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T_2-OBJECTS IN CATEGORIES OF FILTER AND LOCAL FILTER CONVERGENCE SPACES

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Abstract

There are various generalizations of the usual T_2 (Hausdorff) axiom of topology to an arbitrary topological category defined in [1]. In this paper, four more new generalizations of Hausdorff spaces are given in the topological categories. Moreover, the relationships among all of these T_2 structures and the ones given in [5] and [7] as well as some invariance properties of them are investigated in the categories of filter and local filter convergence spaces.
Keywords: Topological Categories, T_2-objects, Filter Convergence Spaces.

Introduction

There are four ways of generalizations of the usual T_2 (Hausdorff) axiom of topology to an arbitrary topological category defined in [1]. The explicit charactreziations of each of these T_2-objects in the categories of Filter and Local Filter convergence Space, FCO and LFCO, are given in [3] and [4].

Nel [7] has showed that a full subcategory, HLFCO of LFCO whose objects determined by the Hausdorff convergence spaces is a cartesian closed initial structured category. We generalize this result to full subcategories, T_2'LFCO, ΔT_2LFCO, and ST$_2$LFCO of LFCO whose objects are determined by the T_2', ΔT_2, and ST$_2$ structures, respectively. We have also shown that each of these subcategories is epireflective. Furthermore, we have shown the following:

1. To define four more new T_2 structures in arbitrary topological categories and to characterize each of them in FCO and LFCO.
2. To investigate the relationships among these various T_2 structures in FCO and LFCO.
3. To determine the invariance properties (i.e closed uder formation of products, subspaces, and quotient spaces) of each of T_2 structures in FCO and LFCO.

General relationships among these various T_2 for arbitrary topological categories will be established in a subsequent paper.

Let E an d B be any categories. The functor $U : E \to B$ is said to be topological if it is concrete (i.e faithful and amnestic (i.e if U(f) = id and f is an isomorphism, then f = id)), has small (i.e sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift [6] p.279.

Recall, [6] p.279, that an object X of E is called indiscrete (resp. discrete) iff every map from an E-object to X (resp. every map from X to an E-object) is an E-morphism.

Let X be a nonempty set and $X^2 = X \times X$ be the cartesian product of X with itself, and $X^2 V_\Delta X^2$ be two distinct copies of X^2 identified along the diagonal. A point (x,y) in $X^2 V_\Delta X^2$ will be denoted by $(x,y)_1$ $(x,y)_2)$ if (x,y) is in the first (resp. the second) component of $X^2 V_\Delta X^2$. Clearly $(x,y)_1 = (x,y)_2$ iff $x = y$.

1.1 Definitions *The principal axis map* $A : X^2 V_\Delta X^2 \to X^3$ *is given by* $A(x,y)_1 = (x,y,x)$ *and* $A(x,y)_2 = (x,x,y)$. *The skewed axis map* $S : X^2 V_\Delta X^2 \to X^3$ *is given by* $S(x,y)_1 = (x,y,y)$ *and* $S(x,y)_2 = (x,x,y)$, *and the fold map,* $\nabla : X^2 V_\Delta X^2 \to X^2$ *is given by* $\nabla(x,y)_i = (x,y)$ *for* $i = 1,2$.

Let $U : E \to$ *Sets, the category of sets, be topological and* X *an object in* E *with* $UX = B$.

