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# EXTENSIONS OF CARISTI- KIRK'S THEOREM

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**Abstract**

We give some extensions and/or improvements, to uniform spaces and to multi-valued mappings, of Caristi-Kirk's theorem.
**Key words and phrases**: Uniform spaces, multi-valued mappings, fixed point theorem, maximal element, weak p-contraction mappings.

## 1. Introduction

It was observed that certain fixed point theorems can be deduced from the following result:

Let $(E,\leq)$ be an ordered set which admits a maximal element.
Let $f:E\to E$ be a mapping such that $x\leq f(x)$ for every $x$ in $E$.

Then $f$ has a fixed point.

This result served as a basis for certain theorems about the existence of maximal elements ([1], [2], [3], [6]), and hence fixed point theorems. Considered spaces are often metric spaces endowed with an order defined via the distance ([2], [3], [4], [5]).

Ekland's variational principle, which concerns the existence of maximal elements ([2], [6]) and its generalizations allowed simple proof of Caristi-Kirk's theorem ([2],[3],[8]).

Recently, V. Conserva ([5]) gave a slight improvement of this theorem in metric spaces.

In this paper we give some extensions and/or improvements to uniform spaces and to multi-valued mappings, of Caristi- Kirk's theorem.

Let us notice that proofs we give here go along the lines of those given in the case of metric or topological vector spaces ([2], [3], [4], [5]).

In the following, for a uniform space E, we consider a family $(d_i)_{i\in I}$ of semi-metrics which defines its uniform structure and such that $\sup_{i\in I}\, d_i(x,y)<+\infty$, for all $x,y$ in $E$.

Let $E$ be a uniform space and $p:E\to\mathbb{R}_+$ a positive real functional on $E$. Define a partial order on $E$ as follows:

$x\leq y$ if and only if $d_i(x,y)\leq p(x)-p(y)$, for all $i\in I$

For an $x$ in $E$ we put $S(x)=\{y\in E/x\leq y\}$.

Let $A$ be a subset of $E$, diam $(A) = \sup_{i\in I}$ $(\sup_{\substack{x\in A\\ y\in A}} d_i(x,y))$ will be called diameter of $A$.

We denote by $2^E$, the set of all nonempty subsets of $E$.

## I-Single Valued Mappings

We will begin with the following result:

**Theorem I-1.** *Let $E$ be a uniform space and $p : E \to \mathbb{R}_+$ a real functional which is lower semi-continuous (l.s.c.). Let $f : E \to E$ be an arbitrary self-mapping of $E$.*

***(I-1):*** *there exists an $x$ in $E$ such that, for all $i \in I$,*
*$d_i(y, f(y)) \leq p(y) - p(f(y))$, for all $y \in S(x)$.*

***(I-2):*** *any Cauchy sequence in $S(x)$ converges in $E$.*

*Then $f$ has a fixed point which is maximal in $(E, \leq)$.*

**Proof.** Construct a sequence $(x_n)_n$ in $E$ inductively as follows: $x_1 = x$; when $x_1, x_2, \cdots, x_n$ have been chosen, let $a_n := \inf p(S(x_n))$ and take $x_{n+1}$ in $S(x_n)$ such that $p(x_{n+1}) \leq a_n + 1/n$.

Then $x_n \leq x_{n+1}$ and for any $y$ in $S(x_n)$ we have $a_{n-1} \leq a_n \leq p(y) \leq p(x_n) \leq a_{n-1} + 1/n - 1$.

In particular, for $n \leq m$ we have $0 \leq p(x_n) - p(x_m) \leq 1/n - 1$.

This shows that $(x_n)_n$ is a Cauchy sequence and that, for all $i \in I, d_i(x_n, x_m)$ converges to zero when $n$ tends to infinity. Hence diam $(S(x_n))$ converges to zero for all $n$.

By hypothesis $(x_n)_n$ converges to an $x_0$ in $E$. On the other hand, by the construction of $(x_n)_n$, we have, for all $i \in I, d_i(x_n, x_{n+k}) \leq p(x_n) - p(x_{n+k})$, for all $k \geq 0$.

