic to some Köthe space. More precisely, if X is a Fréchet space, $\{e_i\}_{i \in I}$ is an absolute basis in X and $\{\|x\|_p\}_{p \in \mathbb{N}}$ is an increasing sequence of seminorms, generating the topology of X, then X is isomorphic to the Köthe space, defined by the matrix $A = (a_{ip})$, where $a_{ip} = \|e_i\|_p$.

An analogous statement is true (see [44], [45]) for countably - Hilbert spaces with unconditional basis, but instead of $K(A)$ one has to consider the space $K^2(A)$ defined by ℓ_2-norms

$$|\xi|_p := \left(\sum_{i \in I} a_{ip}^2 |\xi_i|^2 \right)^{1/2}, \qquad p \in \mathbb{N}.$$

1.1.

Further if X and Y are locally convex spaces we denote by $X \times Y$ their Cartesian product and by $X \hat{\otimes} Y$ their projective tensor product (see [40], p.119). In the case $X = K(A), Y = K(B)$, where $A = (a_{ip}), B = (b_{jp})$, $i, j \in \mathbb{N}$, the spaces $X \times Y$ and $X \hat{\otimes} Y$ are naturally isomorphic respectively to the spaces $K(C), K(D)$, where

$$C = (c_{kp}), \qquad c_{kp} = \begin{cases} a_{ip} & k = 2i-1 \\ b_{ip} & k = 2i \end{cases}, \; i \in \mathbb{N}$$

$D = (d_{\nu p}), d_{\nu p} = a_{ip} b_{jp}$, $\nu = (i,j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.

1.2.

Let X and Y be locally convex spaces with unconditional bases respectively $\{x_\nu, \nu \in \mathcal{N}\}, \{y_\mu, \mu \in \mathcal{M}\}$. An operator $T : X \to Y$ is said to be:

(a) quasidiagonal (*qd*) (with respect to those fixed bases), if

$$Tx_\nu = \gamma_\nu y_{\sigma(\nu)},$$

where $\sigma : \mathcal{N} \to \mathcal{M}$ and γ_ν is a sequence of numbers;

(b) permutable, if it is *qd* and $\gamma_\gamma \equiv 1$;

(c) diagonal, if it is *qd* and $\mathcal{N} = \mathcal{M}, \sigma(\nu) = \nu$.

In the case there exists an isomorphism T of the type (a), (b), (c) the bases $\{x_\nu\}$ and $\{y_\mu\}$ are called respectively a) **quasiequivalent** (see [29]), b) **permutably equivalent**, c) **preequivalent**. If T is *qd* and $\gamma_\nu \equiv 1$, $\sigma(\nu) \equiv \nu$, $\mathcal{N} = \mathcal{M}$ then bases $\{x_\nu\}$ and $\{y_\mu\}$ are called d) **equivalent**. Isomorphisms at the types (a), (b), (c), (d) will be denote respectively by

$$X \overset{qd}{\simeq} Y, \quad X \overset{p}{\simeq} Y, \quad X \overset{d}{\simeq} Y, \quad X \overset{e}{\simeq} Y. \tag{1.1}$$

If T is an isomorphic imbedding of the type (a) or (b) respectively then it is denoted by

$$X \overset{qd}{\tilde{\to}} Y \quad \text{or} \quad X \overset{p}{\tilde{\to}} Y. \tag{1.2}$$

Usually the relations (1.1) and (1.2) are used for Köthe spaces only with respect to their canonical bases. The canonical bases of two Köthe spaces are equivalent if and only if these spaces coincide as sets (see for example [29], Lemma 4).

1.3.

The following lemma was considered implicitly in [32], p.126-129.

Lemma 1.1. *Let X and Y be Fréchet spaces with unconditional bases $\{x_\nu, \nu \in \mathcal{N}\}, \{y_\mu, \mu \in \mathcal{M}\}$ respectively. If $X \overset{qd}{\tilde{\to}} Y$ and $Y \overset{p}{\tilde{\to}} X$ then $X \overset{qd}{\simeq} Y$ (it is possible to write everywhere p instead of qd).*

Proof. Let $\tau : \mathcal{N} \to \mathcal{M}$ and $\sigma : \mathcal{M} \to \mathcal{N}$ are injective mappings such that linear operators $T : X \to Y$ and $S : Y \to X$, defined by $Tx_\nu = t_\nu y_{\tau(\nu)}, Sy_\mu = s_\mu X_{\sigma(\mu)}$ are imbeddings. Using Cantor-Bernstein set theory construction we get partitions $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2, \mathcal{N}_1 \cap \mathcal{N}_2 = \phi, \mathcal{M}_1 \cap \mathcal{M}_2 = \phi$, such that $\tau(\mathcal{N}_1) = \mathcal{M}_1, \sigma(\mathcal{M}_2) = \mathcal{N}_2$. Then the mapping

$$\rho : \mathcal{N} \to \mathcal{M} \mid \rho(\nu) = \tau(\nu) \text{ if } \nu \in \mathcal{N}_1, \ \rho(\nu) = \sigma^{-1}(\nu) \text{ if } \nu \in \mathcal{N}_2,$$

is bijective and generates the isomorphism $R : X \to Y, Rx_\nu = r_\nu y_{\rho(\nu)}$, where $r_\nu = t_\nu$ if $\nu \in \mathcal{N}_1, r_\nu = s_\nu^{-1}$ if $\nu \in \mathcal{N}_2$. □

1.4.

The space $E_\tau(a) = K(\exp(\lambda_p a_i)), \lambda_p \nearrow \tau, a = (a_i), a_i > 0$, is called finite power series space if $\tau < \infty$ and infinite power series space if $\tau = \infty$ (respectively finite or infinite centre of the absolute scale of spaces $\ell_1(\exp \lambda a_i)$ in terms of [29], §5). Since $E_\tau(a) \overset{d}{\simeq} E_0(a)$ for $\tau < \infty$ we can consider in the following only finite power series spaces with $\tau = 0$. The space $E_\tau(a)$ is Montel iff $a_i \to \infty$. For Montel power series spaces holds the following.

Proposition 1.1. *(see [29], Prop. 18). Let $a = (a_i), a_i \nearrow \infty$ and $b = (b_i), b_i \nearrow \infty$, are given and $\tau = 0, \infty$. Then $E_0(a) \not\simeq E_\infty(b)$ always and*

$$E_\tau(a) \simeq E_\tau(B) \Leftrightarrow E_\tau(a) \overset{e}{\simeq} E_\tau(b) \Leftrightarrow a_i \simeq b_i .$$

The isomorphisms of power series spaces which are not Montel was studied in [32]. We will consider these results in 3.2.

§2. Power Köthe Spaces

2.1.

Let $\mathcal{E}$ be the class of the Köthe spaces of the kind

$$E(\lambda, a) = K\left(\exp\left[\left(-\frac{1}{p} + \lambda_i p\right) a_i\right]\right),$$

where

$$a = (a_i)_{i\in\mathbb{N}},\ a_i > 0,\ \lambda = (\lambda_i)_{i\in\mathbb{N}},\ \ 0 < \lambda_i \le 1.$$

Spaces of that kind will be called first type power Köthe spaces.

It is easy to see that $E(\lambda, a) \overset{e}{\simeq} E(\tilde{\lambda}, \tilde{a})$ if $\tilde{a} = (\tilde{a}_i), \tilde{a}_i = 1 + a_i, \tilde{\lambda}_i = \max\{\lambda_i, 1/\tilde{a}_i\}$. Indeed if $\tilde{\lambda}_i = 1/\tilde{a}_i$ then $\tilde{\lambda}_i \tilde{a}_i = 1$; hence we obtain for any $i, p \in \mathbb{N}$ the inequalities

$$\left(-\frac{1}{p} + \lambda_i p\right) a_i \le \left(-\frac{1}{p} + \tilde{\lambda}_i p\right) \tilde{a}_i \le \left(-\frac{1}{p} + \lambda_i p\right) a_i + 2p,$$

which imply immediately $E(\lambda, a) \simeq E(\tilde{\lambda}, \tilde{a})$. Therefore without loss of generality we assume

$$a_i \ge 1 \quad \text{and} \quad \lambda_{i_k} \to 0 \Rightarrow a_{i_k} \to \infty. \tag{2.1}$$

The space $E(\lambda, a)$ is said to be *finite, infinite* or *mixed* respectively in the cases:

(i) $\lambda_i \to 0$;

(ii) $\underline{\lim}\lambda_i > 0$;

(iii) $\underline{\lim}\lambda_i = 0,\ \overline{\lim}\lambda_i > 0$.

It is easy to see that (i) is equivalent to $E(\lambda, a) \overset{e}{\simeq} E_0(a)$, and (ii) is equivalent to $E(\lambda, a) \overset{e}{\simeq} E_\infty(a)$. In the following we denote by $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ the classes of spaces corresponding to (i), (ii) and (iii).

For any given infinite set $L = \{i_k\} \subset \mathbb{N}$ we shall consider the corresponding basis subspace:

$$E_L(\lambda, a) := \overline{span}\{e_i : i \in L\} \simeq E(\lambda_L, a_L),$$

where $\lambda_L = (\lambda_{i_k}),\ \ a_L = (a_{i_k})$. Then it is easy to see that

(1) $\lambda_{i_k} \to 0 \Leftrightarrow E_L(\lambda, a) \overset{e}{\simeq} E_0(a_L)$.

(2) $\lambda_{i_k} \ge \delta > 0 \Leftrightarrow E_L(\lambda, a) \overset{e}{\simeq} E_\infty(a_L)$.

(3) $a_{i_k} \le C < \infty \Rightarrow E_L(\lambda, a) \overset{e}{\simeq} E_0(a_L) \overset{e}{\simeq} E_\infty(a_L) \overset{e}{\simeq} \ell_1$.

By these relations in particular follows.

Proposition 2.1. *The space $E(\lambda, a)$ is isomorphic to $E_0(c) \times E_\infty(d)$ with some c and d if and only if there exist $N_1, N_2 \subset N$ such that $N = N_1 \cup N_2, N_1 \cap N_2 = \phi$ and $\lambda_i \geq \delta > 0$ for $i \in N_1; \lambda_i \to 0$ if $i \to \infty, i \in N_2$.*

Let $\tilde{\mathcal{E}}$ be the class of all locally convex spaces, which are isomorphic to some spaces in the class $\mathcal{E}$. Further we call spaces in $\tilde{\mathcal{E}}$ *first type power spaces.* Respectively we consider also classes $\tilde{\mathcal{E}_1}, \tilde{\mathcal{E}_2}, \tilde{\mathcal{E}_3}$ which corresponds to $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$. It is obvious that $E_0(a) \times E_\infty(b) \in \tilde{\mathcal{E}_3}$ for arbitrary a, b.

Lemma 2.1. $E_0(a) \otimes E_\infty(b) \in \tilde{\mathcal{E}_3}$.

Proof. Taking a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ we correspond to any $k \in \mathbb{N}$ a pair (i_k, j_k). Put $c_k = a_{i_k} + b_{j_k}$, $\lambda_k = b_{j_k}/(a_{i_k} + b_{j_k})$: then we get (see 1.1).

$$E_0(a) \otimes E_\infty(b) = K\left(\exp\left(-\frac{1}{p}a_i + pb_j\right)\right) \overset{p}{\simeq} K\left(\exp\left[\left(-\frac{1}{p}(1-\lambda_k) + \lambda_k p\right)c_k\right]\right). \quad (2.2)$$

Since

$$-\frac{1}{p} + \lambda_k p \leq -\frac{1}{p}(1-\lambda_k) + \lambda_k p \leq -\frac{1}{p+1} + \lambda_k(p+1)$$

we obtain by (2.2) that $E_0(a) \otimes E_\infty(b) \overset{p}{\simeq} E(\lambda, c)$. □

Lemma 2.2. *The following conditions are equivalent:*

(a) $E(\lambda, a) \overset{p}{\simeq} E(\mu, b)$;

(b) $E(\lambda, a) \overset{q}{\simeq} E(\mu, b)$;

(c) there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a_i \asymp b_{\sigma(i)}$ and for any subsequence i_k

$$\lambda_{i_k} \to 0 \Leftrightarrow \mu_{\sigma(i_k)} \to 0 \quad (2.3)$$

Proof. Evidently (a) $\Rightarrow$ (b). We will show that (c) $\Rightarrow$ (a). Since $a_i \asymp b_i$ implies obviously $E(\lambda, a) = E(\lambda, b)$ it is enough to prove that under the conditions $a = b$ and $\sigma(i) \equiv i$ the corresponding operator $I : E(\lambda, a) \to E(\mu, b)$ is an isomorphism. First we check that I is continuous, i.e.

$$\forall p\, \exists q\, \exists c : \exp\left(-\frac{1}{p} + \mu_i p\right) a_i \leq c \, \exp\left(-\frac{1}{q} + \lambda_i q\right) a_i. \quad (2.4)$$

By (2.3) there exists a function $\varphi(t) : (0,1] \to (0,1]$, such that $\varphi(t) \downarrow 0$ as $t \downarrow 0$ and for any $\delta \in (0,1]$ $\mu_i \geq \delta \Rightarrow \lambda_i \geq \varphi(\delta)$. Fix an arbitrary p and put

$$\mathbb{N}_1 = \mathbb{N}_1(p) = \left\{i : \mu_i \geq \frac{1}{2p^2}\right\}, \qquad \mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1.$$

For $q = q(p) = \max\{p/\epsilon, 2p\}$, where $\epsilon = \varphi(1/2p^2)$, we obtain the inequalities

$$\begin{aligned} -\frac{1}{p} + \mu_i p &\le -\frac{1}{p} + p \le -\frac{1}{q} + q\epsilon \le -\frac{1}{q} + q\lambda_i \quad \text{for} \quad i \in \mathbb{N}_1; \\ -\frac{1}{p} + \mu_i p &\le -\frac{1}{2p} \le -\frac{1}{q} \le -\frac{1}{q} + q\lambda_i \quad \text{for} \quad i \in \mathbb{N}_2, \end{aligned}$$

which imply (2.4) even with $c = 1$. Since the condition (c) is symmetric with respect to sequences (a_i) and (b_i), the same arguments give us that the operator I^{-1} is continuous, i.e. we get that (c) $\Rightarrow$ (a).

Let us now prove that (b) $\Rightarrow$ (c). Suppose that there exist a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and sequence of numbers $\gamma_i \ne 0$ such that the linear operator $T : E(\lambda, a) \to E(\mu, b)$, defined by $Te_i = \gamma_i e_{\sigma(i)}$ is an isomorphism. We will show that then (c) holds with the same σ. If we assume that (2.3) does not hold for some subsequence i_k, then we obtain the existence of a subsequence $j_s = i_{k_s}$, such that $a_{j_s} \to \infty$ and either $\lambda_{j_s} \to 0$ but $\mu_{\sigma(j_s)} \ge \delta > 0$ or $\mu_{\sigma(j_s)} \to 0$ but $\lambda_{j_s} \ge \delta > 0$. It is enough to consider the first case. Let us put $L = \{j_s\}, M = \sigma(L)$; then

$$E_0(a_L) \simeq E_L(\lambda, a) \simeq E_M(\mu, b) \simeq E_\infty(b_M),$$

where

$$a_L = (a_{j_s}), \quad b_M = (b_{\sigma(j_s)}).$$

This is a contradiction, since by Proposition 1.1 in the Montel case finite and infinite power series spaces are not isomorphic. Hence (2.3) is proved for any subsequence i_k.

To prove $a_i \asymp b_{\sigma(i)}$ we again start from the contrary. Assume that $a_i \asymp b_{\sigma(i)}$ is not true. Then there exists a subsequence i_k such that

(i) eihter $a_{i_k}/b_{\sigma(i_k)} \to \infty$, or $b_{\sigma(i_k)}/a_{i_k} \to \infty$.

(ii) either $\lambda_{i_k} \to 0$, or $\lambda_{i_k} \ge \delta > 0$.

We consider one of the four possible cases we get since the rest can be treated analogously. Let $a_{i_k}/b_{\sigma(i_k)} \to \infty$ and $\lambda_{i_k} \to 0$. Then it follows $\mu_{\sigma(i_k)} \to 0$ and we have for $L = \{i_k\}, \mu = \sigma(L)$

$$E_0(a_L) \simeq E_L(\lambda, a) \simeq E_M(\mu, b) \simeq E_0(b_M),$$

where $a_L = (a_{i_k})$, $b_M = (b_{\sigma(i_k)})$. However by Proposition 1.1 the isomorphism $E_0(a_L) \simeq E_0(b_\mu)$ implies $a_{i_k} \asymp b_{\sigma(i_k)}$ contrary to the assumption. □

Remark. By the proof of Lemma 2.2 it follows that if in (c) the mapping σ is only an injection (and all other conditions hold), then the operator $S : E(\lambda, a) \to E(\mu, b)$ defined by $Se_i = e_{\sigma(i)}$ is an isomorphic imbedding.