1.2 Definitions *1. X is* $T_\emptyset$ *iff X does not contain an indiscrete subspace with (at least) two points [9] p.316.*
2. X is $T'_\emptyset$ *iff the initial lift of the U-source* $\{ id : B^2 V_\Delta B^2 \to U(B^2 V_\Delta B^2)' = B^2 V_\Delta B^2$ *and* $\nabla : B^2 V_\Delta B^2 \to UD(B^2) = B^2\}$ *is discrete, where* $(B^2 V_\Delta B^2)'$ *is the final lift of the U-sink* $\{i_1, i_2 : U(X^2) = B^2 \to B^2 V_\Delta B^2\}$, *[1] p.338.*
3. X is $\bar{T}_\emptyset$ *iff the initial lift of the U-source* $\{A : B^2 V_\Delta B^2 \to U(X^3) = B^3$ *and* $\nabla : B^2 V_\Delta B^2 \to UD(B^2) = B^2\}$ *is discrete, [1] p.338.*
4. X is preT'_2 *iff the initial lift of the U-source* $\{S : B^2 V_\Delta B^2 \to U(X^3) = B^3\}$ *and the final lift of the U-sink* $\{i_1, i_2 : U(X^2) = B^2 \to B^2 V_\Delta B^2\}$ *agree [1] p.338.*
5. X is Pre$\bar{T}_2$ *iff the initial lift of the U-sources* $\{S : B^2 V_\Delta B^2 \to U(x^3) = B^3\}$ *and* $\{A : B^2 V_\Delta B^2 \to U(X^3) = B^3\}$ *agree [1] p.338.*
6. X is T'_2 *iff X is* $T'_\emptyset$ *and Pre*T'_2 *[1] p.338.*
7. X is $\bar{T}_2$ *iff X is* $\bar{T}_\emptyset$ *and Pre*$\bar{T}_2$ *[1] p.338.*
8. X is ΔT_2 *iff the diagonal,* Δ, *is closed in* X^2 *[1] p.338 or [4].*
9. X is ST_2 *iff the diagonal,* Δ, *is strongly closed in* X^2 *[1] p.338. or [4].*
10. X is KT_2 *iff is* $T'_\emptyset$ *and Pre*$\bar{T}_2$.
11. X is LT_2 *iff X is* $\bar{T}_\emptyset$ *and Pre*T'_2.
11. X is MT_2 *iff X is* $T_\emptyset$ *and Pre*T'_2.
12. X is NT_2 *iff X is* $T_\emptyset$ *and Pre*$\bar{T}_2$.

1.3 Remark For the category of topological spaces, we have:
1. All of the $T_\emptyset$'s in 1.2 are equivalent and they reduce to the usual $T_\emptyset$ (see [1] p.338 Theorem 2.2.11(1) and [9] p. 316).
2. All of the T_2's in 1.2 are equivalent and they reduce to T_2, the Hausdorff condition. This follows from part (1) and [1] p.338.

Let A be a set and K a function whose value at each x in A is a set $K(x)$ of filters on A "convergent to x" such that:
1. $[x] \in K(x)$ for each $x \in A$ (where $[x] = \{B \subset A : x \in B\}$.
2. $\beta \supset \alpha \in K(x)$ implies $\beta \in K(x)$ for any filter β on A.

Then (A, K) is called a filter convergence space [8], [3] or [7]. A map $f : (A, K) \to (B, L)$ between filter convergence spaces is called continuous iff $\alpha \in K(x)$ implies $f(\alpha) \in L(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$. The category of filter convergence spaces and continuous maps is denoted by FCO (see [8] p.354). A filter convergence space (A, K) is said to be a local filter convergence space if $\alpha \cap [x] \in K(x)$ whenever $\alpha \in K(x)$ [7] p.1374. The corresponding full subcategory of FCO is denoted by LFCO. Both FCO and LFCO are topological categories. If α and β are filters on A, then $\alpha \cup \beta = \{U \subset A : V \cap W \subset U$ for some $V \in \alpha$ and $W \in \beta\}$ is a filter on A [2] p.98

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We now, for later use, give the charcaterizations of each of the separation properties defined in 1.2 for FCO and LFCO.

2.1 Theorem *1. A local filter convergence (resp. filter convergence space) (A,K) is* $T_\emptyset$ *iff for any* $x, y \in X, [x] \in K(y)$ *and* $[y] \in K(x)$ *(resp.* $[x] \cap [y] \in K(x) \cap K(y)$*) implies* $x = y$ *[9] p.318.*
2. (A,K) in FCO or LFCO is $\bar{T}_\emptyset$ *iff for each distinct pair of poins x and y in A,* $[x] \notin K(y)$ *or* $[y] \notin K(x)$ *[3].*
3. All objects of FCO and LFCO are $T'_\emptyset$ *[3].*
4. (A,K) in FCO or LFCO is pre$\bar{T}_2$ *iff for any pair x and y in A if* $K(x) \cap K(y) \neq \{[\phi]\}$*, where* ϕ *is the empty set, then* $K(x) = K(y)$ *and for any proper filters* α *and* β *in* $K(x)$ *if* $\alpha \cup \beta$ *is proper, then* $\alpha \cap \beta \in K(x)$ *[3].*
5. (A,K) in FCO or LFCO is preT'_2 *iff (A,K) is discrete i.e for each x in A,* $K(x) = \{[x], [\phi]\}$ *[3].*
6. (A,K) in FCO or LFCO is $\bar{T}_2$ *iff for any* $x \neq y$ *in A,* $K(x) \cap K(y) = \{[\phi]\}$ *and for any proper filters* α *and* β *in* $K(x)$ *if* $\alpha \cup \beta$ *is proper, then* $\alpha \cap \beta \in K(x)$ *[3].*
7. (A,K) in FCO or LFCO is T'_2 *iff (A,K) is discrete [3].*
8. (A,K) in FCO or LFCO is ΔT_2 *iff for each* $x \neq y, [x] \notin K(y)$ *[4].*
9. (A,K) in FCO or LFCO is ST_2 *iff for any* $x \neq y$ *in A,* $K(x) \cap K(y) = \{[\phi]\}$ *[3] or [4].*
10. Let (A,K) be in FCO or LFCO and F be a nonempty subset of A. F is strongly closed iff for any $a \in A$ *if* $a \notin F$*, then* $[a] \notin K(c)$ *for all* $c \in F$ *and if* $\alpha \in K(a)$*, then* $\alpha \cup [F]$ *is improper [4].*
11. (A,K) in FCO or LFC O is T_1 *iff for each* $x \neq y$ $[x] \notin K(y)$ *[3].*

