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ON THE ADDITIVE GROUP STRUCTURE OF NONSTANDARD MODELS OF Z

Ç. Gencer, M. Terziler

Abstract

Let $\hat{Z}$ denote the inverse limit of finite cyclic groups and $F_g Z$ the group $< F \times Z, + >$ where F is a vector space over Q and $+$ is defined by $(a, x) + (b, y) = (a + b, x + y + g(a, b))$ for some $g : F \times F \to Z$. In this paper we show that any nonstandard model $Z^\star$ of Z is isomorphic to $F_\beta Z$ for some $\beta : F \to \hat{Z}$ where $F = Z^\star / Z$.

1. Introduction

Consider the set T of first-order sentences written in the language $\{+, \times\}$ which are true in the ring Z of integers. Gödel's Compactness Theorem implies that there are rings different from Z that satisfy all the sentences of T. These rings are called nonstandard models of T. The ring Z embeds in every nonstandard model of T (just look at the subring generated by 1). In this article we will investigate the structure of the additive group of nonstandard models of T. To explain our result we need some definitions.

Let F be a divisible torsion-free abelian group (i.e. F is a vector space over Q) and let G be an additive group containing Z. Let $\beta : F \to G$ be a map such that $g(a, b) = \beta(a + b) - \beta(a) - \beta(b) \in Z$ for all $a, b \in F$. Let $F_\beta Z$ be the group whose underlying set is $F \times Z$ and whose addition is defined by $(a, x) + (b, y) = (a + b, x + y + g(a, b))$.

Let $Z^\star$ be a nonstandard model of T. It is well-known (and we will produce the proof below) that $F = Z^\star / Z$ is a torsion-free, divisible abelian group. It is very easy to show (using some elementary cohomological considerations) that there is a map $h : F \to Z^\star$ such that for all $a, b \in F$, $g(a, b) = h(a + b) - h(a) - h(b) \in Z$ and that $F_h Z \approx Z^\star$. Furthermore the isomorphism between $F_h Z$ and $Z^\star$ can be chosen so that the element $(0, n)$ of $F_h Z$ is sent to $n \in Z \leq Z^\star$ for all $n \in Z$. Indeed, by using the axiom of choice, for each $a \in F$ choose a representative $h(a) \in Z^\star$ of a. We may assume that $h(0) = 0$. Thus $a = h(a) + Z$ for all $a \in F$. Then

$$h(a + b) + Z = a + b = (h(a) + Z) + (h(b) + Z) = h(a) + h(b) + Z,$$

so that $g(a,b) = h(a+b) - h(a) - h(b) \in Z$. Define now $\theta : F_h Z \to Z^\star$ by $\theta(a,n) = -h(a) + n$. It is a matter of a few lines of trivial computation to check that θ is an isomorphism and that $\theta(0,n) = n$ for all $n \in Z$. All this is done in [P].

Let $\hat{Z}$ denote the inverse limit of all the finite cyclic groups. The additive group Z can be embedded in $\hat{Z}$ naturally as explained in the next section. In this paper we show that if $Z^\star$ is a nonstandard model of T, then the additive group of $Z^\star$ is isomorphic to $F_\beta Z$ for some $\beta : F \to \hat{Z}$ where $F = Z^\star/Z$. We furthermore show that the isomorphism can be chosen so as to send $(0,n) \in F_\beta Z$ onto $n \in Z \leq Z^\star$.

2. Preliminaries

Pure Injective Groups. A subgroup H of an abelian group G is called pure if $nG \cap H = nH$ for all integers $n > 1$.

A sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is called exact if α is injective, β is surjective and if $\ker(\beta) = Im(\alpha)$. Such an exact sequence is called pure-exact if $\alpha(A)$ is pure in B.

An abelian group Y is called pure-injective if for any purexact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ and every $\phi : A \to Y$, there is a $\sigma : B \to Y$ such that $\sigma o \alpha = \phi$.

Inverse Limits. Let I be a partially ordered directed set, i.e. there is a reflexive and transitive binary relation $\leq$ on I such that for each $i, j \in I$,

1) $i \leq j$ and $j \leq i$ imply $i = j$, and

2) there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.

Let $(A_i)_{i \in I}$ be a family of groups and $(\pi_{i,j} : A_j \to A_i)_{i \leq j}$ be a family of homomorphisms such that $\pi_{i,i} = Id_{A_i}$ and $\pi_{i,j} o\ \pi_{j,k} = \pi_{i,k}$ for all $i \leq j \leq k$. Such a system $(A_i, \pi_{i,j})_{i \leq j}$ is called an inverse system. Given an inverse system $(A_i, \pi_{i,j})_{i \leq j}$, the set

$$A^\star = \{a = (a_i)_{i \in I} \in \Pi_{i \in I} A_i : \pi_{i,j}(a_j) = a_j) = a_i \text{ for all } i \leq j\}$$

is a subgroup of the product $\Pi_{i \in I} A_i$. The group $A^\star$ is called the inverse limit of the given inverse system. (See $[F, \delta 12]$).

p-Adic Integers. Let p be a prime integer and let Q_p denote the ring of rational numbers whose denominator is prime to p. For two distinct elements $x, y \in Q_p$, let $d_p(x,x) = 0$ and $d_p(x,y) = 1/p^n$ where $n \in Z$ is such that for some $a, b \in Z$, we have $x - y = p^n a/b, (p,a) = 1$, $(p,b) = 1$. Then d_p is a metric of Q_p. The completion $\bar{Q}_p$ of Q_p with respect to this metric is also a ring. We denote the additive group of $\bar{Q}_p$ by J_p. The group J_p is abelian, torsion-free and is isomorphic to the inverse limit of the groups $Z/p^n Z$ where the homomorphisms $\Pi_{m,n} : Z/p^n \to Z/p^m Z$ for $m \leq n$ are defined by $\Pi_{m,n}(x + p^n Z) = x + p^m Z$. (See e.g. [F, p. 62]).