2.2.

Let $\mathcal{F}$ be the class of all Köthe spaces of the kind

$$F(\lambda, a) = K(\exp \varphi_p(\lambda_i) a_i),$$

where $a = (a_i), a_i > 0, \lambda = (\lambda_i), 1 \le \lambda_i < \infty$, and

$$\varphi_p(z) = \min \left\{ p - \frac{1}{p},\ z - \frac{1}{p} \right\}, \quad z \in [1, \infty),\ p \in \mathbb{N}.$$

Spaces of that kind are said to be *second type power Köthe spaces.* The space X is called second type power space if it is isomorphic to a space of the class $\mathcal{F}$. The class of all second type power spaces will be denoted by $\tilde{\mathcal{F}}$.

Naturally three cases arise:

(i) $\overline{\lim}\lambda_i < \infty \overset{def}{=} F(\lambda, a) \in \mathcal{F}_1 \Leftrightarrow F(\lambda, a) \simeq E_0(a)$

(ii) $\lim \lambda_i = \infty \overset{def}{=} F(\lambda, a) \in \mathcal{F}_2 \Leftrightarrow F(\lambda, a) \simeq E_\infty(a)$

(iii) $\underline{\lim}\lambda_i < \infty,\ \overline{\lim}\lambda_i = \infty \overset{def}{=} F(\lambda, a) \in \mathcal{F}_3$ (mixed case).

Respectively we denote by $\tilde{\mathcal{F}}_j$ the class of spaces, isomorphic to some space in $\mathcal{F}_j$, $j = 1, 2, 3$. Evidently we have $\tilde{\mathcal{F}}_1 = \tilde{\mathcal{E}}_1, \tilde{\mathcal{F}}_2 = \tilde{\mathcal{E}}_2$. The intersection of the classes $\tilde{\mathcal{E}}_3$ and $\tilde{\mathcal{F}}_3$ contains the class $\tilde{\mathcal{E}}_1 \times \tilde{\mathcal{E}}_2$ (see below). The classes of essentially mixed type spaces

$$\mathcal{F}_3' = \tilde{F}_3 \setminus (\tilde{\mathcal{E}}_1 \times \tilde{\mathcal{E}}_2), \quad \mathcal{E}_3' = \tilde{\mathcal{E}}_3 \setminus (\tilde{\mathcal{E}}_1 \times \tilde{\mathcal{E}}_2)$$

are with very different (in some sense dual) properties.

In some sense the class $\mathcal{F}_3'$ contains "much more" nonisomorphic spaces than the class $\mathcal{E}_3'$ as it is shown in the next examples (in them we use some results which will be proved later - see §§5-6).

Example 2.1. Let α be an irrational number and let (a_i) be a fixed sequence of real numbers, $a_i \ge 1$. For any function $h : (0,1) \to (0,1)$, such that $h(x) \to 0$ as $x \to 0$ we consider the space $X_h = E(h(\{i\alpha\}), a)$ where $\{i\alpha\}$ denote the fractional part of the number $i\alpha$, $i = 1, 2, \ldots$. By Lemma 2.2 all spaces of the kind X_h are isomorphic regardless of the choice of the function h. The space X_h is essentially mixed type, i.e. it is not isomorphic to a finite or infinite power series spaces or to their Cartesian product (see Prop. 1.1).

Example 2.2. Let α be again an irrational number, $a = (i), h : (0,1) \to (0, \infty)$, such that $h(x) \to \infty$ as $x \to 0$. Then there exist a continuum of pairwise nonisomorphic spaces of the kind $Y_h = F(h(\{i\alpha\}), a)$ (see Prop. 6.7).

Characterization of quasi-diagonal isomorphisms for the spaces in the class $\mathcal{F}$ is more complicated then for the spaces of the class $\mathcal{E}$.

Lemma 2.3. *If $a_i \to \infty$, then $F(\lambda, a) \overset{qd}{\simeq} F(\mu, b)$ if and only if there exist a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, sequence of numbers (t_i) and a positive constant $\Delta < \infty$ such that the following conditions hold:*

(a) $b_{\sigma(i)} \asymp a_i$

(b) if $I \subset \mathbb{N}$ is a subsequence such that for $i \in I$ we have

$$\lambda_i \to \ell \in [1, \infty], \qquad \mu_i \to m \in [1, \infty], \qquad b_{\sigma(i)}/a_i \to \gamma \in (0, \infty),$$

then $\ell \neq \infty \Leftrightarrow m \neq \infty$ and

$$\lim_{i \in I} \frac{\ln |t_i|}{a_i} = \ell - m\gamma \in [-\Delta, \Delta] \quad \text{if} \quad \ell \neq \infty \tag{2.5}$$

$$-\Delta \leq \underline{\lim} \frac{\ln |t_i|}{a_i} \leq \overline{\lim} \frac{\ln |t_i|}{a_i} \leq \Delta \quad \text{if} \quad \ell = \infty \tag{2.6}$$

Proof. Necessity. Suppose $F(\lambda, a) \overset{qd}{\simeq} F(\mu, b)$, i.e. there exist a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence of numbers (t_i) such that the operator $T : F(\lambda, a) \to F(\mu, b)$ defined by $Te_i = t_i\, e_{\sigma(i)}$ is isomorphism. Then

$$\exists p_0 \, \forall p \geq p_0 \, \exists r = r(p), \; q = q(p), \qquad r < p < q, \, \exists C_p :$$

$$\frac{1}{C_p} \exp(\varphi_r(\lambda_i) a_i) \leq |t_i| \, \exp(\varphi_p(\mu_{\sigma(i)}) b_{\sigma(i)}) \leq C_p \, \exp(\varphi_q(\lambda_i) a_i). \tag{2.7}$$

Taking another triple $r' < p' < q'$, $q < r'$, we get an analogous double inequality. Dividing and taking the logarithm we obtain

$$\begin{aligned} -\ln(C_p C_{p'}) + [\varphi_{r'}(\lambda_i) - \varphi_q(\lambda_i)] a_i &\leq [\varphi_{p'}(\mu_{\sigma(i)}) - \varphi_p(\mu_{\sigma(i)})] b_{\sigma(i)} \\ &\leq \ln(C_p C_{p'}) + [\varphi_{q'}(\lambda_i) - \varphi_r(\lambda_i)] a_i. \end{aligned}$$

It is easy to see that these inequalities imply $b_{\sigma(i)} \asymp a_i$.

On the other hand by (2.7) it follows that there exists a constant $\Delta < \infty$ such that

$$-\Delta \leq \frac{\ln |t_i|}{a_i} \leq \Delta \tag{2.8}$$

Let J be a subsequence in $\mathbb{N}$, such that for $i \in J$ $\lambda_i \to \ell$, $\mu_{\sigma(i)} \to m$, $b_{\sigma(i)}/a_i \to \gamma$. If $\ell < \infty$, then using the same argument as in Lemma 2.2 we obtain $m < \infty$. Consider a triple $r < p < q$ such that $r > \max(\ell, m)$ and (2.7) holds. After taking the logarithm, dividing by a_i and passing to the limits as $i \in J$ we get

$$\ell - \frac{1}{r} - \left(m - \frac{1}{p}\right)\gamma \leq \underline{\lim}_{i \in J} \frac{\ln t_i}{a_i} \leq \overline{\lim}_{i \in J} \frac{\ln |t_i|}{a_i} \leq \left(\ell - \frac{1}{q}\right) - \left(m - \frac{1}{p}\right)\gamma.$$

When $r \to \infty$ (then of course $p \to \infty$, $q \to \infty$) we obtain (2.5).

In the case $\ell = \infty$ we have $m = \infty$ and (2.8) implies (2.6).

Sufficiency. Let conditions (a), (b) hold and $\sigma(i) = i$ (this is not a loss of generality). We will prove that the operator $T : F(\lambda, a) \to F(\mu, b)$, defined by the formula $Te_i = t_i e_i$, is an isomorphism. Suppose, on the contrary, that either T or T^{-1} is not continuous. If T is not continuous then $\exists p\, \forall q\, \exists J \subset \mathbb{N},\ |J| = \infty$:

$$|t_i|\ \exp(\varphi_p(\mu_i)b_i) \geq \exp(\varphi_q(\lambda_i)a_i) \qquad \forall\, i \in J. \tag{2.9}$$

If $\sup\{\lambda_i, i \in J\} = \infty$ we can choose a subsequence $(i_s), i_s \in J$, such that $\lambda_{i_s} \to \infty$, $\mu_{i_s} \to \infty$. Then by (2.9) we obtain

$$\frac{\ln |t_{i_s}|}{a_{i_s}} + \left(p - \frac{1}{p}\right) \frac{b_{i_s}}{a_{i_s}} \geq q - \frac{1}{q},$$

which contradict to (b) for big enough q.

If $\sup\{\lambda_i, i \in J\} < \infty$ we can choose a subsequence $(i_s), i_s \in J$, such that $\lambda_{i_s} \to \ell, \mu_{i_s} \to m, b_{i_s}/a_{i_s} \to \gamma$. By (2.9) we obtain

$$\underline{\lim} \frac{\ln |t_{i_s}|}{a_{i_s}} \geq \ell - m\gamma + \frac{\gamma}{p} - \frac{1}{q}.$$

Taking q big enough (such that $\frac{1}{q} < \frac{\gamma}{p}$) we get a contradiction to (b). □

§3. Diametral Dimension of Power Köthe Spaces. Isomorphisms of finite and infinite types power spaces.

3.0.

In this paragraph it is introduced a modification of diametral dimensions which is convenient for considerations of Cartesian products and tensor products.

In 3.2 we consider also an invariant, essentially due to Mitiagin [32]. This invariant was an initial point in construction of general invariants [47], [50] (see for details in the next paragraphs).

3.1.

Diametral dimensions are the most convenient linear topological invariants for investigation of Köthe spaces.

Let X be a locally convex space, U, V be absolutely convex neighbourhoods of zero in X, and $d_n(V,U)$, $n = 0,1,2,\dots$, be Kolmogorov's n-diameters (see [23], [29]), i.e.

$$d_n(V,U) = \inf_{L_n} \sup_{x\in V} \inf_{y\in L_n} \|x-y\|_U,$$

where $\|x\|_U$ is the seminorm, generated by U, L_n is varying in the class of all n-dimensional subspaces of X. By diametral dimensions we call the classes of sequences (see [6], [16], [29]).

$$\begin{aligned} \Gamma(X) &= \{\gamma = (\gamma_n) : \forall U\, \exists V \mid \gamma_n d_n(V,U) \to 0\}, \\ \Gamma'(X) &= \{\gamma = (\gamma_n) : \exists U\, \forall V \;\; \gamma_n / d_n(V,U) \to 0\}, \end{aligned}$$

where U and V are varying in some basis of neighbourhoods of zero in X (in [6] are used respectively the notations $\delta(X)$ and $\delta'(X)$), we are following the notations of [16], [29]).

Let us remark that the invarint $\Gamma(X)$ is essentially weaker than the invariant $\Gamma'(X)$. This follows by the result of Crone and Robinson [12] proving that $\Gamma'(\cdot)$ distinguished any two nuclear Köthe spaces with regular basis and by the example of Dragilev [18], which showed that the invariant $\Gamma(\cdot)$ does not have this property. Therefore in the following we consider only the invariant $\Gamma'(\cdot)$.

For our purposes (especially for investigation of isomorphisms of Cartesian and tensor products) it is more convenient to use for the class of Köthe spaces $X = K(a_{ip})$ so called inverse diametral dimension $\gamma(X)$ - the set of all real functions $\varphi : (0,\infty) \to \mathbb{R}$ defined by

$$\gamma(X) = \{\varphi : \exists p\, \forall q\, \exists c \mid |\varphi(t)|\} \lessapprox \left|\left\{ i : \frac{a_{iq}}{a_{ip}} \le c \exp t \right\}\right|.$$

The sign $\lessapprox$ means that the inequality is asymptotic, i.e. it is true for enough big t. From here on $|A|$ means the number of elements of the set A, if A is finite, and $+\infty$, if A is infinite.

It is easy to see, that for Köthe spaces $X = K(a_{ip}), Y = K(b_{ip})$ we have

$$\Gamma'(X) = \Gamma'(Y) \Leftrightarrow \gamma(X) = \gamma(Y).$$

Now we shall show how to find $\gamma(X)$ for spaces in the class $\mathcal{E}$. Consider for any sequence $a = (a_i)$ its counting function

$$m_a(t) = |\{i : a_i \leq t\}|, \qquad t > 0.$$

We shall write $m_a \approx m_b$ iff the functions $m_a(t)$ and $m_b(t)$ are weakly equivalent i.e.

$$m_a \approx m_b \Leftrightarrow \exists\, c > 0 : m_a(t) \lessapprox m_b(ct), m_b(t) \lessapprox m_a(ct).$$

The following relations hold:

$$\begin{aligned}
\gamma(E_0(a)) &= \{\varphi : \exists\, A \;\; |\varphi(t)| \lessapprox m_a(At)\}, \\
\gamma(E_\infty(a)) &= \{\varphi : \forall\, \epsilon \;\; |\varphi(t)| \lessapprox m_a(\epsilon t)\}, \\
\gamma(E_0(a) \times E_\infty(b)) &= \{\varphi : \exists\, A\, \forall\, \epsilon \;\; |\varphi(t)| \lessapprox m_a(At) + m_b(\epsilon t)\}, \\
\gamma(E_0(a) \otimes E_\infty(b)) &= \{\varphi : \exists\, A\, \forall\, \epsilon \;\; |\varphi(t)| \lessapprox m_a(At) \cdot m_b(\epsilon t)\},
\end{aligned}$$

$$\gamma(E_\infty(a)) \subset \gamma(E(\lambda, a)) \subset \gamma(E_0(a)).$$

Let us notice also that

$$E_\alpha(a) \times E_\alpha(b) \simeq E_\alpha(c), \quad E_\alpha(a) \times E_\alpha(b) \simeq E_\alpha(d), \quad \alpha = 0, \infty,$$

where sequences c and d have respectively the properties

$$m_c(t) \approx m_a(t) + m_b(t), \qquad m_d(t) \approx m_a(t) \cdot m_b(t).$$

Since

$$\gamma(E_\alpha(a)) = \gamma(E_\alpha(b)) \Leftrightarrow m_a \approx m_b \Leftrightarrow a_i \asymp b_i,$$

we get by Proposition 1.1 that the invarinat $\gamma(\cdot)$ is complete on the class of Montel power series spaces (Mitiagin [29], [30]). We shall show (see Corrolaries 5.3-5.5) that $\gamma(\cdot)$ is also a complete invariant on some (quite narrow) classes of tensor products of the kind $E_0(a) \otimes E_\infty(b)$. However this phenomenon is rather an exception; on the whole class $\mathcal{E}$ the invariant $\gamma(\cdot)$ is not complete, as the next example show.

Example 3.1. ([16], [29], [30]). If $0 < \alpha < \beta < \beta'$ then the spaces $X = E_0(i^\alpha), Y = E_0(i^\alpha) \times E_\infty(i^\beta), Z = E_0(i^\alpha) \times E_\infty(i^{\beta'})$ are pairwise nonisomorphic, but $\gamma(X) = \gamma(Y) = \gamma(Z)$.

Lemma 3.1. *For $\gamma(E_0(a)\otimes E_\infty(b))\subset\gamma(E_0(a')\otimes E_\infty(b'))$ it is necessary and sufficient that*

$$\exists\,\Delta\;\forall\,\epsilon\;\exists\,\delta : m_a(t)\;m_b(\delta t)\lessapprox m_{a'}(\Delta t)m_{b'}(\epsilon t).$$

Proof. Since ($\Leftarrow$) is evident, it is enough to prove ($\Rightarrow$). Suppose the contrary the statement of Lemma 3.1 holds but right hand side is false. Then

$$\forall\,s\;\exists\epsilon_s\;\forall\,j\;\; m_a(t)m_b(t/j)\geq m_{a'}(st)m_{b'}(\epsilon_s t)\;\text{ for }\; t=t_j^{(i)},\quad j,s=1,2,\ldots,$$

and we may assume that

$$t_i^{(s)}<\cdots<t_j^{(s)}<\cdots\quad\text{ and }\quad t_j^s\to\infty\;\text{ as }\; j\to\infty.$$

Consider the functions

$$\delta_s(t')=\left\{\frac{1}{j},\;t_j^{(s)}<t<t_{j+1}^{(s)},\quad j=0,1,\ldots,\quad t_0^{(s)}=0\right\}$$

and choose any function $\delta(t)$, tending to zero as $t\to\infty$ more slowly than all the functions $\delta_s(t)$, $s=1,2,\ldots$. Then the function $\varphi(t)=m_a(t)(\delta(t)t)$ belongs to $\gamma(E_0(a)\otimes E_\infty(b))$, but on the other hand the inequality $\varphi(t)\leq m_{a'}(st)m_{b'}(\epsilon t)$ does not hold for any s if $\epsilon=\epsilon_s$. Hence $\varphi\notin\gamma(E_0(a')\otimes E_\infty(b'))$, i.e. we get a contradiction which proves the lemma. □

3.2.