2.2 Corollary *(A,K) in FCO or LFCO is* KT_2 *iff (A,K) is pre*$\bar{T}_2$.
Proof. It follows from (3) and (4) of 2.1 and definition 1.2. □

2.3 Corollary *(A,K) in FCO or LFCO is LT_2 (MT_2) iff (A,K) is discrete.*
Proof. Combine (2) and (5) (resp. (1) and (5)) of 2.1 and 1.2. □

2.4 Corollary *(A,K) in FCO or LFCO is NT_2 iff (1) for any pair x and y in A if $K(x) \cap K(y) \neq \{[\phi]\}$, where ϕ is the empty set, then $K(x) = K(y)$ (2) for any proper filters α and β in K(x) if $\alpha \cup \beta$ is proper, then, $\alpha \cap \beta \in K(x)$, (3) for any distinct pair of points x and y in A, $[x] \cap [y] \notin K(x)$ or $[x] \cap [y] \notin K(y)$.*
Proof. Combine parts (1) and (4) of 2.1 and definition 1.2. □

2.5 Remark (1) For the category of FCO, we have $\bar{T}_\emptyset$ implies $T_\emptyset$, $T_\emptyset$ implies $T'_\emptyset$, $T'_2 = LT_2 = MT_2$ implies $\bar{T}_2$, T_2 implies ST_2, and NT_2 implies KT_2 but the converse of each implication is not true.
(2) For the category of LFCO, we have $\bar{T}_\emptyset = T_\emptyset$ implies $T'_\emptyset$, $T'_2 = LT_2 = MT_2$ implies $\bar{T}_2 = NT_2, NT_2$ implies ST_2, ST_2 implies ΔT_2, and NT_2 implies KT_2 but the converse of each implication is not true.
(3) In LFCO, we have some relationships among our notions of T_2's and the ones in [5] and [7]. Theirs T_2 is equivalent to our ST_2 but each of our $T'_2 = LT_2 = MT_2$ and $\bar{T}_2 = NT_2$ implies theirs T_2, and theirs T_2 implies each of our ΔT_2's and KT_2's.
(4) If (A,K) is an indiscrete i.e $\forall x \in A, K(x) = F(A)$, (A has at least two elements), then (A,K) is clearly KT_2 but not NT_2 and ΔT_2.
(5) In FCO and LFCO, by 2.1, $\Delta T_2 = T_1$ implies each of $\bar{T}_\emptyset, T'_\emptyset$ and $T_\emptyset$.
Let (A,L) and (X,K) be objects of FCO or LFCO.

2.6 Theorem *Let $F : (A, L) \to (X, K)$ be a monomorphisms.*
(1) If (X,K) is ST_2, then (A,L) is ST_2.
(2) If f is also initial and (X,K) is KT_2 or $\bar{T}_2$, then (A,L) is KT_2 or $\bar{T}_2$, respectively.
Proof. (1) follows easily from Theorem 2.1 and f is being mono
(2) Suppose for any a and b in A, $L(a) \cap L(b) \neq \{[\phi]\}$. We show that $L(a) = L(b)$.Let $\alpha \in L(a)$. Then $f(\alpha) \in K(f(a))$. $L(a) \cap L(b) \neq \{[\phi]\}$ implies there exists a proper filter β in $L(a) \cap L(b)$. It follows that $f(\beta)$ is proper and in $K(f(a)) \cap K(f(b))$. Since X is KT_2, by 2.2 $K(f(a)) = K(f(b))$ and consequently $f(\alpha) \in K(f(b))$. Hence, $\alpha \in L(b)$ since f is initial (see [3] or [7]). Similarly, if $\alpha \in L(b)$, then $\alpha \in L(a)$ and so $L(a) = L(b)$. Suppose for $a \in A$ and α, β proper filters in $L(a)$ with $\alpha \cup \beta$ is proper. Note that $f(\alpha)$ and $f(\beta)$ are in $K(f(a))$ and $f(\alpha) \cup f(\beta) = f(\alpha \cup \beta)$ (since f is mono) is proper.X is KT_2 implies $f(\alpha \cap \beta) \in K(f(a))$ and consequently $\alpha \cap \beta \in L(a)$ since f is initial. By 2.2, (A, L) is KT_2. The proof for $\bar{T}_2$ follows. □