Hence, allowing $k$ tend to infinity we have $d_i(x_n, x_0) \leq p(x_n) - p(x_0)$, for all $n$ and for all $i \in I$. This means that $x_n \leq x_0$ for all $n$. Therefore $x_0 \in S(x)$ and

$$d_i(x_0, f(x_0)) \leq p(x_0) - p(f(x_0)), \textit{for all } i \in I \textit{ i.e. } x_0 \leq f(x_0)$$

Let now $(y_n)_n$ be a sequence such that $x_n \leq y_n$ for all $n$. Then $\lim_n y_n = x_0$, for diam $(S(x_n))$ converges to zero for all $n$.

Finally, suppose that $y$ in $E$ is such that $x_0 \leq y$. Then we also have $x_n \leq y$ for all $n$ and it follows that $y = x_0$ (take $y_n := y$ for all $n$ in the preceding sequence), i.e. $x_0$ is maximal and then $f(x_0) = x_0$. □

We have the following corollary:

**Corollary I-2.** *Let $E$ be a sequentially complete uniform space and $p : E \to \mathbb{R}_+$ a l.s.c. real functional. Let $f$ be an arbitrary self-mapping of $E$. Suppose that there exists an $x$ in $E$ such that $d_i(y, f(y)) \leq p(y) - p(f(y))$, for all $y$ in $S(x)$ and for all $i \in I$.*

*Then $f$ has a fixed point which is a maximal element in $(E, \leq)$.*
*As a consequence of this result we have*

**Corollary I-3.** *(Caristi-Kirk's theorem). Let $(E,d)$ be a complete metric space and $p : E \to \mathbb{R}_+$ a l.s.c. real functional. Let $f$ be a self-mapping of $E$ such that $p(x, f(x)) \leq p(x) - p(f(x))$, for all $x$ in $E$. Then $f$ has a fixed point.*

Analyzing the proof of Theorem I-1, we can state the following

**Theorem I-4.** *Let $E$ be a uniform space and $p : E \to \mathbb{R}_+$ a l.s.c. real functional. Let $f$ be an arbitrary self-mapping of $E$. Suppose that:*

***(I-3):*** *there exists an $x$ in $E$ such that, for all $i \in I$, $d_i(y, f(y)) \leq p(y) - p(f(y))$, for every $y$ in $S(x)$*

***(I-4):*** *any nondecreasing sequence in $S(x)$ is relavitely compact.*

*Then $f$ has at least one fixed point which is maximal in $(E, \leq)$.*

**Proof.** Let $(x_n)_n$ be a sequence defined as follows: $x_1 = x$; when $x_1, x_2, \cdots, x_n$ have been chosen let $a_n := \inf p(S(x_n))$ and take $x_{n+1}$ in $S(x_n)$ such that $p(x_{n+1}) \leq a_n + 1/n$. The sequence $(x_n)_n$ is increasing. One shows that, for all $i \in I, d_i(x_n, x_m) \leq p(x_n) - p(x_m) \leq 1/n - 1$, for $n \leq m$. Hence $(x_n)_n$ is a Cauchy sequence; moreover diam $(S(x_n))$ converges to zero for all $n$.

By hypothesis, $(x_n)_n$ is relatively compact. Therefore there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which converges to an $x_0$ in $E$. Since $(x_n)_n$ is a Cauchy sequence, it also converges to $x_0$. By the same argument as in the proof of Theorem I-1 we get that $x_0 \in S(x)$ and that $x_0$ is maximal. Thus by hypothesis, for all $i \in I$ $d_i(x_0, f(x_0)) \leq p(x_0) - p(f(x_0))$, i.e. $x_0 \leq f(x_0)$. Thus $f(x_0) = x_0$.

□

**Remark I-5.** Instead of condition (I-4) in Theorem I-4, if we suppose that $S(x)$ is complete for each $x$ in $E$, the conclusion of the theorem still holds.

If we suppose likewise that $E$ is sequentially complete, the condition (I-4) is no more needed. Corollary I-2 can again be obtained as a consequence.

## II-Multi-Valued Mappings

Now we give an extension of Theorem I-1 to the case of certain multi-valued mapping, namely those which in some way are p-contractive. Thus we improve some of the results of M-H. Shih ([8]) which are of Caristi- Kirk type.

We slightly soften a definition of M-H. Shih ([8]).