In this section we describe shortly (slightly modifying) the results of Mitiagin [25], [26], which influenced the author in constructing the general invariants, studying in the next paragraph.

Let $\alpha=0$ or ∞ and $a=(a_i)$ where $a_i\geq 1$. In [31], [32] Mitiagin investigated isomorphism of non-Montel power series spaces of the kind $E_\alpha(a)$, using the counting function

$$M_a(t,\tau)=|\{i:\tau<a_i\leq t\}|,\qquad 0<\tau<t<\infty. \tag{3.1}$$

Considerably later [34] more detailed account appeared. If $a=(a_i)$ and $b=(b_i)$ are two sequence, we write by definition

$$M_a\approx M_b\overset{def}{=}\exists\,\Delta : M_a(t,\tau)\leq M_b(\Delta t,\tau/\Delta),\quad M_b(t,\tau)\leq M_a(\Delta t,\tau/\Delta).$$

The characteristic (3.1) is invariant in the following sense:

Proposition 3.1. *(see [32], Proposition 9). If $E_\alpha(a) \simeq E_\alpha(b)$, then $M_a \approx M_b$.*

The invariant (3.1) is the strongest in the class of power series spaces. This have been proved in [32] implicitely as a part of the proof of the main theorem on quasiequivalence (application for isomorphic classification is given in [32] by Example 17); later Mitiagin gave the detailed proof in [34]. More precisely, the following statement is true

Proposition 3.2. *If $M_a \approx M_b$, then $E_\alpha(a) \overset{p}{\simeq} E_\alpha(b)$.*

An important point of the proof of Proposition 3.2 in [32] is a combinational lemma (see [32], Lemma 10), based on the Hall-Koenig theorem ([22], Ch. 5).

Here we give another proof of this fact (without Hall-Koenig considerations), which will be obtained immediately from Lemma 1 and the following two elementary assertions.

Proposition 3.3. *Let the sequences $a = (a_i), b = (b_j)$, such that $a_i \geq 1,\ b_j \geq 1$ and*

$$\lim_{i\to\infty} a_i = \infty, \qquad \lim_{j\to\infty} b_j = \infty,$$

satisfy the following condition:

$$M_a(t,\tau) \leq M_b,\left(\Delta t,\ \frac{\tau}{\Delta}\right), \qquad 1 \leq \tau \leq t < \infty. \tag{3.2}$$

with some constant $\Delta > 1$. Then there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that the inequalities

$$\frac{a_i}{\Delta} < b_{\sigma(i)} \leq \Delta a_i, \qquad i \in \mathbb{N} \tag{3.3}$$

hold.

Proof. Without loss of generality we can assume that both sequences are non-decreasing. We define the injection $\sigma : \mathbb{N} \to \mathbb{N}$ by the following algorithm: $\sigma(1) = 1$ and

$$\sigma(i) = \min\left\{j : \frac{a_i}{\Delta} < b_j;\quad j \neq \sigma(k),\ k < i\right\}.$$

What need to be shown is the inequality

$$b_{\sigma(i)} \leq \Delta a_i, \qquad i \in \mathbb{N} \tag{3.4}$$

Ad absurdum, let the number $i_1 \in \mathbb{N}$ exists such that

$$b_{\sigma(i_1)} > \Delta a i_1. \tag{3.5}$$

For the numbers

$$\tilde{\sigma}(i) := \min\left\{j : \frac{a_i}{\Delta} < b_j\right\}$$

we have by (3.2)

$$b_{\tilde{\sigma}(i)} \leq \Delta a_i.$$

The relation $\sigma(i) > \tilde{\sigma}(i)$ means that a number $i' < i$ exists such that

$$\sigma(i' + \alpha) = \tilde{\sigma}(i') + \alpha, \qquad \alpha = 0, 1, \ldots, i - i' - 1. \tag{3.6}$$

Let

$$i_0 := \max\{i < i_1 : \tilde{\sigma}(i) = \sigma'(i)\}$$

(the definition is correct, since $\tilde{\sigma}(1) = \sigma(1)$). Then, since (3.6),

$$\sigma(i_0 + \alpha) = \tilde{\sigma}(i_0) + \alpha, \qquad \alpha = 0, 1, \ldots, i_1 - i_0 - 1,$$

and, by assumption (3.5)

$$(b_{\sigma(i_0)}, \Delta\, a_{i_1}] \cap \{b_j\} = \phi.$$

Therefore $M_a(a_{i_0}, a_{i_1}) = i_1 - i_0$, but $M_b(\frac{a_{i_0}}{\Delta}, \Delta\, a_{i_1}) = i_1 - i_0 - 1$, what is in contradiction with (3.2). Therefore (3.4) and, consequently, (3.3) is true. □

Proposition 3.4. *Let for arbitrary sequences $a = (a_i), b = (b_j), a_i \geq 1, b_j \geq 1$, the condition (3.2) holds. Then there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that the inequalities*

$$\frac{a_i}{\Delta^2} < b_{\sigma(i)} \leq \Delta^2\, a_i, \qquad i \in \mathbb{N},$$

hold.

Proof. Let us introduce the sets $(s \in \mathbb{Z}_+)$:

$$\begin{aligned}
\mathcal{N}_s &= \{i : \Delta^{s-1} < a_i \leq \Delta^s\}, \\
\tilde{\mathcal{N}}_s &= \{i : \Delta^{s-2} < a_i \leq \Delta^{s+1}\}, \\
\mathcal{M}_s &= \{j : \Delta^{s-1} < b_j \leq \Delta^s\}, \\
\tilde{\mathcal{M}}_s &= \{j : \Delta^{s-2} < a_i \leq \Delta^{s+1}\}.
\end{aligned}$$

Let

$$S = \{s \in \mathbb{Z}_+ : |\mathcal{M}_s| = \infty\}, \quad I = \bigcup_{s \in S} \tilde{N}_s, \quad J = \bigcup_{s \in S} M_s.$$

Then both the sequences

$$\tilde{a} = \{a_i\}_{i \in \mathbb{N} - I}, \qquad \tilde{b} = \{b_j\}_{j \in \mathbb{N} \setminus J}$$

have no limit points and satisfy the condition

$$M_{\tilde{a}}(t, \tau) \leq M_{\tilde{b}}\left(\Delta t, \frac{\tau}{\Delta}\right).$$

Therefore, by Proposition 3.3, an injection $\lambda : \mathbb{N} \setminus I \to \mathbb{N} \setminus J$ exists such that

$$\frac{1}{\Delta} a_i \le b_{\lambda(i)} \le \Delta a_i, \qquad i \in \mathbb{N} - I.$$

On the other hand we construct an injection $\gamma s : \tilde{\mathcal{N}_s} \to \mathcal{M}_s$ for every $s \in S$. By the same taken we have got the many-valued mapping:

$$\gamma(i) = \{j \in \mathbb{N} : \exists s \mid j = \gamma_s(i)\}, \qquad i \in I,$$

such that $\gamma(i) \ne \phi$, $i \in I$ and $\gamma(i) \cap \gamma(i') = \phi$, $i \ne i'$. Therefore we can obtain an injection $\mu : I \to J$ simply by choosing of one element in each set $\gamma(i)$, $i \in I$. Obviously, it satisfies the condition:

$$\mu(\mathcal{N}_s) \subset \tilde{\mathcal{M}}_s$$

which implies that

$$\frac{1}{\Delta^2} a_i < b_{\mu(i)} \le \Delta^2 a_i.$$

Thus the injection $\sigma : \mathbb{N} \to \mathbb{N}$, defined as μ on I and λ on $\mathbb{N} \setminus I$ is that what required. □

§4. Generalized Linear Topological Invariants

4.0.

In this paragraph we consider the invariant characteristics, introduced by the author in [47] (see also [50]-[52]). These characteristics generalize the invariant of Mitiagin [31], [32], described in items 3.2. The original proof of invariance of these characteristics (see [47], [51]) used basicaly the arguments of [32] (see for example the proof of Lemma 4.1). Nowadays simple proofs of invariance are obtained by using of a geometric arguments. More precisely, new geometric invariants are considered which are based on the asymptotic behaviour of n-diameters of pairs of "synthetic" neighbourhoods of zero, built in an invariant way for a given pair, triple and so on of neighbourhoods of zero, for example

$$d_n(\lambda_1 V_1 \cap \lambda_3 V_4, V_2), \ \ d_n(V_2, conv(\lambda_1 V_1 \cup \lambda_3 V_3)), \ \ d_n(\lambda_1 V_1 \cap \lambda_3 V_3, \overline{conv}(\lambda_2 V_2 \cup \lambda_4 V_4)), \ldots$$

where $\overline{conv} L$ denotes the closed convex hull of the set L. It turns out that these geometric invariants are equivalent in some sense to the previous characteristics for Köthe spaces and this equivalence prove the desired invariance. Moreover this geometric method gives us the possibility to improve some estimates connected with the invariance of these characteristics. Here the asymptotic behaviour of n-diameters with respect to n and parameters λ is important.

4.1.

The next lemma proves the invariance of a chain of characteristics for Köthe spaces, if necessary one can extend it.

Lemma 4.1. *([47], [50]-[52]). Let $K(a_{ip}) \simeq K(b_{ip})$. Then*

(a) $\forall p\, \exists p'\, \forall q'\, \exists q\, \exists c$:

$$\left|\left\{i : \frac{a_{iq}}{a_{ip}} \le t\right\}\right| \le \left|\left\{i : \frac{b_{iq'}}{b_{ip'}} \le ct\right\}\right|,$$

(b) $\forall p'\, \exists p\, \forall q\, \exists q'\, \forall r'\, \exists r\, \exists c$:

$$\left|\left\{i : \frac{a_{iq}}{a_{ip}} > t,\ \frac{a_{ir}}{a_{iq}} \le \tau\right\}\right| \le \left|\left\{i : \frac{b_{iq'}}{b_{ip'}} > \frac{t}{c},\ \frac{b_{ir'}}{b_{iq'}} \le c\tau\right\}\right| \tag{4.1}$$

(b') $\forall p\, \exists p'\, \forall q'\, \exists q\, \forall r\, \exists r'\, \exists c$:

$$\left|\left\{i : \frac{a_{iq}}{a_{ip}} \le t,\ \frac{a_{i\tau}}{a_{iq}} > \tau\right\}\right| \le \left|\left\{i : \frac{b_{iq'}}{b_{ip'}} \le ct,\ \frac{b_{ir'}}{b_{iq'}} > \tau/c\right\}\right|$$

(c) $\forall p'\, \exists p\, \forall q\, \exists q'\, \forall r'\, \exists r\ \ \forall s\, \exists s'\, \exists c$:

$$\left|\left\{i : \frac{a_{iq}}{a_{ip}} > t,\ \frac{a_{ir}}{a_{iq}} \le \tau,\ \frac{a_{is}}{a_{ir}} > \sigma\right\}\right| \le \left|\left\{i : \frac{b_{iq'}}{b_{ip'}} > \frac{t}{c},\ \frac{b_{ir'}}{b_{iq'}} \le c\tau,\ \frac{b_{is'}}{b_{ir'}} > \frac{\sigma}{c\tau}\right\}\right| \tag{4.2}$$

(c') $\forall p\, \exists p'\, \forall q'\, \exists q\, \forall r\, \exists r'\, \forall s'\, \exists s\, \exists c$:

$$\left|\left\{i : \frac{a_{iq}}{a_{ip}} \le t,\ \frac{a_{ir}}{a_{iq}} > \tau,\ \frac{a_{is}}{a_{ip}} \le \sigma\right\}\right| \le \left|\left\{i : \frac{b_{iq'}}{b_{ip'}} \le ct,\ \frac{b_{ir'}}{b_{iq'}} > \frac{\tau}{c},\ \frac{b_{is'}}{b_{ip'}} \le ct\,\sigma\right\}\right|$$

On the other hand the inequalities, which can be obtained by changing places of matrices $\{a_{ip}\}$ and $\{b_{ip}\}$, hold.

Remark. The assertion of Lemma 4.1 is true for many modifications of Köthe spaces. For example, it holds for spaces of sequences

$$K^m(a_{ip}) = \{x = (\xi_i) : |x|_p < \infty \quad \forall p\},$$

where

$$|x|_p = (\Sigma\, |\xi_i|^m\, (a_{ip})^m)^{1/m} \quad \text{if} \quad m < \infty$$

and

$$|x|_p = \sup_i\ |\xi_i| a_{ip} \quad \text{if} \quad m = \infty$$

4.2.

First we shall prove (c) and note how to change the proof of (c) in order to get a proof of (c′).

The assertions (b), (b′) can be derived as partial cases of (c), (c′); in item 4.3 an independent proof is given, based on a new geometric approach.

The simplest one-inequality characteristic is equivalent to classical diametral dimension ($\Gamma'(x)$ or $\gamma(x)$: really, the claim (a) of Lemma implies $\gamma(K(a_{ip})) \subset \gamma(K(b_{ip}))$, which, because of symmetry, gives $\gamma(K(a_{ip})) = \gamma(K(b_{ip}))$.

Proof. Let $\{e_i\}$ be the canonical basis in the spaces $X = K'(a_{ip})$, $Y = K(b_{ip})$. If $T : Y \to X$ is an isomorphism, then the vectors $h_i = Te_i$, $i = 1, 2, \dots$, form an absolute basis in X and any element $x \in X$ can be expand with respect to both bases $\{e_i\}, \{h_i\}$:

$$x = \Sigma\, \xi_i\, e_i = \Sigma\, \eta_i\, h_i.$$

The system of norms

$$\|x\|_p = \Sigma\, |\eta_i|\, b_{ip}, \qquad x \in X, \;\; p = 1, 2, \dots$$

is equivalent to the original system of norms in the Köthe space X:

$$|x|_p = \Sigma\, \xi_i\, a_{ip}, \qquad p = 1, 2, \dots .$$

The systems of sets $\{\|x\|_p \le 1/p\}$ and $\{|x|_p \le 1/p\}$ form bases of neighbourhoods of zero in X. Therefore
$\forall\, p'\; \exists p\; \forall\, q\; \exists\; q'\; \forall\, r'\; \exists\, r\; \forall\, s\; \exists\, s' \mid \forall\, x \in X$

$$p'\|x\|_{p'} \le p\,|x|_p \le q\,|x|_q \le q'\,\|x\|_{q'} \le r'\,\|x\|_{r'} \le r\,|x|_r = s|x|_s \le s'\|x\|_{s'}. \tag{4.3}$$

Moreover one may assume that $p' \le p \le \dots \le s \le s'$. We shall show that (4.2) holds with

$$C = 4\left(\frac{s'}{p'}\right)^3. \tag{4.4}$$

We denote by M_ν the set of indices $i \in \mathbb{N}$ satisfying the ν-th inequality in the left-hand side of (4.2), and by N_ν the set of indices $i \in \mathbb{N}$ satisfying ν-th imequality in the right-hand side of (4.2), $v = 1, 2, 3$.