2.7 Theorem *Let $f : (A, L) \to (X, K)$ be the initial lift (see [3] or [7] p.1374) and (X, K) be ΔT_2, $T_2' = LT_2 = MT_2$, or NT_2, then (A, L) is ΔT_2, $T_2' = LT_2 = MT_2$ or NT_2, respectively, iff f is mono.*

Proof. If f is mono, then the result follows easily. Suppose (A, L) is ΔT_2 but f is not mono, i.e, $\exists x \neq y$ in A such that $f(x) = f(y)$. Note that $f([x]) = [f(x)] = [f(y)] \in K(f(y))$ implies $[x] \in L(y)$ which is a contradiction since (A, L) is ΔT_2. If (A, L) is $T_2' = LT_2 = MT_2$, then clearly f is mono. Suppose (A, L) is NT_2 but f is not mono. Then $\exists x \neq y$ such that $f(x) = f(y)$. Note that $f([x] \cap [y]) = [f(x)] \cap [f(y)] = [f(x)] \in K(f(x)) \cap K(f(y))$. Since f is initial, $[x] \cap [y] \in L(x) \cap L(y)$, a contradiction. By 2.4 and 2.6 , we get the result. □

2.8 Corollary *Suppose (X, K) is one of the our "T_2" spaces. Then every subspace of (X, K) (i.e $f : (A, L) \to (X, K)$ is mono and initial) is "T_2" space.*

Proof. It follows from 2.6 and 2.7. □

2.9 Theorem *(1) The cartesian product $(X = \{\Pi X_i : i \in I, K)$ is $\bar{T}_2, ST_2, \Delta T_2, KT_2$ or NT_2 iff each (X_i, K_i) is $\bar{T}_2, ST_2, \Delta T_2, KT_2$ or NT_2, respectively.*
(2) The finite cartesian product $(X = \{\Pi X_i : i \in I =$ finite $\}, K)$ is $T_2' = LT_2 = MT_2$ iff each (X_i, K_i) is $T_2' = LT_2 = MT_2, i = 1, 2, \cdots, n$.

Proof. (1) Suppose (X, K) is "Hausdorff". Then it is easy to see that each (X_i, K_i) is isomorphic to some slice in (X, K) and consequently by 2.8, (X_i, K_i) is "Hausdorff" for all i.

Conversely, suppose each (X_i, K_i) is ΔT_2 but (X, K) is not ΔT_2 i.e by 2.1(8) $\exists x \neq y$ in X such that $[x] \in K(y)$. It follows that $\exists k \in I$ such that $x_k \neq y_k$ in X_k and $\pi_k([x]) = [x_k] \in K_k(\pi_k(y) = y_k)$, a contradiction. Hence (X, K) mus be ΔT_2.

Suppose each (X_i, K_i) is ST_2. For any $x \neq y$ in X, let $\alpha \in K(x) \cap K(y)$. Note that $\pi_i\alpha \in K_i(\pi_i x = x_i) \cap K_i(\pi_i y = y_i)$. Since $x \neq y, \exists k \in I$ such that $x_k \neq y_k$ in X_k. In particular, by 2.1(9), $\pi_k\alpha = [\phi]$ which implies $\alpha = [\phi]$. Hence by 2.1(9), (X, K) is ST_2.

Suppose each (X_i, K_i) is $\bar{T}_2$. By 2.1 (6) and above result, we need to only show that for any x in X and any proper filters α and β in $K(x)$ if $\alpha \cup \beta$ is proper, then $\alpha \cap \beta \in K(x)$. Note that $\pi_i\alpha$ and $\pi_i\beta$ are in $K_i(x_i = \pi_i x)$ and each $\pi_i\alpha \cup \pi_i\beta$ is proper (since $\pi_i\alpha \cup \pi_i\beta \subset \pi_i(\alpha \cup \beta)$) for all i. Since each (X_i, K_i) is $\bar{T}_2$, $\pi_i(\alpha \cap \beta) = \pi_i\alpha \cap \pi_i\beta \in K_i(\pi_i x = x_i)$ for all i and consequently $\alpha \cap \beta \in K(x)$.