**Definition II-1.** *Let $A$ be a subset of $E$. A multi-valued mapping $f : E \to 2^E$ is said to be a weak p-contraction on $A$, if there exists a real functional $p : E \to \mathbb{R}_+$ such that for each $x$ in $A$ and $y \in f(x), d_i(x,y) \leq p(x) - p(y)$, for all $i \in I$.*

*$f$ is said to be a p-contraction on $A$, if for each $x$ in $A$ and all $y$ in $f(x)$, $d_i(x,y) \leq p(x) - p(y)$, for all $i \in I$.*

*$f$ is said to be a weak p-contraction (respectively a p-contraction) in the sense of Shih, if $A = E, E$ being a metric space.*

*We have the following result:*

**Theorem II-2.** *Let $E$ be a uniform space and $f : E \to 2^E$ a closed multi-valued mapping. Suppose that:*

*(II-1): there exists an x in E such that f is a weak p-contraction on S(x);*

***(II-2)**: any Cauchy sequence in S(x) converges in E.*

*Then f has a fixed point.*

**Proof.** Endow $E$ with the partial order corresponding to $p$ and construct a sequence $(x_n)_n$ as follows: $x_1 = x$ and for $n > 1$, take $x_{n+1}$ in $f(x_n)$ such that $x_n \leq x_{n+1}$ (this is possible for $f$ is a weak p-contraction on $S(x)$). One shows that $(x_n)_n$ is a Cauchy sequence. Therefore by hypothesis, there exists $x^* \in f(x^*)$, i.e. $x^*$ is a fixed point of $f$.

Moreover, we have, for all $i \in I$,

$$d_i(x_n, x_{n+k}) \leq p(x_n) - p(x_{n+k}), \text{ for } k \geq 0.$$

Tending $k$ to infinity we obtain

$$d_i(x_n, x^*) \leq p(x_n), \text{ for all } i \in I, n = 0, 1, 2 \cdots$$

As a consequence we get the following: □

**Corollary II-3.** *Let $E$ be a sequentially complete uniform space and $f : E \to 2^*$ a closed multi-valued mapping. Suppose that $f$ is a weak p-contraction. Then $f$ has a fixed point.*

**Corollary II-4** *(M-H. Shih ([8])). Let (E,d) be a complete metric space and $f : E \to 2^E$ a closed multi-valued mapping. Suppose that $f$ is a weak p-contraction. Then $f$ has a fixed point.*

*We will need the following statement, in uniform spaces, of Ekeland's variational principle. A. Brondsted ([3]) stated it differently (in uniform spaces). Here we give a statement directly applicable to our case.*

**Theorem II-5.** *Let E be a sequentially complete uniforme space and $p : E \to \mathbb{R}$ a l.s.c. real functional which is bounded below. Then there exists an x in E such that:*

$$(II-3) : \forall\, y \neq x, \exists\, i_o \in I : p(y) > p(x) - d_{i_o}(x,y).$$

*Now we can state the following*

**Theorem II-6** *Let E be a uniform space and $f : E \to 2^E$ a multi-valued mapping. Suppose that:*

***(II-4)**: there exists an x in E such that f is a weak p-contraction on S(x) with p being I.s.c. and S(x) complete. Then f has a fixed point.*

**Proof.** By Theorem **II-5**, there exists v $\in S(x)$ such that for every $w \neq v$, there exists $i_o \in I$ such that p(w)-p(v) > $-d_{i_o}$ (w,v). We assert that $v \in f(v)$. Indeed, if not, then $p(w) - p(v) > d_{i_o}(w,v)$, for each w in f(v). Whence a contradicition to the p-contractness of f on S(x). □

**Corollary II-7.** *Let E be a sequentially complete uniform space and $f : E \to 2^E$ a multi-valued mapping. Suppose that f is a weak p-contraction with p being I.s.c. Then f has a fixed point.*

*From the previous corollary we deduce the following result:*

**Corollary II-8** *(M-H. Shih ([8]). Let (E,d) be a complete metric space and $f : E \to 2^E$ a multi-valued mapping. Suppose that f is a weak p-contraction with p being I.s.c. Then f has a fixed point.*

**Remark II-9.** Replacing weak p-contractness by p-contractness we get special cases of the results above and in particular some results of M-H. Shih ([8]).

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## CARISTI- KIRK THEOREMİNİN BİR GENELLEŞTİRİLMESİ


### Özet

Bu makalede Caristi- Kirk teoreminin düzgün uzaylara ve çok-değerli temsillere genişletilmeleri verilmiştir.



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