Consider the projectors $P_\nu : X \to X$, $Q_\nu : X \to X$ defined by

$$P_\nu x = \sum_{i \in M_\nu} \xi_i\, e_i, \quad Q_\nu x = \sum_{i \in N_\nu} \eta_i\, h_i, \qquad \nu = 1, 2, 3.$$

Since (4.2) is equivalent to the inequality

$$\dim P_1 P_2 P_3(X) \le \dim Q_1 Q_2 Q_3(X)$$

it is enough to prove that the operator

$$S : P_1P_2P_3(X) \to Q_1Q_2Q_3(X),$$

defined by the formula $Sx = Q_1Q_2Q_3(x)$, $x \in P_1P_2P_3(X)$, is an injection. Suppose the contrary, i.e. there exists $z \in P_1P_2P_3(X)$, such that $Q_1Q_2Q_3(z) = 0$ but $z \neq 0$. Then $z = (I - Q_1)z + Q_1(I - Q_2)z + Q_1Q_2(I - Q_3)z$ and using that $z = P_1z = P_2z = P_3z$ we get

$$z = P_3(z) = P_3(I - Q_1)P_1z + P_3Q_1(I - Q_2)P_2z + P_3Q_1Q_2(I - Q_3)P_2z. \tag{4.5}$$

Directly by the definitions of projectors the inequalities follow:

$$|P_1x|_p \leq \frac{1}{t}|x|_q, \quad |P_2x|_r \leq \tau\,|x|_q, \quad |P_3x|_\tau \leq \frac{1}{\sigma}|x|_s, \tag{4.6}$$

$$\|(I - Q_1)x\|_{q'} \leq \frac{t}{c}\|x\|_{p'}, \|(I - Q_2)x\|_{q'} \leq \frac{1}{c\tau}\|x\|_{\tau'}, \|(I - Q_3)x\|_{s'} \leq \frac{\sigma}{c\tau}\|x\|_{\tau'}.$$

In addition, for any $\pi = 1, 2, \ldots$ we have

$$|P_\nu x|_\pi \leq |x|_\pi, \ \|Q_\nu x\|_\pi \leq \|x\|_\pi, \qquad x \in X, \nu = 1, 2, 3. \tag{4.7}$$

Using (4.3)-(4.7) we obtain the estimate:

$$\begin{aligned}
|z|_q &\leq |(I - Q_1)P_1z|_q + |Q_1(I - Q_2)P_2z|_q + \frac{r}{q}|P_3Q_1Q_2(I - Q_3)P_2z|_r \\
&\leq \frac{q'}{q}\|(I - Q_1)P_1z\|_{q'} + \frac{q'}{q}\,\|(I - Q_2)P_2z\|_{q'} + \frac{r}{q\sigma}\,|Q_1Q_2(I - Q_3)P_2z|_s \\
&\leq \frac{q't}{qc}\|P_1z\|_{p'} + \frac{q'}{qc\tau}\,\|P_2z\|_{r'} + \frac{rs'}{q\sigma s}\,\|(I - Q_3)P_2z\|_{s'} \\
&\leq \frac{q'tp}{qcp'}\,|P_1z|_p + \frac{q'r}{qc\tau r'}\,|P_2z|_r + \frac{rs'\sigma}{q\sigma scr}\,\|P_2z\|_{r'} \\
&\leq \frac{q'p}{qcp'}|z|_q + \frac{q'r}{qcr'}|z|_q + \frac{\tau s'r}{qscr'}|z|_q \leq \frac{3}{4}|z|_q.
\end{aligned}$$

Hence the assumption $z \neq 0$ leads to the contradiction $1 \leq \frac{3}{4}$.

The proof of (c′) is analogous, but it is necessary to use instead of (4.5) the representation

$$z = P_1P_2(I - Q_1)z + P_1P_2Q_1(I - Q_2)z + P_1P_2Q_1Q_2(I - Q_3)P_3z$$

and to estimate $\|z\|_{q'}$ instead of $|z|_q$. □

4.3.

Let V_i, U_i be absolutely convex absorbing sets in the linear space $X, i = 1,2,3$, satisfying

$$V_1 \subset U_1, \qquad U_2 \subset V_2, \qquad V_3 \subset U_3. \tag{4.8}$$

Then the set

$$\alpha\, V_1 \cap V_3, \qquad conv(V_1 \cup \beta V_3)$$

are also absolutely-convex and absorbing in X. Consider n-diameters

$$d_n(\alpha V_1 \cap V_3, V_2), \quad d_n(V_2, conv(V_1 \cup \beta V_3)). \tag{4.9}$$

Using the simplest properties of u-diameters (see [8], p.73) we get under conditions (4.8) the inequalities

$$d_n(\alpha V_1 \cap V_3, V_2) \le d_n(\alpha U_1 \cap U_3, U_2), \tag{4.10}$$

$$d_n(U_2, conv(U_1 \cup \beta U_3)) \le d_n(V_2, conv(V_1 \cap \beta V_3)). \tag{4.11}$$

The use of characteristics (4.9) in the proof of statement (b), (b$'$) of Lemma 4.1 is based on (4.10), (4.11) and the next two (obvious!) lemmas. Before stating them we put for $a = (a_i), a_i > 0$,

$$\begin{aligned} B_p(a) &= \left\{ \xi = (\xi_i) : \left(\sum_{i=1}^{\infty} |\xi_i|^p \, a_i^p \right)^{1/p} \le 1 \right\}, \qquad 1 < p < \infty, \\ B_\infty(a) &= \left\{ \xi = (\xi_i) : \sup_{1 \le i \le \infty} |\xi_i| \, a_i \le 1, \qquad \xi_i \to 0 \right\} \end{aligned}$$

Lemma 4.2. *If $a = (a_i)$ and $b = (b_i)$, then*

$$\begin{aligned} B_p(c) &\subset B_p(a) \cap B_p(b) \subset B_p(2^{-\frac{1}{p}} c) \\ B_p(2^{1-\frac{1}{p}} d) &\subset conv\{B_p(a) \cup B_p(b)\} \subset B_p(d) \end{aligned}$$

where $c = (c_i)$, $d = (d_i)$ and $c_i = \max\{a_i, b_i\}$, $d_i = \min\{a_i, b_i\}$.

Lemma 4.3. *(see, for example [29]). If $\frac{a_n}{b_n} \downarrow 0$, then for any $1 \le p,\ q \le \infty$ we have*

$$d_{n-1}(B_q(a), B_p(b)) = \frac{a_n}{b_n}, \qquad n = 1, 2, \dots .$$

Now we shall prove a statement, which implies immediately (b) and (b$'$).

Proposition 4.1. *Let X be a linear space and $\{h_\nu, h'_\nu\}_{\nu\in\mathbb{N}}, \{g_\mu, g'_\mu\}_{\mu\in M}$ be two biorthogonal systems in X and X be a normal space with respect to systems $\{h_\nu, h'_\nu\}$ and $\{g_\mu, g'_\mu\}$ (i.e. if $x \in X$ and $|\xi_\nu| \le |h'_\nu(x)|\ \forall \nu \in N$, then there exists $y \in X$, such that $\xi_\nu = h'_\nu(y)\ \forall \nu \in N$, and analogously for the system $\{g_\mu, g'_\mu\}$). Suppose also that in X are defined two triples of ℓ_p-norms $(1 \le p \le \infty$:*

$$\|x\|_i = \left(\sum_\nu |h_\nu(x)|^p\, a_i(\nu)^p\right)^{1/p}, \tag{4.12}$$

$$|x|_i = \left(\sum_\mu |g_\mu(x)|^p\, b_i(\mu)^p\right)^{1/p}, \tag{4.13}$$

$$\|x\|_1 \le |x|_1, \quad |x|_2 \le \|x\|_2, \quad \|x\|_3 \le |x|_3, \tag{4.14}$$

where $a_i(\nu) > 0$, $b_i(\mu) > 0$, $\nu \in N$, $\mu \in M$, $i = 1,2,3$. Then for any $t, \tau > 0$ the estimates

$$\left|\left\{\mu : \frac{b_2(\mu)}{b_1(\mu)} \ge \tau,\ \frac{b_3(\mu)}{b_2(\mu)} \le t\right\}\right| \le \left|\left\{\nu : \frac{a_2(\nu)}{a_1(\nu)} \ge 2^{-\frac{1}{p}}\tau,\ \frac{a_3(\nu)}{a_2(\nu)} \le 2^{\frac{1}{p}}t\right\}\right|, \tag{4.15}$$

$$\left|\left\{\nu : \frac{a_2(\nu)}{a_1(\nu)} \le t,\ \frac{a_3(\nu)}{a_2(\nu)} \ge \tau\right\}\right| \le \left|\left\{\mu : \frac{b_2(\mu)}{b_1(\mu)} \le 2^{1-\frac{1}{p}}t,\ \frac{b_3(\mu)}{b_2(\mu)} \ge 2^{\frac{1}{p}-1}\tau\right\}\right| \tag{4.16}$$

hold.

Proof. Put

$$V_i = \{x \in X : |x|_i \le 1\}, \qquad U_i = \{x \in X : \|x\|_i \le 1\}.$$

Then by (4.14) the sets V_i, U_i satisfy the relations (4.8), therefore the inequalities (4.10) and (4.11) hold. These inequalities imply directly

$$\left|\left\{n : \frac{1}{d_{n-1}(\alpha U_1 \cap U_3, U_2)} \le t\right\}\right| \le \left|\left\{n : \frac{1}{d_{n-1}(\alpha V_1 \cap V_3, V_2)} \le t\right\}\right|,$$

$$\left|\left\{n : \frac{1}{d_{n-1}(V_2, conv(V_1 \cup \beta V_3))} \le t\right\}\right| \le \left|\left\{n : \frac{1}{d_{n-1}(U_2, conv(U_1 \cup \beta U_3))} \le t\right\}\right|.$$

By Lemma 4.2 and Lemma 4.3 we get from these relations

$$\left|\left\{\mu : \frac{\max\{b_1(\mu)/\alpha, b_3(\mu)\}}{b_2(\mu)} \le t\right\}\right| \le \left|\left\{\nu : \frac{\max\{a_1(\nu)/\alpha, a_3(\nu)\}}{a_2(\nu)} \le 2^{\frac{1}{p}}t\right\}\right|$$

$$\left|\left\{\nu : \frac{a_2(\nu)}{\min\{a_1(\nu), a_3(\nu)/\beta\}} \le t\right\}\right| \le \left|\left\{\mu : \frac{b_2(\mu)}{\min\{b_1(\mu), b_3(\mu)/\beta\}} \le 2^{1-\frac{1}{p}}t\right\}\right|.$$

Puting $\alpha = 1/t\tau$, $\beta = \tau t$ we obtain by these the inequalities (4.15) and (4.16). □

It is easy to see that the claims (b), (b′) of Lemma 4.1 are consequences of Proposition 4.1.

4.4.

Corollary 4.1. *If* $E(\lambda, a) \simeq E(\mu, b)$, *then* $M_a(t,\tau) \approx M_b(t,\tau)$. *In particular, if* $a_i \nearrow \infty$ *then* $m_a(t) \approx m_b(t)$.

Let $\varphi(p,q,r,t,\tau)$ denote the left-hand side of the inequality (4.1), written for the matrix $a_{ip} = \exp(-\frac{1}{p} + \lambda_i p) a_i$ of the space $E(\lambda, a)$, and $\psi(p,q,r,t,\tau)$ be the same for the space $E(\mu, b)$. Then for $q > 2p, r > 2q$ we have

$$M_a(t/2p, 2q\tau) \leq \varphi(p,q,r,e^t,e^\tau) \leq M_\alpha(2pt, \tau/2r)$$

and analogously,

$$M_b(t/2p', 2q'\tau) \leq \psi(p',q',r',e^t,e^\tau) \leq M_b(2p't, \tau/2r),$$

where $q' > 2p', r' > 2q'$.

Using norms with enough taken apart indices $p << p' << \cdots << r'$ we derive by the statement (b$'$) of Lemma 4.1

$$M_a(t/2q, 2q\tau) \leq M_b\left(2p'(t + \ln c), \frac{\tau - \ln c}{2r}\right),$$

hence we get

$$M_a(t,\tau) \leq M_b(\Delta t, \tau/\Delta) \quad \text{if} \quad \Delta \geq 4qp'(1 + \ln c).$$

(Let us remind that we always assume $a_i \geq 1, b_i \geq 1$ - see item 2.1).

By the symmetry with respect to a and b the corollary is proved.

Corollary 4.2. *If* $F(\lambda, a) \simeq F(\mu, b)$ *then* $M_a(t,\tau) \approx M_b(t,\tau)$. *In particular, if* $a_i \nearrow \infty$ *and* $b_i \nearrow \infty$, *then* $m_a(t) \approx m_b(t)$.

The proof of this corollary is analogous to the proof of Corollary 4.1.

§5. Applications of Invariants to First Type Power Spaces

5.0.

Here the general invariants, considered in §4, are used to introduce more specific invariants on the class of first type power spaces. The main result of this paragraph is Lemma 5.2, which shows, that on the class of spaces $X \in \mathcal{E}$ satisfying the additional condition $X \overset{p}{\simeq} X^2$, these specific invariants give a complete isomorphic classification.

In items 5.3-5.4 the consequences of Lemma 5.2. are considered. In particular, the question of isomorphic classification of tensor products of finite and infinite power series spaces is studied.

Finally in items 5.5-5.6 there are applications to spaces of infinitely-differentiable and analytic functions with values in locally convex spaces.

5.1.

For spaces in the class $\mathcal{E}$ the invariant characteristics, considered in Lemma 4.1, can be reduced to a more simple form.

Lemma 5.1. *If* $E(\lambda,a)\simeq E(\mu,b)$*, then* $\forall B\,\exists A\,\forall\delta\,\exists\epsilon$*:*

(a) $|\{i:\mu_i>\delta,\ t/B<b_i\le Bt\}|\le|\{i:\lambda_i>\epsilon,t/A<a_i\le At\}|$

(b) $|\{i:\mu_i<\epsilon,\ t/B<b_i\le Bt\}|\le|\{i:\lambda_i<\delta,t/A<a_i\le At\}|$

(c) $\forall\epsilon'\,\exists\delta'$

$$|\{i:\epsilon'<\mu_i\le\epsilon,\ t/B<b_i\le Bt\}|\le|\{i:\delta'<\lambda_i\le\delta,t/A<a_i\le At\}| \tag{5.1}$$

and also relations (a′), (b′), (c′) hold, which are obtained from (a), (b), (c) by changing places of λ,a *and* μ,b*.*

Proof. Since the proofs of (a), (b) and (c) are analogous, we shall prove only the most complicated relation (c).

Let $\varphi(\pi,t,\tau,\sigma)$ be the left-hand side of the inequality (4.1), written for the matrix

$$a_{ip}=\exp\left(-\frac{1}{p}+\mu_i p\right)b_i$$

of the Köthe space $K(a_{ip})=E(\mu,b)$, where $\pi=(p,q,r,s)$. Let $\psi(\pi,t,\tau,\sigma)$ be the analogous characteristic for the space $E(\lambda,a)$. We consider the functions

$$\begin{aligned}\Phi(t) &= \Phi(\pi,\delta,\gamma,t)\overset{def}{=}\varphi(\pi,\exp((1+\delta),t,\exp,(1-\delta)t,\exp\gamma t),\\ \Psi(t) &= \Psi(\pi,\delta,\gamma,t)\overset{def}{=}\psi(\pi,\exp,(1+\delta)t,\exp,(1-\delta)t,\exp\gamma t),\end{aligned}$$

where $\gamma>2$. By Lemma 4.1 (c) we obtain
$\forall p'\,\exists p\,\forall q\,\exists q'\,\forall r'\,\exists r\,\forall s\,\exists s'\,\exists c$:

$$\begin{aligned}\Phi(\pi,\delta,\gamma,t) &\le \psi(\pi',\exp[(1+\delta)t-\ln c],\exp[(1-\delta)t+\ln c],\exp[(\gamma-1+\delta)t-\ln c])\\ &\le \Psi(\pi',\delta/2,\gamma/2,t),\quad t\ge\tau_0,\end{aligned} \tag{5.2}$$

where $\pi'=(p',q',r',s')$ and the constant τ_0 depends on the parameters π,π'.

It is easy to see that for any $m>1$ we have

$$\begin{aligned}&\left|\left\{i:\frac{1}{mqr}<\lambda_i\le\frac{1}{qr},\ 4pt<a_i\le\frac{q}{4}t\right\}\right|\\ &\quad\le\Psi(t)\le\left|\left\{i:\frac{1}{4qs}<\lambda_i\le\frac{4}{pr},\ \frac{pt}{4}<a_i\le 4qt\right\}\right|,\end{aligned} \tag{5.3}$$

if the indices p, q, r, s are taken enough apart:

$$q > 2p, \qquad r > 4q, \qquad s > \frac{m\gamma rq}{p}. \tag{5.4}$$

Let us consider now the inequality (5.2) only for sets of indices $\pi = (p, q, r, s)$, $\pi' = (p', q', s', r')$ satisfying (5.4). If we use first the left-hand inequality (5.3) for the function $\phi(t)$, then the inequality (5.2) and, finally, the right-hand inequality (5.3) for the function $\psi(\pi', \frac{\delta}{2}, \frac{\gamma}{2}, t)$, we obtain the inequality

$$\left|\left\{i : \epsilon' < \mu_i < \epsilon,\ 4pt < b_i \le \frac{q}{4}t\right\}\right| \le \left|\left\{i : \delta' < \lambda_i \le \delta,\ \frac{p'}{4}t < a_i \le 4q't\right\}\right|$$

where $\epsilon = 1/qr, \epsilon' = 1/mqr, \delta = 4/p'r', \delta' = 1/4q's'$, and $t \ge \tau_0$. Changing t with $t/\sqrt{pq}$ and puting

$$B = \frac{1}{4}\sqrt{\frac{q}{p}}, \qquad A = \sup\left\{\frac{4\sqrt{pq}}{p'}, \frac{4q'}{\sqrt{pq}}\right\}$$

we get (5.1). The remaining arbitrariness of parameters π, π', m ensure that the statement (c) of Lemma 5.1 is true but for $t \ge t_0$, where t_0 depends on $\delta, \epsilon, \epsilon', \delta'$..