Suppose each (X_i, K_i) is KT_2 and for any $x \neq y$ in $X, K(x) \cap K(y) \neq \{[\phi[\}$. Then there exists a proper filter $\beta \in K(x) \cap K(y)$ and consequently $\pi_i\beta \in K_i(x_i) \cap K_i(y_i)$. Since each (X_i, K_i) is KT_2, $K_i(x_i) = K_i(y_i)$ for all i. Let α be in $K(x)$. Then $\pi_i\alpha \in K_i(x_i) = K_i(y_i)$ for all i and consequently $\alpha \in K(y)$. By interchanging the role of x and y, we get $K(y) \subset K(x)$ if $K(x) \cap K(y) \neq \{[\phi]\}$. Hence $K(x) = K(y)$ and by 2.2 and above result $\alpha \cap \beta \in K(x), (X, K)$ is KT_2.

Suppose each (X_i, K_i) in FCO is NT_2. By 2.4 and above result, we need to only show that for any x and y in X, if $[x] \cap [y] \in K(x) \cap K(y)$, then $x = y$. Note that $\pi_i([x] \cap [y]) = [x_i] \cap [y_i] \in K_i(x_i) \cap K_i(y_i)$ for all i and consequently $x_i = y_i$ for all i i.e $x = y$.

(2) Suppose each (X_i, K_i) is $T_2' = LT_2$ for i $= 1, 2, \cdots, n = I$. Let $x \in X$ and $\alpha \in K(x)$. By definition of the product structure, we get $\pi_i \alpha \in K_i(\pi_i x = x_i)$ for all $i \in I$ and, by 2.1(7) and 2.3, $\pi_i \alpha = [\phi]$ or $[x_i]$. If $\pi_i \alpha = [\phi]$, then $\alpha = [\phi]$. If $\pi_i \alpha = [x_i]$, then let $\beta = \bigcup_{i=1}^{n} \pi_i^{-1}[x_i] = \bigcup_{i=1}^{n} \pi_i^{-1}[\pi_i \alpha]$. We now show that $\beta = [x]$. We need to show that $\{x\} \in \beta$, then Note that $\{x\} = \bigcap_{i=1}^{n} \pi_i^{-1}\{x_i\} = \{x_1\}x\{x_2\}x \cdots \{x_n\} \in \beta$. If $\cup \in \beta$, then there exists $\cup_i \in \alpha, i \in I$ such that $\cup \supset \bigcap_{i=1}^{n} \pi_i^{-1}(\cup_i) \supset \cup_1 \cap \cup_2 \cap \cdots \cap \cup_n$. Since α is a filter, $\cup_1 \cap \cup_2 \cap \cdots \cap \cup_n \in \alpha$ and consequently $\cup \in \alpha$. Hence, $\beta \subset \alpha$. Thus, $\alpha = [x]$ and by 2.1, (X, K) is $T_2' = LT_2 = MT_2$. If (X, K) is T_2', then it is easy to see that (X_i, K_i) is T_2' for $i \in I$.

We now generalize Nell's, [7], result for some of our "T_2" structures. Nel has showed that a full subcategory, HLFCO (objects of HLFCO are Hausdorff filter convergence spaces) of LFCO is a cartesian closed initially structured category. Let $T_2'LFCO, ST_2LFCO$, and ΔT_2LFCO be subcategories of LFCO, where objects of these subcategories are determined by the corresponding "T_2" local filter convergence spaces. □

2.10 Lemma *The full subcategories $T_2'LFCO, ST_2LFCO$, and ΔT_2LFCO of LFCO are cortesian closed initially structured categories.*

Proof. It follows easily from 2.1 and the results 1.13, 1.14, and 2.5 of [7]. □

2.11 Remark Each of the full subcategories T_2LFCO (T_2FCO) af $LFCO$ (FCO), where T_2 is $\bar{T}_2, ST_2, \Delta T_2, KT_2$, or NT_2, is epireflective (i.e each subcategories are closed under formation of subspaces and products, and isomorphism -closed) by 2.8 and 2.9.