It remains to show that the restriction $t \ge t_0$ in relations (a), (b), (c) can be really eliminated: for this purpose we use the freedom in the choice of parameters ϵ, δ'. By Corollary 4.1 we can assume that B and $A = A(B)$ are such that the estimate

$$M_a(Bt, t/B) \le M_b(At, t/B)$$

holds for each $t \ge 1$ (we take in account (2.1)).

If we suggest that (b) holds for $t \ge t_0 = t_0(\delta)$, then after taking

$$\tilde{\epsilon} = \min\{\epsilon, \min\{\mu_j : b_j \le Bt_0\}\}$$

instead of ϵ we get (b) for all $t \ge 1$. If (c) is true for $t \ge t_0$, $t_0 = t_0(\delta, \epsilon')$, we can assume that $\epsilon = \epsilon(\delta)$ is already chosen in such a way that (b) holds for all $t \ge 1$, then it is enough to take

$$\tilde{\delta}' = \min\{\delta', \min\{\lambda_i : a_i \le At_0\}\}$$

instead of δ'. In the case of (a) we need to take

$$\tilde{\epsilon} = \min\{\epsilon, \min\{\lambda_i : a_i \le At_0\}\}$$

instead of ϵ. □

5.2.

The next lemma is partially inverse to the previous statement in some extent. It shows particularly that the invariant considered in Lemma 5.1 is complete on some subclass of the class $\mathcal{E}$ (see Theorem 5.1 below).

Lemma 5.2. *Let $X = E(\lambda, a), Y = E(\mu, b)$ and $\forall B \,\exists A \,\forall \delta \,\exists \epsilon$ such that the conditions (a), (b), (c) in Lemma 5.1 are fulfilled. Then there exists an integer k such that $Y \overset{p}{\to} X^k$.*

Proof. Let $B > 1$ and $A = A(B)$, $\epsilon = \epsilon(\delta)$, $\delta' = \delta'(\epsilon')$ be chosen in such a way that conditions (a), (b), (c) in Lemma 5.1 are fulfilled. We consider the function $\varphi(t) = \min\{\epsilon(t), \delta'(t)\}$ and bild a sequence $\delta_s \uparrow 0$ such that $\delta_0 = 1$, $\delta_{s+1} = \varphi(\delta_s)$, $s \in \mathbb{Z}_+$. Let us choose $m \in \mathbb{N}$ such that $A \le B^m$ and put $\Delta = B^2$. We introduce the following families of subsets of $\mathbb{N}$:

$$N_{r,s} = \{i : \delta_{s+1} < \lambda_i \le \delta_s;\ \Delta^{r-1} < a_i \le \Delta^r\}, \qquad r, s \in \mathbb{Z}_+,$$

$$M_{r,s} = \{j : \delta_{s+1} < \mu_j \le \delta_s;\ \Delta^{r-1} < b_j \le \Delta^r\}, \qquad r, s \in \mathbb{Z},$$

$$\begin{aligned} \tilde{M}_{r,s} &= \{j : \delta_{s+2} < \mu_j \le \delta_{s-1};\ \Delta^{r-m-1} < b_j \le \Delta^{r+m}\} \\ &= \bigcup_{\alpha=1}^{2m+2} \bigcup_{\beta=1}^{4} M_{r-m-2+\alpha,\, s-2+\beta}, \quad r, s \in \mathbb{Z}_+. \end{aligned}$$

Thus, by construction, we have

$$|N_{r,s}| \le |\tilde{M}_{r,s}| = \sum_{\alpha=1}^{2m+2} \sum_{\beta=1}^{4} |M_{r-m-2+\alpha, s-2+\beta}|, \quad r, s \in \mathbb{Z}_+. \tag{5.5}$$

It is obvious that $Y^k \overset{p}{\simeq} E(\tilde{\mu}, \tilde{b})$, where $\tilde{\mu}_\ell = \mu_j$, $\tilde{b}_\ell = b_j$, if

$$\ell = k(j-1) + \nu, \ j \in \mathbb{N}, \quad \nu \in \mathbb{N}_k := \{1, 2, \ldots, k\}.$$

Let us take any bijection $\gamma : \mathbb{N}_{2m+2} \times \mathbb{N}_3 \to \mathbb{N}_k$, where $k = 8m + 8$ and construct the following subsets of $\mathbb{N}$:

$$M'_{r,s} = \bigcup_{\alpha=1}^{2m+2} \bigcup_{\beta=1}^{4} \{\ell = k(j-1) + \gamma(\alpha, \beta) : j \in M_{r-m-2+\alpha, s-2+\beta}\}, \quad r, s \in \mathbb{Z}_+.$$

Then $|\tilde{M}_{r,s}| = |M'_{r,s}|$ and, due to (5.5), there exists an injection $\sigma_{r,s} : N_{r,s} \to M'_{r,s}$. Since $M'_{r,s} \cap M'_{r',s'} = \emptyset$, $(r,s) \ne (r',s')$, the injection $\sigma : \mathbb{N} \to \mathbb{N}$ is defined by $\sigma(i) = \sigma_{r,s}(i)$, $i \in N_{r,s}$, $r, s \in \mathbb{Z}_+$. According to Lemma 2.2 the formula $T_{e_i} = e_{\sigma(i)}$ generates a permutational isomorphic imbedding $T : E(\lambda, a) \to E(\tilde{\mu}, \tilde{b})$: in fact, by the construction of σ, $a_i \asymp \tilde{b}_{\sigma(i)}$, and for any subsequence $\{i_j\} \subset \mathbb{N}$ the condition $\lambda_{i_j} \to 0$ is equivalent to $\tilde{\mu}_{i_j} \to 0$. □

Remark. The above-mentioned proof is a very simplification of the corresponding proof in [55] (see, also [51]); it can be realized due to the new version of Lemma 5.1 (with usual inequalities instead of asymptotic ones in [55]).

5.3.

Here we present results on isomorphisms of spaces of the kind $E(\lambda, a)$, which are consequences of Lemma 5.1 and Lemma 5.2. We begin with

Corollary 5.1. *If $X = E(\lambda, a) \simeq Y = E(\mu, b)$, then there exists an integer k such that $X \overset{p}{\tilde{\to}} Y^k, Y \overset{p}{\tilde{\to}} X^k$.*

Indeed, by Lemma 5.1 the isomorphism $X \simeq Y$ implies the conditions (a), (b), (c) in Lemma 5.1 and the symmetric to them conditions (a′), (b′), (c′). By Lemma 5.2 from these conditions it follows that there exists a constant k such that

$$X \overset{p}{\tilde{\to}} Y^k, \qquad Y \overset{p}{\tilde{\to}} X^k.$$

The next statement shows that for a wide subclass of spaces of the class $\mathcal{E}$ the problem of isomorphism is equivalent to the essentially simplier problem to study quasidiagonal (or even permutational) isomorphisms.

Theorem 5.1. *If $X = E(\lambda, a), Y = E(\mu, b)$ and $X \overset{p}{\simeq} X^2$, then $X \simeq Y \Leftrightarrow X \overset{p}{\simeq} Y$.*

Proof. By Corollary 5.1 there exists k such that $X \overset{p}{\tilde{\to}} Y^k, Y \overset{p}{\tilde{\to}} X^k$. Then obviously we have $Y^k \overset{p}{\tilde{\to}} X^{k^2}$. On the other hand $X \overset{p}{\simeq} X^2$ implies $X \overset{p}{\simeq} X^k \overset{p}{\simeq} X^{k^2}$, therefore we obtain $X^k \overset{p}{\simeq} Y^k, Y^k \overset{p}{\simeq} X^k$. By Lemma 1.1 it follows that $X^k \overset{p}{\simeq} Y^k$, which implies $X \overset{p}{\simeq} Y$. □

The last conclusion in the previous proof follows from the following fact, which is a simple consequence of the criteria of the quasidiagonal isomorphism for Köthe spaces [10].

Proposition 5.1. *Let $m \in \mathbb{N}$. Then*

$$K(A) \overset{qd}{\simeq} K(B) \Leftrightarrow K(A)^m \overset{qd}{\simeq} K(B)^m.$$

Corollary 5.2. *If $X = E_0(a) \otimes E_\infty(b)$, $Y = E_0(a') \otimes E_\infty(b')$ and either $E_0(a) \simeq E_0(a)^2$ or $E_\infty(b) \simeq E_\infty(b)^2$, then $X \simeq Y \Leftrightarrow X \overset{p}{\simeq} Y$.*

Indeed, suppose for example $E_0(a) \simeq E_0(a)^2$. Then by the quasiequivalence of unconditional bases in the space $E_0(a)$ it follows that $E_0(a) \overset{p}{\simeq} E_0(a)^2$. Therefore $X \overset{p}{\simeq} [E_0(a) \times E_0(a)] \otimes E_\infty(b) \overset{p}{\simeq} X^2$, and taking into account Lemma 2.1 we get by Theorem 5.1 that $X \overset{p}{\simeq} Y$.

For tensor products of the kind $E_0(a) \otimes E_\infty(b)$ the invariants, described in Lemma 5.1, become simplier as the next lemma shows.

Lemma 5.3. *Let*

$$X = E_0(a) \otimes E_\infty(b) \simeq Y = E_0(a') \otimes E_0(b').$$

Then $\forall A \, \exists B$:

(a) $m_a(At)[m_b(At) - m_b(t/A)] \le m_{a'}(Bt)[m_{b'}(Bt) - m_{b'}(t/B)]$;

(b) $\forall \epsilon \, \exists \delta : [m_a(At) - m_a(t/A)] m_b(\delta t) \le [m_{a'}(Bt) - m_{a'}(t/B)] m_{b'}(\epsilon t)$,

(c) $\forall \epsilon \, \exists \delta \, \forall \delta' \, \exists \epsilon' : [m_a(At) - m_a(t/A)][m_b(\delta t) - m_b(\delta' t)]$
$\le [m_{a'}(Bt) - m_{a'}(t/B)][m_{b'}(\epsilon t) - m_{b'}(\epsilon' t)]$

and the relations (a′), (b′), (c′), which appear after changing the roles of a, b *and* a', b' *in the above relations, hold.*

Proof. Let $(i_k, j_k), k \in \mathbb{N}$ be an ordering of the set $\mathbb{N} \times \mathbb{N}$ into a sequence, $c = (c_k), c_k = a_{i_k} + b_{j_k}; \lambda = (\lambda_k), \lambda_k = b_{j_K}/c_k$. Then $X \overset{p}{\simeq} E(\lambda, c)$.

Let us denote

$$\gamma(\delta, A, t) = |\{(i,j) : b_j > \delta a_i, \; t/A < a_i + b_j \le At\}|.$$

Assuming that $0 < \delta \le 1/A^2$ we obtain the following estimates:

$$\gamma(\delta, A, t) \le M_a(At)\left[M_b(At) - M_b\left(\frac{\delta t}{2A}\right)\right] \le M_a(Ct)[M_b(Ct) - M_b(t/C)],$$

where $C = 2A/\delta$, and

$$\begin{aligned} \gamma(\delta, A, t) \;&\ge\; |\{(i,j) : b_j > 2\delta a_i, \; t/A < a_i + b_j \le At\}| \\ &\ge\; M_a\left(\frac{At}{2}\right)\left[M_b\left(\frac{At}{2}\right) - M_b\left(\frac{2t}{A}\right)\right] \ge M_a(Ct)\left[M_b(Ct) - M_b\left(\frac{t}{C}\right)\right], \end{aligned}$$

where $C = A/2$.

These estimates show, that statement (a) of the present lemma follows by statement (a) of Lemma 5.1.

Analogously statement (b) can be obtained as a consequence of Lemma 5.1. (b) with the help of the following inequalities:

$$\begin{aligned} m_b\left(\frac{\delta t}{A}\right)\left[m_a\left(\frac{At}{2}\right)-m_a\left(\frac{t}{A}\right)\right] &\leq \left|\left\{k:\lambda_k\leq\delta,\ \frac{t}{A}<c_k\leq At\right\}\right| \\ &\leq m_b(2\delta At)\left[m_a(At)-m_a\left(\frac{t}{4A\delta}\right)\right]. \end{aligned}$$

Finally (c) can be obtained by Lemma 5.1 (c) and estimates for the function

$$\gamma(\delta,\delta',A,t)=\left|\left\{k:\delta'<\lambda_k\leq\delta,\ \frac{t}{A}<c_k\leq At\right\}\right|$$

as follows:

$$\begin{aligned} \gamma(\delta,\delta',A,t) &\leq |\{(i,j):\delta' a_i<b_j\leq 2\delta a_i,\ \frac{t}{A}<a_i+b_j\leq At\}| \\ &\leq \left[m_a(At)-m_a\left(\frac{t}{2A}\right)\right]\left[m_b(2\delta At)-m_b\left(\frac{\delta' t}{A}\right)\right]; \\ \gamma(\delta,\delta',A,t) &\geq |\{(i,j):2\delta' a_i<b_j\leq\delta a_i,\ \frac{t}{A}<a_i+b_j\leq At\}| \\ &\geq \left[m_a\left(\frac{At}{2}\right)-m_a\left(\frac{2t}{A}\right)\right]\left[m_b\left(\frac{2\delta t}{A}\right)-m_b\left(\frac{\delta' At}{2}\right)\right] \\ &\geq \left[m_a(Ct)-m_a\left(\frac{t}{C}\right)\right][m_b(\epsilon t)-m_b(\epsilon' t)], \end{aligned}$$

where $\delta'<<1/A^2, C=A/2, \epsilon=2\delta/A, \epsilon'=\delta' A/2<<\epsilon$. □

The next theorem shows that for tensor products, considered in Corollary 5.2, the invariant of Lemma 5.3 is not only essentially simplified but complete (i.e. it distinguishes nonisomorphic spaces).

Theorem 5.3. *Suppose the sequences $a=(a_i), a'=(a'_i)$ satisfy the conditions*

$$a_i\uparrow\infty,\quad a_{2i}\asymp a_i;\quad a'_i\uparrow\infty,\quad a'_{2i}\asymp a'_i \tag{5.6}$$

and let sequences $b=(b_j), b'=(b'_j)$ are such that $b_j\uparrow\infty$, $b'_j\uparrow\infty$. Then the isomorphism

$$X=E_0(a)\otimes E_\infty(b)\simeq Y=E_0(a')\times E_\infty(b') \tag{5.7}$$

is equivalent to the set of the following conditions:

(a) $\exists\, B\ \forall\,\epsilon\ \exists\,\delta: m_a(t)m_b(\delta t)\leq m_{a'}(Bt)m_{b'}(\epsilon t)$;

(b) $\exists B\, \forall \epsilon\, \exists \delta\, \forall \delta'\, \exists \epsilon'$:

$$m_a(t)[m_b(\delta t) - m_b(\delta' t)] \le m_{a'}(Bt)[m_{b'}(\epsilon t) - m_{b'}(\epsilon' t)]$$

and conditions (a′), (b′), which appear from (a) and (b) after changing the places of sequences a, b and a', b'.

On the other hand the isomorphism

$$X = E_0(b) \otimes E_\infty(a) \simeq Y = E_0(b') \otimes E_\infty(a') \tag{5.8}$$

is equivalent to the set of the following conditions:

(c) $\exists B : m_a(t) m_b(t) \le m_{a'}(Bt) m_{b'}(Bt)$

(d) $\forall A\, \exists B\, \forall \epsilon\, \exists \delta$:

$$[m_b(At) - m_b(t/A)] m_a(\delta t) \le [m_{b'}(Bt) - m_{b'}(t/B)]\;\; m_{a'}(\epsilon t)$$

and symmetric conditions (c′), (d′).