Let (X, K) be in FCO or LFCO, and F be a nonempty subset of X. Let $q : (X, K) \to (X/F, L)$ be the quotient map that identifying F to a point, *.

Theorem *If (X,K) is $T_2' = LT_2 = MT_2$, then $(X/F, L)$ is T_2'.*

Proof. Suppose (X, K) is T_2' and for any $a \in X/F$, $\alpha \in L(a)$ Since the map q is quotient, there exists $\beta \in K(x)$ such that $\alpha \supset q\beta$ and $qx = a$. But (X, K) is T_2' implies

$\beta = [x]$ or $[\phi]$, and consequently $\alpha = [a]$ or $[\phi]$. Hence by 2.1 and 2.3, $(X/F, L)$ is T_2'. □

2.13 Lemma *Let α and β be proper filters on X. Then $q\alpha \cup q\beta$ is proper iff either $\alpha \cup \beta$ is proper or $\alpha \cup [F]$ and $\beta \cup [F]$ are proper.*

Proof. Suppose $q\alpha \cup q\beta$ is proper. We show that if $\alpha \cup [F]$ is improper, then $\alpha \cup \beta$ is proper. Suppose it is not proper i.e there exists U in α and V in β such that $\mathsf{U} \cap V = \phi$. Since $\alpha \cup [F]$ is improper, $\exists W \in \alpha$ such that $W \cap F = \phi$. Note that $\mathsf{U} \cap W \in \alpha$ and $\mathsf{U} \cap W = q(\mathsf{U} \cap W)$ in $q\alpha$. It follows easily that $\phi = q(\mathsf{U} \cap W) \cap q(V) \in q\alpha \cup q\beta$, a contradiction. Similarly, if $\beta \cup [F]$ is improper, then $\alpha \cup \beta$ is proper.

Conversely, suppose $\alpha \cup \beta$ is proper, but $q\alpha \cup q\beta$ is improper. Then $\exists \mathsf{U} \in q\alpha$ and $\exists V \in q\beta$ such that $\phi = U \cap V \supset q(U_1) \cap q(V_1)$ for some $U_1 \in \alpha$ and $V_1 \in \beta$. It follows that $\phi = U_1 \cap V_1 \in \alpha \cup \beta$, a contradiction. Suppose $\alpha \cup [F]$ and $\beta \cup [F]$ are proper. Suppose also that $q\alpha \cup q\beta = [\phi]$. Then $\exists \mathsf{U} \in q\alpha$ and $\exists V \in q\beta$ such that $\phi = U \cap V \supset q(U_1) \cap q(V_1)$ for some $U_1 \in \alpha$ and $V_1 \in \beta$. But this is a contradiction since $U_1 \cap F \neq \phi \neq V_1 \cap F$, $* \in q(U_1 \cap F)$ and $* \in q(V_1 \cap F)$. □

2.14 Theorem *If (X, K) is ST_2 and F is strongly closed, then $(X/F, L)$ is ST_2.*

Proof. Let a and b be any disinct pair of points in X/F and α be in $L(a) \cap L(b)$. If α is improper, then we are done. Suppose α is proper. q is the quotient map implies $\exists \beta \in K(x)$ and $\exists \delta \in K(y)$ such that $\alpha \supset q\beta, \alpha \supset q\delta$, and $qx = a, qy = b$. Note that $q\beta \cup q\delta$ is proper and by 2.13, either $\beta \cup \delta$ is proper or $\beta \cup [F]$ and $\delta \cup [F]$ are proper. The first case can not occur since $x \neq y$ and (X, K) is ST_2. Since $a \neq b$, we may assume $x \in F$. We have $\beta \in K(x)$ and since F if strongly closed, by 2.1 (10), $\beta \cup [F]$ is improper. This shows that the second case also can not hold. Therefore, α must be improper and, by 2.1(9), we have the result. □

2.15 Theorem *Let (X, K) be $\bar{T}_2$. If F is strongly closed and for each $a \in F, K(a) = \{[a], [\phi]\}$, then $(X/F, L)$ is $\bar{T}_2$.*

Proof. Suppose (X, K) is $\bar{T}_2$. By 2.1(6) and 2.14, we only need to show that for any $a \in X/F$ and α and β in $L(a)$, if $\alpha \cup \beta$ is proper, then $(\alpha \cap \beta) \in L(a)$. Suppose $a \neq *$ and $\alpha, \beta \in L(a)$ implies $\exists \alpha_1, \beta_1 \in K(x)$ such that $\alpha \supset q\alpha_1, \beta \supset q\beta_1$, and $x = qx = a$. Since $\alpha \cup \beta$ is proper, $q\alpha_1 \cup q\beta_1$ is proper and by 2.13, we have either $\alpha_1 \cup \beta_1$ is proper or $\alpha_1 \cup [F]$ and $\beta_1 \cup [F]$ are proper. If the first case holds, then clearly $\alpha \cap \beta \in L(a)$ since (X, K) is $\bar{T}_2$. The second case can not hold since F is strongly closed ($x \in F$ and $\alpha_1 \cup [F]$ is improper) by 2.1 (10).

Suppose $a = *$ and $\alpha_1 \in K(d)$ and $\beta_1 \in K(c)$ for some $c, d \in F$. It follows from the assumption that $\alpha_1 = [d]$ or $[\phi]$ and $\beta_1 = [c]$ or $[\phi]$. If $\alpha_1 = [d]$ and $\beta_1 = [c]$, then

$\alpha = [*] = \beta$ and consequently $\alpha \cap \beta \in L(a)$. This shows that $(X/F, L)$ is $\bar{T}_2$. □

2.16 Theorem *If (X,K) is* ΔT_2 *and F is strongly closed, then* $(X/F, L)$ *is* ΔT_2.

Proof. By 2.1, we need to show that for any distinct pair of points a and b in $X/F, [a] \notin L(b)$. Suppose $[a] \in L(b)$ for some $a \neq b$ and $a \neq *$. Since q is quotient, $\exists \alpha \in K(x)$ such that $[a] \supset q\alpha$ and $qx = b$. By Lemma 3.16 of [2], we get $[a] \supset \alpha$ and consequently $[a] \in K(x)$, a contradiction (X is ΔT_2). Suppose $a = *$. Then $[*] \supset q\alpha$ for some α in $K(x)$ and $x = qx = b$. By Lemma 3.16 of [2], $\alpha \cup [F]$ is proper, a contradiction, since $x \notin F$ and $\alpha \cup [F]$ is improper (F is strongly closed by 2.1 (10)). □

2.17 Theorem *Let (X,K) be* T_1*, F be strongly closed, and for each a in F, K(a)* $=\{[a], [\phi]\}$. *If (X,K) is* KT_2 *or* NT_2*, then (X/F,L) is* KT_2 *or* NT_2*, respectively.*

Proof. By 2.16 and assumption, $(X/F, L)$ is $T_1 = \Delta T_2$. By 2.5, $(X/F, L)$ and (X, K) are $\bar{T}_\emptyset$ and $T'_\emptyset$. It follows from 2.15 and assumption that $(X/F, L)$ is $\bar{T}_2$ and consequently, by 1.2 and 2.5, $(X/F, L)$ is KT_2 and NT_2. This completes the proof. □

2.18 Remark (1) Let (X, K) be in FCO or LFCO. For $\phi \neq F \subset X$, let R be the equivalence relation $(FxF) \cup \{(x,x) : x \in X\}$. Note that the quotient space $(X/R, L)$ is the space (X, K) with F is identified to a point *, and is written $(X/F, L)$ which we had before. Let I be a finite set and for $i \in I, F_i$ be nonempty subsets of X such that $\forall i \neq j, F_i \cap F_j = \phi$. Let R be the equivalence relaton $(F_1xF_1) \cup \cdots \cup (F_nxF_n) \cup \{(x,x) : x \in X\}$. Note that the quotient map $q : (X, K) \to (X/R, L)$ is the composition of the quotient maps $q_1 : (X, K) \to (X/F_1 = Y_1, L_1), q_2 : (Y_1, L_1) \to (Y_1/F_2 = Y_2, L_2) \cdots$, and $q_n : (Y_{n-1}, L_{n-1}) \to (Y_{n-1}/F_n, L_n) = (X/R, L)$. Then

- (i) By induction, Theorem 2.12 holds for this quotient space $(X/R, L)$.
- (ii) Theorems 2.14 and 2.16 hold for this quotient space $(X/R, L)$ provided that each of F_i is strongly closed for $i = 1, 2, \cdots, n$.
- (iii) Theorem 2.15 holds for this quotient space $(X/R, L)$ if for a in $F_i, K(a) = \{[a], [\phi]\}$ and all of F_i are strongly closed, $i = 1, 2, \cdots, n$.

(2) In TOP, the category of topological spaces, in general, if X is T_2 and F is closed, then X/F is not necessarily T_2. However, in FCO or LFCO, by 2.12, the result is true provided that F is closed.