Proof. The conditions (5.6) are equivalent to the following conditions:

$$m_a(t) < \infty,\; m_{a'}(t) < \infty, \quad \exists \Delta \mid 2m_a(t) \le m_a(\Delta t),\;\; 2m_{a'}(t) \le m_{a'}(\Delta t). \tag{5.9}$$

Therefore for $A > \Delta^2$ we have the inequalities

$$m_a(At) \ge m_a(At) - m_a(t/A) \ge m_a(At) - m_a\left(\frac{A}{\Delta} t\right) \ge m_a(\Delta t),$$

which imply immediately the equivalence of conditions (b), (c) of Lemma 5.3 and conditions (a), (b) of the theorem. It is easy to see that condition (a) of Lemma 5.3 is a consequence of condition (b) of the theorem. Thus the three conditions (a), (b), (c) of Lemma 5.3 are equivalent to the two conditions (a), (b) of the theorem. Since the conditions (5.6) imply the isomorphism $X \overset{p}{\simeq} X^2$ we obtain by Lemmas 5.2, 5.3 that conditions (a), (b) together with the symmetric to them conditions (a′), (b′) are equivalent to the isomorphism $X \simeq Y$.

The second part of the theorem can be proved in an analogous way. □

The next two corollaries show that the diametral dimension is a complete invariant on a quite wide classes of tensor products.

Corollary 5.3. *If the sequences a, a', b, b' satisfy conditions (5.6) then the isomorphism (5.7) is equivalent to $\gamma(X) = \gamma(Y)$, i.e. $X \simeq Y \Leftrightarrow \exists B\, \forall \epsilon\, \exists \delta$:*

$$m_a(t) m_b(\delta t) \le m_{a'}(Bt) m_{b'}(\epsilon t), \quad m_{a'}(t) m_{b'}(\delta t) \le m_a(Bt) m_b(\epsilon t).$$

Indeed, since the sequences b, b' satisfy also (5.9), we obtain that conditions (b) and (c) of Theorem 5.3 coincide. This proves the corollary.

Corollary 5.4. *If the sequences a, a' satisfy conditions*

$$a_{i^2} \asymp a_i, \qquad a'_{i^2} \asymp a'_i \tag{5.10}$$

and the sequences b, b' satisfy conditions

$$b_i \uparrow \infty, \quad b'_i \uparrow \infty, \quad b_{i+1} \asymp b_i, \quad b'_{i+1} \asymp b'_i \tag{5.11}$$

$$a_i = O(b_i), \qquad a'_i = O(b'_i)$$

then the isomorphism (5.7) is equivalent to any one of the following conditions:

$$\gamma(X) = \gamma(Y) \Leftrightarrow \gamma(E_0(a)) = \gamma(E_0(a')) \Leftrightarrow a_i \asymp a'_i.$$

Proof. The conditions of Corollary are equivalent to the following relations

$$\exists\,\Delta : m_a(t)^2 \le m_a(\Delta t), \quad m_b(\Delta t) - m_b(t) \ge 1, \quad m_b(t) \le m_a(\Delta t)$$

and the same conditions for a', b'.

If $\delta' < \Delta\delta$ then these conditions imply the inequalities

$$m_a(At) \lesssim m_a(At)[m_b(\delta t) - m_b(\delta' t)] \le m_a(At) m_a(\delta t) \lesssim (m_a(At))^2 \le m_a(\Delta A t)$$

which show, that conditions (b) and (c) of Theorem 5.3 are equivalent here to the condition:

$$\forall\, A\, \exists\, B \qquad m_a(At) \le m_{a'}(Bt).$$

Therefore $X \simeq Y \Leftrightarrow m_a(t) \asymp m_{a'}(t) \Leftrightarrow a_i \asymp a'_i \Leftrightarrow \gamma(E_0(a)) = \gamma(E_0(a')) \Leftrightarrow \gamma(X) = \gamma(Y)$, which proves the corollary. □

Corollary 5.5. *If the sequences a, a', b, b' satisfy the conditions (5.9), (5.10) and*

$$a_i = o(b_i), \qquad a'_i = o(b'_i),$$

then the isomorphism (5.8) is equivalent to any one of the following conditions:

$$\gamma(X) = \gamma(Y) \Leftrightarrow \gamma(E_0(a)) = \gamma(E_0(a')) \Leftrightarrow a_i \asymp a'_i.$$

The proof is analogous to the previous.

Corollary 5.6. *(See, [13], [50], [51]).*
$E_0(i^\alpha)\otimes E_0(i^\beta)\overset{p}{\simeq}E_0(i^{\alpha'})\otimes E_0(i^{\beta'})\Leftrightarrow 1/\alpha+1/\beta=1/\alpha'+1/\beta'$.
Proof. If $a=(i^\alpha), b=(i^\beta), a'=(i^{\alpha'}), b'=(i^{\beta'})$, then $t^{1/\alpha}-1\le m_a(t)\le t^{1/\alpha}$ and we have analogous relations for a', b, b'. Therefore the couple of conditions of Corollary 5.3 are equivalent to the following relations:

$$\exists C: t^{1/\alpha+1/\beta}\le C\,t^{1/\alpha'+1/\beta'},\ t^{1/\alpha'+1/\beta'}\le C\,t^{1/\alpha+1/\beta}$$

which are equivalent to $1/\alpha+1/\beta=1/\alpha'+1/\beta'$. Hence the proof is over by Corollary 5.3. □

Example 5.1. Suppose

$$X=E_0(e^{i^\alpha})\otimes E_\infty(e^{i^\beta}),\qquad Y=E_0(e^{i^{\alpha'}})\otimes E_\infty(e^{i^{\beta'}}).$$

It is easy to see that the conditions (a), (b), (c) of Lemma 5.3 are equivalent to the condition $\gamma(X)=\gamma(Y)$, which is equivalent to the equality $1/\alpha+1/\beta=1/\alpha'+1/\beta'$. On the other hand it is known (see [49], [53]) that

$$X\simeq Y\Leftrightarrow \alpha=\alpha',\ \ \beta=\beta'.$$

This example shows, that the invariant considered in Lemma 4.1 is not complete in the class of first type power spaces. In papers [46], [48] is considered another approach to the isomorphic classification of spaces, based on Riesz theory of compact operators in locally convex spaces. In many cases (in particular for Example 5.1) this approach is more efficient.

The next example represents very simple spaces for which it remains unknown whether they are isomorphic.

Example 5.2. Let $X=E_0(a)\otimes E_\infty(b)$, $Y=E_0(a')\otimes E_\infty(b')$,

$$a_i=e^{\alpha i},\quad b_i=e^{\beta i},\quad a_i'=e^{\alpha' i},\quad b_i'=e^{\beta' i},$$

where $\alpha,\beta,\alpha',\beta'$ are positive constants.

It is simple to show that

$$\gamma(X)=\gamma(Y)\Leftrightarrow \alpha\beta=\alpha'\beta'. \tag{5.12}$$

Therefore $x\not\simeq y$ if $\alpha\beta\ne\alpha'\beta'$. But neither the invariants considered above, nor those from [49], [53] do not distinguish these spaces if (5.12) holds but $(\alpha,\beta)\ne(\alpha',\beta')$.

Problem. Are the spaces X, Y isomorphic if $\alpha\beta=\alpha'\beta'$, but $(\alpha,\beta)\ne(\alpha',\beta')$?

The proof of the isomorphism of such spaces, represented in [55], has a gap.

5.4.

In this point we give the results on quasi-equivalence of bases in nuclear spaces of the class $\mathcal{E}$, which follow by Theorem 5.1.

We begin with the well-known Dragilev-Bessaga lemma, which play a significant role in the study of the quasi-equivalence problem in nuclear Fréchet spaces.

Lemma 5.3. *(See Dragilev, Bessaga [7, [16], [21]). If $\{x_k\}$ and $\{y_k\}$ are two bases in a nuclear Fréchet space X, which topology is generated by the norms $\|x\|_p, p = 1, 2, \dots$, then there exists sequence of indices $n_k \to \infty$ and sequence of positive numbers γ_k, such that*

$$K(\|x_k\|_p) \overset{e}{\simeq} K(\gamma_k \|y_{n_k}\|_p).^{\dagger}$$

Theorem 5.4. *If the space $X = E(\lambda, a)$ is nuclear and $X \simeq X^2$ then all bases in X are quasi-equivalent.*

Proof. Let $\{x_k\}$ be an arbitrary basis in X and $\{e_i\}$ be the canonical basis in X. By Lemma 5.3 there exist a sequence of integers $\{k_i\}, k_i \to \infty$, and a sequence of positive numbers $\{\gamma_i\}$ such that the basis $\{\gamma_i x_{k_i}\}$ is equivalent to the canonical basis of the Köthe space $Y = E(\mu, b)$, where $b = (a_{k_i}), M = (\lambda_{k_i}), Y \simeq X$. By Theorem 5.1 $X \simeq Y$ implies $X \overset{p}{\simeq} Y$. Thus the canonical bases in X and Y are permutationally equivalent, hence the bases $\{x_k\}$ and $\{e_i\}$ are quasi-equivalent. □

Corollary 5.7. *If the space $X = E_0(a) \otimes E_\infty(b)$ is nuclear and either $E_0(a) \simeq E_0(a)^2$ or $E_\infty(b) \simeq E_\infty(b)^2$, then all bases in X are quasi-equivalent.*

Remark. More recent Kondakov-Zahariuta's result on weak quasiequivalence of bases in arbitrary Köthe spaces allows to remove the restriction of nuclearity everywhere in this item (see for details [27]).

5.5.

In this and the next points we consider some applications to concrete functional spaces.

We denote by $C_X^\infty = C_X^\infty[0,1]$ the space of all C^∞-functions on the segment $[0,1]$ with values in a Fréchet space X. This is a locally convex space with the topology, defined by the system of seminorms

$$\|x\|_{s,p} = \max\{\|x^{(s)}(t)\|_p : t \in [0,1]\}, \qquad s = 0, 1, \dots, ; \quad p = 1, 2, \dots$$

†The bases with this property are called weakly quasi-equivalent.

where $\{\|\cdot\|_p, p=1,2,\ldots\}$ is the system of seminorms, generating the topology in X. It is known that

$$C_X^\infty \simeq C_{\mathbb{R}}^\infty \otimes X,$$

where the complete projective tensor product is considered.

The results of the previous point can be applied to obtain isomorphic classification of spaces of C^∞-functions with values in spaces of the class $\mathcal{E}$.

It is known that $C_{\mathbb{R}}^\infty[0,1] \simeq E_\infty(\ln i)$ (see [29], §7). Therefore if $X = E_\infty(a)$ is a Montel infinite type power series space, then the space C_X^∞ is also isomorphic to a power series space $E_\infty(c)$, where the sequence $c=(c_i)$ satisfies $M_c(t) \approx e^t m_a(t)$. If the space $X = E_\infty(a)$ is nuclear (i.e. $\exists\, \Delta : m_a(t) \leq (e^{\Delta t})$), then $m_c(t) \approx e^t$ and therefore $C_X^\infty \simeq C_{\mathbb{R}}^\infty$. In the case X is not nuclear we have $C_X^\infty \not\simeq C_{\mathbb{R}}^\infty$; furthermore, if $a_i = 0(\ln i)$ and $a_{2_i} \asymp a_i$, then $C_X^\infty \simeq X = E_\infty(a)$.

The case, when the space $X = E_0(a)$ is a Montel finite power series space, is more interesting because then even if $E_0(a)$ is nuclear the space C_X^∞ depend on a. In order to state the precise result we put

$$n_a(A,t) = \begin{cases} 1 & \text{if } \{a_i\} \cap (t/A, At) \neq \phi \\ 0 & \text{if } \{a_i\} \cap (t/A, At) = \phi. \end{cases}$$

Theorem 5.5. *If $X = E_0(b), Y = E_0(b')$ and the sequences $b=(b_i), b'=(b'_i)$ satisfy the conditions*

$$b_i/\ln i \to \infty, \qquad b'_i/\ln i \to \infty \tag{5.13}$$

then the isomorphism $C_X^\infty \simeq C_Y^\infty$ is equivalent to the condition

$$\forall\, A\, \exists\, B : n_b(A,t) \lesssim n_{b'}(B,t), \qquad n_{b'}(A,t) \lesssim n_b(B,t). \tag{5.14}$$

In particular $b=(b_i), b_i \uparrow \infty$ and $b_{i+1} \asymp b_i$ then $C_X^\infty \simeq C_{\mathbb{R}}^\infty \otimes E_0(i) \simeq C_{A_1}^\infty$, where A_1 is the space of functions of one complex variable, analytic in the unit disc.

Proof. We have

$$C_X^\infty \simeq E_0(b) \otimes E_\infty(\ln i), \quad C_Y^\infty \simeq E_0(b') \otimes E_\infty(\ln i).$$

Therefore by Theorem 5.3 (considering $a = a' = (\ln i)$) we get that the isomorphism $C_X^\infty \simeq C_Y^\infty$ is equivalent to the conditions (c), (d), (c′), (d′) of Theorem 5.3.

The conditions (5.13) are equivalent to the following:

$$m_B(t) \lesssim e^{\sigma t}, \quad m_{b'}(t) \lesssim e^{\sigma t}, \qquad \forall\, \sigma > 0. \tag{5.15}$$

Therefore the conditions (c), (c′) are trivially satisfied. The condition (d), (d′) of Theorem 5.3 are equivalent to (5.14). Indeed, since $m_a(t) \approx e^t$ we obtain (using (5.14) with $\sigma < \delta/A$) the estimates

$$\begin{aligned} n_b(A,t)e^{\delta t} &\leq [m_b(At) - m_b(t/A)]e^{\delta t} \leq n_b(A,t)m_b(At)e^{\delta t} \\ &\lesssim n_b(A,t)e^{(\sigma A+\delta)t} \leq n_b(A,t)e^{2\delta t}, \end{aligned}$$

which proves the theorem. □

Remark. The above considerations show that on the class of all spaces of the kind $C_X^\infty \simeq E_0(b) \otimes E_\infty(\ln i)$, where b satisfies condition (5.13), the invariant of Theorem 5.3 is essentially stronger then the diametral dimension $\gamma(X)$. Indeed, it follows by Theorem 5.3 that this class of spaces contains a continuum of pairwise nonisomorphic spaces. On the other hand the diametral dimension $\gamma(E_0(b) \otimes E_\infty(\ln i))$ does not depend on b and coincide with $\gamma(E_\infty(\ln i))$.

At last we give a result on quasi-equivalence of bases in spaces C_X^∞.

Corollary 5.7. *If $X = E(\lambda, a)$ is a nuclear first type power space then all bases in the space C_X^∞ are quasi-equivalent.*

This corollary is a simple consequence of Theorem 5.4.

5.6.

Now we consider applications to spaces of analytic functions.

First we give a solution of Problem 2.2 from point 0.2.

Corollary 5.8. *([13], [50], [51]). The spaces $A(U^{n-k} \times \mathbb{C}^k)$, $0 < k < n$, are isomorphic and all bases in these spaces are quasi-equivalent.*

The isomorphism of the spaces follows directly by Corollary 5.3 since

$$A(U^{n-k} \times \mathbb{C}^k) \simeq A(U^{n-k}) \otimes A(\mathbb{C}^k) \simeq E_0(i^{\frac{1}{n-k}}) \otimes E_\infty(i^{\frac{1}{k}}).$$

The quasi-equivalence property is a partial case of Corollary 5.7.

Further we denote by $A(D, X)$ the space of analytic functions in a domain $D \subset \mathbb{C}^n$ with values in a locally convex space X, considered with their natural locally convex topology.

In the case when $X = E(\lambda, a)$ - nuclear first type power space, we get as in Corollary 5.8 that all base in the space $A(U^n, X) \simeq E_0(i^{1/n}) \otimes E(\lambda, a)$ are quasi-equivalent.

It is also interesting to consider the difference between the cases, when X is finite or infinite power series space.

If $X = E_0(a)$ - finite power series space, then the space $A(U^n, X)$ is also a finite power series space because

$$A(U^n, X) \simeq E_0(i^{1/n}) \otimes E_0(a) \simeq E_0(c),$$

where the sequence $c = (c_i)$ has the property $m_c(t) \approx t^n m_a(t)$. Hence the isomorphism

$$X = A(U^n, E_0(a)) \simeq Y = A(U^n, E_0(b))$$

is equivalent to the relation

$$\gamma(X) = \gamma(Y). \tag{5.16}$$

This is equivalent to the condition

$$\exists\, \Delta \mid m_a(t) \lessapprox \Delta m_b(\Delta t), \quad m_b(t) \lessapprox \Delta m_a(\Delta t).$$

On the other hand the isomorphic classification of spaces of the kind $A(U^n, E_\infty(a))$ is more interesting as it is shown in the next result.