(3) Let $X = \{x, y, z\}$ be three-point set, and define two structures K and L on X as follow: $L = F(X)$, the set of all filters on X i.e L is indiscrete; and $K(x) =$

$\{[x],[y],[z],[x]\cap[y],[x]\cap[z],[\phi]\}K(y) = \{[y],[\phi]\}$, and $K(z) = \{[z],[\phi]\}$. Define $f : (X,K) \rightarrow (X,L)$ by $f(x)=x, f(y)=z$, and $f(z)=y$. Clearly f is mono and continuous, and (X,L) is KT_2 but (X,K) is not KT_2 (let $\alpha=[x]\cap[y]$ and $\beta=[x]\cap[z]$. Note that $\alpha\cup\beta=[x]$ is proper but $\alpha\cap\beta$ is not in $K(x)$). Hence by 1.14 of [7], a subcategory, KT_2 LFCO of LFCO is not quotient-reflective, and by part (1) it is not coreflective. Hence, by 1.13 of [7], it is not an initially structured category. Therefore, the result of Nel [7] can not be extended to KT_2 LFCO.

Let X be an object of E, a topological category over sets, F a nonempty subset of $U(X)$, and $q : X \rightarrow X/F$ be the final lift that identifying F to a point, *. Recall, [1] or [4], that F is closed iff * is closed in X/F (see [1] or [4]]. F is strongly closed iff X/F is T_1 at *. Let (X,K) be in FCO or LFCO. Note that a point p in X is (strongly) closed iff, by [4], $\forall x\neq p, [x]\notin K(p)$ or (and) $[p]\notin K(x)$.

2.19 Corollary *(1) (X,K) is ΔT_2 iff points are strongly closed.*
(2) If (X,K) is ΔT_2, then points are closed.
(3) Let F be a nonempty subset of X. If (X,K) is $T_2' = LT_2 = MT_2$, then F is always (strongly) closed. subset of X.
(4) (X,K) is ΔT_2 and NT_2 iff it is $\bar{T}_2$.

Proof. (1) (X,K) is ΔT_2 iff, by 2.1, $\forall x\neq y$ in $X,[x]\notin K(y)$ iff, by [4], points are strongly closed.
(2) It is a special case of (1).
(3) By 2.12, $(X/F,L)$ is T_2'. By 2.5, $(X/F,L)$ is T_1 and $\bar{T}_\emptyset$ and consequently by [4], F is (strongly) closed subset of X.
(4) It is clear. □

2.20 Remark (1) Let $X=\{x,y\}$ be two-point set, and $K(x)=\{[x],[y],[x]\cap[y],[\phi]\}, K(y)=\{[y],[\phi]\}$. Clearly, the points x and y are closed but they are not strongly closed. Hence by 2.19 (1), (X,K) is not ΔT_2 and consequently, the converse of 2.19 (2) is not true. Furthermore, in FCO or LFCO, T_1 is equivalent to ΔT_2, (see 2.5) and in TOP, the category of topological spaces, the result: T_1 is equivalent to points are closed, is not true, in general.
(2) Let $X=\{x,y\}$ be two-point set, and $K(x)=\{[x],[y],[\phi],[x]\cap[y]\}=K(y).(X,K)$ is clearly KT_2 but the points x and y are not (strongly) closed. However, in TOP, every point is closed in a Hausdorff space.
(3) Recall, in [1], that there are four various generalizations of the usual T_3 (regular) topological spaces to an arbitrary topological category over sets. In FCO or LFCO, if (X,K) is T_3' or ST_3', then it follows easily from Definition 2.2.15 of [1] p.340 by letting F to be a point that (X,K) is T_2'. By 2.12, $(X/F,L)$ is T_2' for any nonempty subset F of X. By 2.5, $(X/F,L)$ is LT_2, $\Delta T_2, ST_2, NT_2$, and KT_2.

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SÜZGEÇ VE YEREL SÜZGEÇ YAKINSAK UZAYLAR KATEGORİLERİNDE T_2-OBJELERİ

Özet

[1] de Topolojide bilinen T_2 (Hausdorff) aksiyomu herhangi bir topolojik kategoriye değişik yollarla genelleştirilmesi tanımlandı. Bu çalışmada, Hausdorff uzayların dört yeni genelleştirilmesi verildi. Ayrıca. bütün bu T_2 yapıları ve [5] ile [7] de verilenler arasındaki bağıntılar ile bunların bazı invaryant özellikleril süzgeç ve yerel süzgeç yakınsak uzaylar kategorilerinde incelendi.

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