Corollary 5.9. *The isomorphism*

$$X = A(U^n, E_\infty(b)) \simeq Y = A(U^n, E_\infty(b'))$$

is equivalent to the conditions (5.17) together with the condition

$$\begin{aligned} \exists\, B\, \forall \epsilon\, \exists \delta\, \forall \delta'\, \exists \epsilon' : \qquad & m_b(\delta t) - m_b(\delta' t) \lesssim B[m_{b'}(\epsilon t) - m_{b'}(\epsilon' t)], \\ & m_{b'}(\delta t) - m_{b'}(\delta' t) \lesssim B[m_b(\epsilon t) - m_b(\epsilon' t)]. \end{aligned} \tag{5.17}$$

This statement follows by Theorem 5.3 (considering $a = a' = (i^{1/n})$), after taking into account that

$$A(U^n, E_\infty(b)) \simeq E_0(i^{1/n}) \otimes E_\infty(b), \quad m_a(t) \approx t^n.$$

The next example shows the existence of a continuum of pairwise nonisomorphic spaces of the kind $A(U^n, E_\infty(a))$ while corresponding spaces of the kind $A(U^n, E_0(a))$ are isomorphic.

Example 5.2. We put $r_k = 2^{2^k}$ and define for any subset $L \subset \mathbb{N}$ a sequence:

$$a(L) = (a_k(L)), \quad a_k(L) = r_k \text{ if } k \in L, \quad a_k(L) = r_{2^k} \text{ if } k \in \mathbb{N} \setminus L.$$

Let us denote by X_L the space $E_\infty(a(L))$, and by Y_L the space $E_0(a(L))$. Then all spaces of the kind $A(U^n, Y_L)$ are isomorphic because the sequences $a = (r_k)$ and $b = a(L)$ satisfy the relation (5.17).

We shall show that among the spaces of the kind $A(U^n, X_L)$ there exists a continuum pairwise nonisomorphic. Let π be the set of all prime numbers $p \geq 3$ and $\mathcal{D}$ be the set of all binary sequences $\alpha = (\alpha_i) : \alpha_i = 0$ or $\alpha_i = 1$. We correspond to any $\alpha \in \mathcal{D}$ the set

$$L_\alpha = \bigcup_{p \in \pi} \{p^i : \alpha_i \neq 0\};$$

then the spaces $A(U^n, X_{L_\alpha})$ and $A(U^n, L_{X_\beta})$ are nonisomorphic if $\alpha \neq \beta$. Indeed, if they are isomorphic then by Corollary 5.9 the sequences $b = (a_k(L_\alpha))$ and $b' = (a_k(L_\beta))$ should satisfy the relation (5.18). In particular for some constants $B < \infty$, $0 < \epsilon' < \delta' < \delta < \epsilon < 1$ we have the inequality

$$m_b(\delta t_k) - m_b(\delta' t_k) \leq B[m_{b'}(\epsilon t_k) - m_b(\epsilon' t_k)], \qquad k \in L_\alpha \setminus L_\beta, \quad k \geq k_0, \tag{5.18}$$

where $t_k = r_k/\sqrt{\delta\delta'}$. By construction the set $L_\alpha \setminus L_\beta$ is infinite. We have for big enough $k \in L_\alpha \setminus L_\beta$ the equality

$$m_b(\delta t_k) - m_b(\delta' t_k) = m_b\left(\sqrt{\frac{\delta}{\delta'}} r_k\right) - m_b\left(\sqrt{\frac{\delta'}{\delta}} r_k\right) = 1, \tag{5.19}$$

because the interval (r_{k-1}, r_k) contains only one element of the sequence $\{b_k\}$ (equal to r_k) but for big enough k the interval

$$\left(\sqrt{\frac{\delta'}{\delta}} r_k, \quad \sqrt{\frac{\delta}{\delta'}} r_k\right)$$

is included into the interval (r_{k-1}, r_{k+1}) and contains the number r_k. In the same time for big enough $k \in L_\alpha \setminus L_\beta$ is true the equality

$$m_{b'}(\epsilon t_k) - m_{b'}(\epsilon' t_k) = m_{b'}\left(\frac{\epsilon r_k}{\sqrt{\delta\delta'}}\right) - m_{b'}\left(\frac{\epsilon' r_k}{\sqrt{\delta\delta'}}\right) = 0 \tag{5.20}$$

because the interval (r_{k-1}, r_k) does not contain points of the sequence $b' = (b'_k)$. The equalities (5.19), (5.20) contradict the relation (5.21), therefore $A(U^n, X_{L_\alpha}) \not\simeq A(U^n, X_{L_\beta})$.

Thus we have built a continuum $\{L_\alpha, \alpha \in \mathcal{D}\}$ such that the spaces of the kind $A(U^n, X_{L_\alpha})$ are pairwise nonisomorphic although all spaces of the kind $A(U^n, Y_{L_\alpha})$ are isomorphic.

Let us now consider analogous results for spaces $A(\mathbb{C}^n, X)$ of entire functions with values in X. It is obvious that $A(\mathbb{C}^n, E_\infty(a)) \simeq E_\infty(c))$ where $m_c(t) \simeq t^n m_a(t)$, and hence this case is of little interest.

On the other hand we have

Corollary 5.10. *The isomorphism*

$$X = A(\mathbb{C}^n, E_0(a)) \simeq Y = A(\mathbb{C}^n, E_0(a'))$$

is equivalent to the conditions (5.16) together with following the relation:

$$\begin{aligned} \forall A \, \exists B \quad : \quad & m_a(At) - m_a(t/A) \lesssim B[m_{a'}(Bt) - m_{a'}(t/B)], \\ & m_{a'}(At) - m_{a'}(t/A) \lesssim B[m_a(Bt - m_a(t/B))]. \end{aligned}$$

Finally we consider the problem: Under what conditions on the space X the space $A(\mathbb{C}^n, X)$ or $A(U^n, X)$ does not depend up to isomorphisms on the dimension of $\mathbb{C}^n$, i.e.

$$A(\mathbb{C}^n, X) \simeq A(\mathbb{C}, X), \qquad \forall\, n \in \mathbb{N} \tag{5.21}$$

or

$$A(U^n, X) \simeq A(U, X), \qquad \forall\, n \in \mathbb{N}. \tag{5.22}$$

First we observe the cases in which the problem can be solved only in terms of diametral dimension.

Proposition 5.2. *Suppose $X = E_0(a)$ (or $X = E_\infty(a)$). Then (5.22) (respectively (5.23)) is equivalent to the relation*

$$\forall\, p\ \exists\, \Delta : t^p m_a(t) \lesssim m_a(\Delta t). \tag{5.23}$$

The condition (5.24) is equivalent to the following: inverse diametral dimension $\gamma(X)$ contains together with any function $\varphi(t)$ also the functions $t^p\varphi(t)$, $p \in \mathbb{N}$.

It is possible to find necessary and sufficient conditions for (5.22) if $X = E_0(a)$ and for (5.23) if $X = E_\infty(a)$ by using the stronger invariants, considered in this paragraph. We consider only one of these cases.

Proposition 5.3. *If $X = E_0(a)$ then (5.22) is equivalent to the condition (5.24) together with the relation*

$$\forall p\, \forall A\, \exists B : t^p[m_a(At) - m_a(t/A)] \lesssim [m_a(Bt) - m_a(t/B)]. \tag{5.24}$$

To prove the proposition is enough to apply Theorem 5.3.

Let us remark that the space $A(\mathbb{C}^n, X)$ does not depend on n if $X = \ell_1$. The space $E_0(a)$ is as "closer" with its properties to the Banach space ℓ_1 as slower grows the sequence (a_i). With the help of Proposition 5.2 we shall build examples of spaces arbitrary "close" to the space ℓ_1, but such that $A(\mathbb{C}^n, X)$ depend on n. Thus Proposition 5.2 shows, that for the relations (5.22), (5.23) is important not only how large is the class $\gamma(X)$, but also some "regularity" of the space X.

Example 5.3. We show the existence of a sequence $a = (a_i)$ with arbitrary slow growth, such that the space $A(\mathbb{C}^n, E_0(a))$ depends on n. Let $c = (c_i)$ be an arbitrary strictly increasing sequence $c_i \uparrow \infty$. We take a sequence $0 < n_0 < n_1 < \cdots < n_k < \cdots$ such that $n_{k+1} - n_k \geq 4^k$ and

$$\frac{c_{n_{k+1}} - 1}{c_{n_k}} \geq 4^k, \quad k = 1, 2, \ldots,$$

and build a sequence $a = (a_i)$:

$$a_i = \begin{cases} c_{n_k} & \text{if } n_k \le i < n_{k+1} - 1 \\ \sqrt{c_{n_k} c_{n_{k+1}-1}} & \text{if } i = n_{k+1} - 1 \end{cases} \quad k = 1, 2, \dots .$$

It is obvious that $a_i \uparrow \infty$ and $a_i \le c_i$. In order to show that $A(\mathbb{C}^n, E_0(a))$ depends on n it is enough to see that (5.25) is not satisfied. We take $p = 1, A = 2, t_k = \sqrt{c_{n_k} c_{n_{k+1}-1}}$ and let B be an arbitrary positive number. Then for $k > \ln B / \ln 2$ the equality

$$m_a(2t_k) - m_a(t_k/2) = m_a(Bt_k) - m_a(t_k/B) = 1$$

holds. Therefore the inequality (5.25) is impossible, which proves that the space $A(\mathbb{C}^n, E_0(a))$ depend on n.

§6. Application of Invariants to Isomorphic Classification of Spaces of Analytic Functions in Unbounded Multicircular Domains

6.0.

We show in this paragraph how the invariant characteristics, considered in §4, can be used to distinguish a continuum of pairwise nonisomorphic spaces of analytic functions in multicircular domains in $\mathbb{C}^n, n \geq 2$. To this end in points 6.2-6.3 we construct quite specific invariants (based on the general considerations of §4), adapted precisely to catch the differences between spaces of analytic functions in unbounded multicircular domains.

Spaces of analytic functions, considered in this paragraph are naturally isomorphic to second type power Köthe spaces. There are not so many results on second type power spaces as for first type power spaces. In particular the problem on quasiequivalence of bases is open even for spaces of analytic functions in unbounded multicircular domains (wtih excepsion of the spaces $A(U^{n-k} \times \mathbb{C}^k)$, which are first type power spaces and have been considered in the previous paragraph).

Nevertheless the invariants, considered in the present paragraph, give us the possibility to obtain essentially stronger results on isomorphic classification. Let us mention that by using of classical invariants it was possible to distinguish only finitely many pairwise nonisomorphic spaces of analytic functions in multicircular domains (see [1], [8], [35]).

In general, we study in this paragraph spaces of analytic functions; only in the final point 6.7 we give some generalizations of the results for second type power spaces.

6.1.

Let $\mathcal{R}^k$ be the set of all (holomorphically) complete multicircular domains in the space $\mathbb{C}^n$.

Any domain $D \in \mathcal{R}^n$ can be described by its characteristic function:

$$h(a) = h_D(\theta) = \sup \left\{ \sum_{i=1}^{n} \theta_i \ln |z_i| : z = (z_1, \dots, z_n) \in D \right\}, \tag{6.1}$$

defined on the simplex

$$\Sigma = \Sigma^{n-1} = \{\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n : \theta_1 + \cdots + \theta_n = 1, \quad \theta_i \geq 0\}.$$

The function $h_D(\theta)$ is convex on the convex set

$$\pi(D) = \{\theta \in \Sigma : h_D(\theta) < \infty\}.$$

The domain D is bounded iff $\pi(D) = \Sigma$, or equivalently, iff the function $h_D(\theta)$ is uniformly bounded.

We consider also the function

$$\chi_D(t) = mes\{\theta \in \pi(D) : h(\theta) > t\} = mes\,\pi(D) - mes\,\pi(D,t),$$

where mes means the $(n-1)$-dimensional Lebesgue measure, $\pi(D,t) = \{\theta \in \pi(D) : h_D(t) \le t\}$ is a convex closed set.

Further we use the following notation: $M = \mathbb{Z}_+^n$ - the set of all multi-indices

$$\mu = (\mu_1, \dots, \mu_n), \quad |\mu| = \mu_1 + \cdots + \mu_n, \quad \theta(\mu) \overset{def}{=} \frac{\mu}{|\mu|} \in \Sigma.$$

The system

$$z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}, \qquad \mu \in M,$$

is an absolute basis in any of the spaces $A(D), D \in \mathcal{R}^n$. We have a sequence of bounded convex functions‡ on Σ

$$h_1(\theta) < h_2(\theta) < \cdots < h_p(\theta) < \cdots$$

such that

$$h(\theta) = h_D(\theta) = \sup h_p(\theta), \qquad \theta \in \Sigma^{n-1}.$$

The system of norms in the space $A(D)$

$$\|x\|_p = \sum_{\mu \in M} |\xi_\mu| \exp(|\mu| h_p(\theta(\mu))), \qquad x = \sum_\mu \xi_\mu z^\mu, \quad p = 1, 2, \dots,$$

generate the original topology in the space $A(D)$ hence it is isomorphic to the Köthe space

$$K(\exp(|\mu| h_p(\theta(\mu)))). \tag{6.2}$$

This space is isomorphic to the second type power Köthe space $F(\lambda, a)$, where

$$\begin{aligned} \lambda = (\lambda_\mu) : \lambda_\mu &= h(\theta(\mu)) \text{ if } h(\theta(\mu)) < \infty, \\ \lambda_\mu &= |\mu| \text{ if } h(\theta(\mu)) = \infty; \end{aligned}$$

$a = (a_\mu), a_\mu = |\mu|, \ \mu \in M$.

Theorem 6.1. *If* $A(D_1) \simeq A(D_2), D_1, D_2 \in \mathcal{R}^n$, *then*

$$\exists c \mid \chi_{D_1}(t) \le c\chi_{D_2}(t/c), \quad \chi_{D_2}(t) \le c\chi_{D_1}(t/c), \qquad t \ge 1.$$

Below we consider a detailed proof of this theorem in the special case, when $int\,\Sigma \subset \pi(D) \subset \Sigma$, only; as a consequence we obtain the following fact.

Corollary 6.1. *There exists a continuum of pairwise nonisomorphic spaces of the kind* $A(D), D \in \mathcal{R}^n$.

‡ According to the formula (6.1) any function h_p corresponds to some bounded domain D_p.

6.2.

Let $X = K(a_{\mu,p})$ be a Köthe space. By its Köthe matrix $A = (a_{\mu,p})$ we build the function:

$$\delta(P,t) = \delta_A(P,t) = \lim_{p_4\to\infty} \overline{\lim}_{\alpha\to\infty} \frac{\left|\left\{\mu \in M : \frac{a_{\mu p_2}}{a_{\mu p_1}} \le e^{\alpha}, \frac{a_{\mu p_4}}{a_{\mu p_2}} > e^{\alpha t}\right\}\right|}{\left|\left\{\mu \in M : \frac{a_{\mu p_3}}{a_{\mu p_2}} \le e^{\alpha}\right\}\right|} \tag{6.3}$$

where $P = (p_1, p_2, p_3)$, $t \ge 1$.

We correspond to the Köthe space X the class $\mathcal{D}(X)$ of all nonnegative continuous functions $\varphi(t), t \ge 1$, such that

$$\exists\, p_1\, \forall\, p_2\, \exists\, p_3\, \exists\, \sigma < \infty \mid \varphi(t) = 0(\delta(P, \sigma t)) \quad \text{as} \quad t \to \infty. \tag{6.4}$$

Using Lemma 4.1 we shall show that $\mathcal{D}(X)$ is a linear topological invariant on the class of all Köthe spaces.

Theorem 6.2. *If the Köthe spaces $X = K(a_{\mu,p})$ and $Y = K(b_{\nu,q})$ are isomorphic, then $\mathcal{D}(X) = \mathcal{D}(Y)$.*

Proof. By symmetry it is enough to prove that $\mathcal{D}(X) \subset \mathcal{D}(Y)$. Let $\varphi \in \mathcal{D}(X)$, i.e. we have (6.4) with $\delta = \delta_A$. We have to show that

$$\exists\, q_1\, \forall\, q_2\, \exists\, q_3\, \exists\, r : \varphi(t) = 0(\delta_B(Q, rt)), \tag{6.5}$$

where $Q = (q_1, q_2, q_3)$.

By Lemma 4.1 (b) we obtain that $\forall\, p_1\, \exists\, q_1\, \forall\, q_2\, \exists\, p_2\, \forall\, p_4\, \exists\, q_4\, \exists\, c$:

$$\left|\left\{\mu : \frac{a_{\mu p_2}}{a_{\mu p_1}} \le e^{\alpha}, \frac{a_{\mu p_4}}{a_{\mu p_2}} > e^{\alpha t}\right\}\right| \le \left|\left\{\nu : \frac{b_{\nu q_2}}{b_{\nu q_1}} \le c e^{\alpha}, \frac{b_{\nu q_4}}{b_{\nu q_2}} > \frac{1}{c} e^{\alpha t}\right\}\right|. \tag{6.6}$$

Because of the invariant property of n-diameters (Lemma 4.1 (a)) we have that $\forall\, q_2\, \exists\, p_2$ $\forall\, p_3\, \exists\, q_3\, \exists\, c$:

$$\left|\left\{\nu : \frac{b_{\nu q_3}}{b_{\nu q_2}} \le e^{\alpha}\right\}\right| \le \left|\left\{\mu : \frac{a_{\mu p_3}}{a_{\mu p_2}} \le c\, e^{\alpha}\right\}\right|. \tag{6.7}$$

Here we can think that $p_2 = p_2(q_2)$ is one and the same in (6.6) and (6.7), because otherwise we can take the biggest of two numbers.

Therefore we obtain the estimate

$$\delta_A(P,t) \le \delta_B(Q,t), \qquad t \ge 1,$$

under relations (6.6) and (6.7) between P and Q. From here it follows (6.5), which proves the theorem. □

The invariant $\mathcal{D}(X)$ is specially adapted for an investigation of the class of spaces $A(D), D \in \mathcal{R}^n$. In point 6.6 it will be shown that this invariant works on a somewhat more general classes of spaces.

6.3.

To compute the invariant $\mathcal{D}(X)$ for spaces $X = A(D), D \in \mathcal{R}^n$, we need some elementary facts on asymptotic behavior of the number of points with integer coordinates in a domain.

We denote by $N(V)$ the number of points with integer coordinates in the domain V and put as usual $tV \stackrel{def}{=} \{tx : x \in V\}$, $t > 0$. It is quite elementary to prove the following

Lemma 6.1. *Let V be a bounded convex set in $\mathbb{R}^n$. Then we have the estimate:*

$$|N(tV) - t^n \mu_n(V)| \leq c_n \, d^{n-1} t^{n-1}, \qquad t \geq t_0,$$

where $\mu_n(V)$ is the Lebesgue measure of V, $d = d(V)$ is the diameter of V, c_n is a constant, which does not depend on V and t, and t_0 depends only on the internal radius of V, i.e. on the constant $\tau(V) = \sup\{\rho(x, \partial V) : x \in V\}$.

We use for computation of invariants directly the next

Lemma 6.2. *Let $W \subset \Sigma$ and $W = W_1 \setminus W_2$, where W_1, W_2 are convex subsets of Σ and $W_2 \subset W_1$. We denote by $w(t)$ the number of points with integer coordinates in the cone with a vertex the origin and the base tW. Then the following estimate holds:*

$$\left| w(t) - \frac{t^n \, \mu_{n-1}(W)}{n\sqrt{n}} \right| \leq B_n \, t^{n-1}, \qquad t \geq t_0,$$

where μ_{n-1} is the $(n-1)$-dimensional Lebesgue measure on Σ, B_n is a constant, which does not depend on W, and the constant t_0 depend only on the internal radius of the set W_2, considered as a subset of Σ.

This lemma is a consequence of the previous, since $w(t) = N(tV_1) - N(tV_2)$, where V_i is the cone with a vertex the origin and the base W_i, $\mu_n(V_i) = \mu_{n-2}(W_i)/n\sqrt{n}$, $i = 1, 2$.

6.4.

Now we begin the computation of the invariant $\mathcal{D}(X)$ for the space $X = A(D), D \in \mathcal{R}^n$.

First we consider the case when the following additional condition on the domain holds:

$$mes(\Sigma - \pi(D)) = 0.$$

Because of the convexity of $\pi(D)$, this condition means that

$$int \, \Sigma \subset \pi(D) \tag{6.8}$$

i.e. the characteristic function $h_D(\theta)$ can take value $+\infty$ only on the boundary of the simplex Σ.

The space $X = A(D)$ is isomorphic to the Köthe space (6.2) with a matrix $a_{\mu,p} = \|e_\mu\|_p = \exp(h_p(\theta(\mu)) \cdot |\mu|)$, where $\{e_\mu = z^\mu,\ \mu \in M\}$ is the canonical basis in the space $A(D)$.

We have to estimate from above and below the numerator in the formula (6.3), where for convenience we write s instead of p_4:

$$\begin{aligned}\rho_s(\alpha,\beta) &= \left|\left\{\mu \in M : \frac{|e_\mu|_{p_2}}{|e_\mu|_{p_1}} \le e^\alpha,\ \ \frac{|e_\mu|_s}{|e_\mu|_{p_2}} > e^\beta\right\}\right| = |\{\mu \in M : \beta/B < |M| \le \alpha/A\}| \\ &= |\{\mu \in M : B/A < |M| \le \alpha/A,\ \theta(\mu) \in V\}|,\end{aligned}$$

where

$$B = B(p_2, s, \mu) = h_s(\theta(\mu)) - h_{p_2}(\theta(\mu)),\ A = B(p_1, p_2, \mu) = h_{p_2}(\theta(\mu)) - h_{p_1}(\theta(\mu)),$$

$$V = \left\{\theta \in \Sigma : h_S(\theta) \ge A\frac{\beta}{\alpha} + h_{p_2}(\theta)\right\}.$$

Let $a_i = a_i(p_1, p_2)$, $b_i = b_i(p_2)$, $c_i = c_i(p_1, p_3)$ be constants, such that the following inequalities hold:

$$0 < a_1 \le h_{p_2}(\theta) - h_{p_1}(\theta) \le a_2 < \infty,$$

$$-\infty < b_1 \le h_{p_2}(\theta) \le b_2 < \infty \ \ 0 < c_1 \le h_{p_3}(\theta) - h_{p_1}(\theta) \le c_2 < \infty.$$

These constants do not depend on s.

We estimate $\rho_s(\alpha,\beta)$ from above:

$$\rho_s(\alpha,\beta) \le \left|\left\{\mu \in M : |\mu| \le \frac{\alpha}{a_1},\ \ \theta(\mu) \in W'\right\}\right|,$$

where

$$W' = \left\{\theta \in \Sigma : h(\theta) \ge a_1\frac{\beta}{\alpha} + b_1\right\}$$

(we take into account that $h(\theta) > h_s(\theta)$). By Lemma 6.2 we obtain the estimate:

$$\begin{aligned}\rho_s(\alpha,\beta) &\le \frac{1}{n\sqrt{n}}\left(\frac{\alpha}{a_1}\right)^n \mu_{n-1}(W') + L_n\,\alpha^{n-1} \\ &\le \frac{1}{n\sqrt{n}}\left(\frac{\alpha}{a_1}\right)^n \chi_D\left(a_1\frac{\beta}{\alpha} + b_1\right) + L_n\,\alpha^{n-1},\end{aligned}$$

where the constant L_n does not depend on s.

We estimate $\rho_s(\alpha,\beta)$ from below:

$$\rho_s(\alpha,\beta) \geq \left|\left\{\mu \in M : \frac{\alpha}{2b} \leq |\mu| \leq \frac{\alpha}{b}, \quad \theta(\mu) \in W''\right\}\right|, \tag{6.9}$$

where

$$W'' = \left\{\theta \in \Sigma : h_s(\theta) \geq 2b\frac{\beta}{\alpha} + \alpha\right\}.$$

Using Lemma 6.2 we obtain the estimate:

$$\begin{aligned}\rho_s(\alpha,\beta) &\geq \frac{1}{n\sqrt{n}}\left(\frac{\alpha}{a_2}\right)^n \mu_{n-1}(W'') - \frac{1}{n}\left(\frac{\alpha}{2a_2}\right)^n \mu_{n-1}(W'') - L_n\, \alpha^{n-1} \\ &\geq \frac{1}{2n\sqrt{n}}\left(\frac{\alpha}{a_2}\right)^n \chi_{D_s}\left(2a_2\frac{\beta}{\alpha} + b_2\right) - L_n\, \alpha^{n-1}, \end{aligned} \tag{6.10}$$

where the constant L_n does not depend on s.

The estimates of the denominator in (6.3)

$$R(\alpha) = \left|\left\{\mu \in M : |\mu| \leq \frac{\alpha}{h_{p_3}(\theta) - h_{p_2}(\theta)}\right\}\right|$$

are more simple (therefore we omit the proof). We obtain

$$T_1\alpha^n - S\alpha^{n-1} \leq R(\alpha) \leq T_2\alpha^n + S\alpha^{n-1} \tag{6.11}$$

where the constants T_1, T_2, S depend on C_1, C_2 and do not depend on S. By estimates (6.9)-(6.11), taking into account that all constants do not depend on α, β and using the continuity of Lebesgue measure (which gives us $\chi_D(t) = \lim_{s\to\infty}\chi_{D_s}(t)$), we obtain for the function

$$\delta(P,t) = \lim_{s\to\infty} \overline{\lim}_{\alpha\to\infty} \frac{\rho_s(\alpha,\alpha t)}{R(\alpha)}$$

the estimates:

$$\frac{1}{c}\chi_D(ct) \leq \delta(P,t) \leq c\chi_D(t/c), \qquad t \geq t_0,$$

where c is a positive constant.

Using these estimates we get immediately that the class $\mathcal{D}(X)$ consists of all nonnegative continuous functions $\varphi(t), t \geq 1$, such that $\exists\, \epsilon \mid \varphi(t) = 0(\chi_D(\epsilon t))$ as $t \to \infty$. Hence it is proved

Proposition 6.2. *Let the domain $D \in \mathcal{R}^n$ satisfies the condition (6.8). Then*

$$\mathcal{D}(A(D)) = \{\varphi \in C([1,\infty)),\ \varphi \geq 0 : \exists\, \epsilon > 0 \mid \varphi(t) = 0(\chi_D(\epsilon t)),\ t \to \infty\}$$

This means that Theorem 6.1 is proved for domains $D \in \mathcal{R}^n$, satisfying condition (6.8).

Remark. In the case $n = 2$ Proposition 6.2 can be formulated directly in terms of estimates of characteristic function h_D. Here it is convenient to consider the function of one variable $g_D(\lambda) = h_D(\lambda, 1-\lambda)$, $0 \le \lambda \le 1$. We consider only the simplest case when

$$g_D(0) = \infty; \quad g_D(\lambda) < \infty \quad \text{if} \quad 0 < \lambda \le 1. \tag{6.12}$$

Obviously then the function $\chi_D(t)$ coincides with the inverse function of the function $g_D(\lambda)$. Therefore the necessary condition of Theorem 6.1 for the pair of domains $D_1, D_2 \in \mathcal{R}^2$, satisfying (6.17), can be written as follows:

$$\exists\, c : \frac{1}{c} g_{D_2}(\lambda/c) \le g_{D_1}(\lambda) \le c g_{D_2}(c\lambda), \quad 0 < \lambda \le \lambda_0 < 1.$$

Corollary 6.1. *If $n \ge 2$ there exists a continuum of pairwise nonisomorphic spaces of the kind $A(D), D \in \mathcal{R}^n$.*

We shall explain this in the case $n = 2$. Let us consider the continuum of domains $\{D_\nu, \nu > 0\}$, determined by the characteristic functions

$$\begin{aligned} g_{D_\nu}(\lambda) &= h_{D_\nu}(\lambda, 1-\lambda) = \exp(\lambda^{-\nu}), \quad 0 < \lambda \le 1, \\ g_{D_\nu}(0) &= +\infty. \end{aligned}$$

Any one of the domains D_ν is uniquely determined by its image in the logarithmic plane - the domain

$$G_\nu = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_i = \ln |z_i|, \ z = (z_i) \in D_\nu\},$$

which boundary is the envelope of the family of lines:

$$\lambda \xi_1 + (1-\lambda)\xi_2 = \exp(\lambda^{-\nu}), \qquad 0 < \lambda \le 1.$$

By the remark to Proposition 6.2 it follows that the spaces $A(D_\nu), \nu > 0$, are pairwise non-isomorphic.

6.5.

If the condition (6.12) is not satisfied then by estimates, analogous to the above, we obtain an estimate of the knid

$$0 < \frac{1}{c} \le \delta(P, t) \le c < \infty,$$

since the numerator in (6.3) is proportional to the expression $\alpha^n \psi(t)$, where

$$\psi(t) = \mu_{n-1}(\Sigma^{n-1} - \pi(D, t)) \ge \sigma > 0.$$

Hence if the condition (6.12) is not fulfilled, then the invariant $\mathcal{D}(A(\mathcal{D}))$ is invalid, since it coincides with the class of all nonnegative continuous bounded functions on $[1, \infty)$.

In order to distinguish such spaces it is necessary to use the stronger invariant $\tilde{D}(X)$, based on the three-inequality invariant characterstic (Lemma 4.1 (c′)); for its definition see [52]. This invariant rejects more strictly the "nonregular" elements of the basis and with its help it is possible to "eliminate" also those elements of the basis (e_μ), for which $\theta(\mu) \in \Sigma \setminus \overline{\pi(D)}$. Thus, by using of that invariant Theorem 6.1 can be proved in general case (see [52]).

6.6.

Here we consider a general class of second type power spaces, for which there are results, analogous to those of point 6.5.

We define the class Φ of all spaces $F(\lambda, a)$, satisfying the conditions:

(Φ.1) for any $\sigma \geq \sigma_0$ there exists a positive limit:

$$u_\lambda(\sigma) \overset{def}{=} \lim_{n \to \infty} \frac{|\{i : \lambda_i \geq \sigma, a_i \leq n\}|}{n},$$

with $u_\lambda(\sigma) \to 0$ as $\sigma \to \infty$.

(Φ.2) $\quad a_{2i} \asymp a_i$ as $i \to \infty$.

Computing the invariant $D(X)$, defined in point 6.2, we get the following statement.

Theorem 6.4. *If the spaces $F(\lambda, a)$ and $F(\mu, b)$ belong to the class Φ, u, v are the corresponding functions, defined by (6.13) and are isomorphic then we have*

(a) $a_i \asymp b_i$

(b) $\exists\, c : \frac{1}{c} v_\mu(c\sigma) \leq u_\lambda(\sigma) \leq c\, v_\mu(\sigma/c), \quad \sigma \geq 1.$

Corollary 6.2. *Let Y_h, Y_g be the pair of spaces, considered in Example 2.2. (let us remind, that functions $h(\theta), g(\theta)$ are defined on $(0,1]$ and $h(\theta) \uparrow \infty, g(\theta) \uparrow \infty$, as $\theta \downarrow 0$). Then $Y_h \simeq Y_g$ if and only if there exists a constant $c, 1 < c < \infty$, such that*

$$\frac{1}{c} g(c\theta) \leq h(\theta) \leq c g(\theta/c), \quad 0 < \theta < 1/c. \tag{6.13}$$

Indeed, let $\lambda_h = \{h(\{\alpha i\})\}, a = (i)$. Since the number α is irrational the sequence of fractional parts $\{\alpha_i\}$ is uniformly distributed on the segment $[0, 1)$, i.e. for any $0 < x < y < 1$ we have

$$\lim_{n \to \infty} \frac{|\{i \leq n : \{\alpha i\} \in (x, y)\}|}{n} = y - x.$$

Consequently the space $Y_h = F(\lambda_h, a)$ belongs to the class Φ and the function $\varphi_{\lambda_h}(\sigma)$ is inverse to the function $h(\theta)$. Therefore Corollary 6.2 follows by Theorem 6.4.

Corollary 6.3. *There exists a continuum pairwise nonisomorphic spaces of the kind considered in Example 2.2.*

Indeed, consider the continuum of spaces Y_{h_γ}, where

$$h_\gamma(\theta) = \exp(\frac{1}{\theta})^\gamma, \qquad 0 < \gamma < \infty.$$

By Corollary 6.2 the spaces Y_{h_γ} are pairwise nonisomorphic.

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LİNEER TOPOLOJİK İNVARİYANTLAR VE ONLARIN GENELLEŞTİRİLMİŞ KUVVET SERİLERİ UZAYLARININ İZOMORFİK KLASİFİKASYONU PROBLEMİNE UYGULAMALARI

Özet

Bu derleme yazısında Kolmogorov ve Pelczynski invariyantlarından yola çıkarak genel lineer topolojik invariyantlar tanımlanmış ve bu invariyantların içlerinde, sonlu ve sonsuz kuvvet seri uzaylarının tensor çarpımı, bazı çok değişkenli analitik fonksiyonlar uzayı, vektör değerli türevlenebilen ve analitik bazı fonksiyon uzaylarınında bulunduğu, genelleştirilmiş kuvvet serileri uzaylarının izomorfik klasifikasyonu problemlerine uygulamaları ele alınmıştır.

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