The Group $\hat{Z}$. Define $\pi_{n,m} : Z/nZ \to Z/mZ$ by $\pi_{n,m}(x + nZ) = x + mZ$ if m divides n. If $1 \leq m \leq n$ and m does not divide n, let $\pi_{m,n}$ be the zero map. Then $(Z/nZ, \pi_{m,n})_{0 < m \leq n}$ is an inverse system. Let $\hat{Z}$ denote the inverse limit of this system.

We can view $\hat{Z}$ as the set of sequences $x = (x_n)_{n\in N}$ such that for each $n, 0 \leq x_n < n$ is an integer and $x_n \equiv x_m (mod m)$ whenever m divides $n[N]$. Presented this way, the addition on $\hat{Z}$ is defined coordinatewise, so that for all $x, y \in Z$, the n^{th} coordinate $(x+y)_n$ of $x+y$ is the unique nonnegative integer $< n$ satisfying $(x+y)_n \equiv x_n + y_n$ (mod n). It is easy to see that $\hat{Z}$ is a torsion-free group. Indeed, assume that $nx = 0$ for some $n > 0$ and $x \in \hat{Z}$. Then for all $m \geq 1$, we have $0 \equiv (nx)_{nm} \equiv nx_{nm}$ (mod nm), so that $x_{nm} \equiv 0$ (mod m). Since m divides nm, we also get $x_m \equiv 0$ (mod m), hence $x_m = 0$.

The additive group Z embeds in $\hat{Z}$ via the map $x \to (x_n)_n$ where x_n is the unique nonnegative integer $< n$ satisfying $x \equiv x_n$ (mod n) (see e.g. [N, page 4]). We will denote this embedding by α.

It can be shown that the additive group $\hat{Z}$ is isomorphic to the product of the groups $J_p : \hat{Z} \approx \Pi_p$ prime J_p. See e.g. [M] about this result.

Algebraically Compact Groups. A group is said to be algebraically compact if it is a direct-summand of every group that contains it as pure subgroup.

A direct product of is algebraically compact if and only if each factor is algebraically compact [F, Corollary 38.2].

The groups J_p are algebraically compact [F, Proposition 39.4]. The last two facts, together with the isomorphism $\hat{Z} \approx \Pi_{p\ prime} J_p$ imply that $\hat{Z}$ is algebraically compact.

By [F, Theorem 38.1], an abelian group is algebraically compact if and only if it is pure-injective. Thus the group $\hat{Z}$ is pure-injective.

Models of T. Let $Z^\star$ be a model of T. We denote by θ the embedding of Z in $Z^\star$. In the remarks below we assume θ is the identity.

Note that if $n \in N$, then the set $\{z \in Z^\star : 0 \leq z < n\}$ has exactly $n-1$ elements, namely $0, 1, \cdots, n-1$, because there is a first-order sentence that expresses this fact. Therefore any element of $Z^\star$ which is bounded below and above by the elements of Z is an element of Z.

As is well-known, $Z^\star/Z$ is a divisible torsion-free group [M]. With the nonlogician in mind, we know prove this result. Recall first that, every positive integer is a sum of the squares of 4 integers. Therefore the relation $\leq$ on Z defined by

$$x \leq y \Leftrightarrow \exists z_1, z_2, z_3, z_4\ y = x + z_1^2 + z_2^2 + z_3^2 + z_4^2$$

is the usual order on Z. It is clear that there is first-order sentence that states that the relation $\leq$ defined as above turns Z into an ordered ring. Therefore, any nonstandard model $Z^\star$ of T satisfies the same sentence, i.e. the ring $Z^\star$ is an ordered ring with the above definition of $\leq$. Since

$$\forall z, n((n > 0 \wedge z > 0) \Rightarrow z \leq nz)$$

is true in Z, this sentence is in T, hence it is also true in $Z^\star$. Now assume that for some $n \in N\backslash\{0\}$ and $z \in Z^\star$ with $z > 0$, we have $nz \in Z$. Since $< 0 < z \leq nz \in Z$, we must have $z \in Z$. One can also prove that the same holds if $z < 0$. Therefore the abelian group $Z^\star/Z$ is torsion-free. To show that it is divisible, we remark that

$$\forall x, n(n \neq 0 \Rightarrow \exists q, r(0 \leq r < n \wedge x = nq + r))$$

is in T. Note that if $n \in N$, then $r \in N$ also. Therefore for any $x \in Z^\star$ and any $n \in N \backslash \{0\}$, there is a $q \in Z^\star$ such that $x \equiv nq \pmod{Z}$. This shows that $Z^\star/Z$ is divisible.

Since $Z^\star/Z$ is torsion-free and divisible, it can be viewed as a vector space over Q.

The proof shows also that Z is a pure subgroup of $Z^\star$, i.e. $nZ^\star \cap Z = nZ$.

3. Proof of the Theorem

Theorem *For every nonstandard model $Z^\star$ of T, the additive group of $Z^\star$ is isomorphic to $F_\beta Z$ for some $\beta : F \to \hat{Z}$ where $F = Z^\star/Z$.*

Proof. Let $Z^\star$ be a nonstandard model of T. Let $F = Z^\star/Z$. Recall that $\alpha : Z \to \hat{Z}$ and $\theta : Z \to Z^\star$ denote the embeddings. Clearly the sequence

$$0 \to Z \xrightarrow{\theta} Z^\star \to Z^\star/Z \to 0$$

is a pure exact sequence. Since $\hat{Z}$ is pure-injective, there is a homomorphism $\mu : Z^\star \to \hat{Z}$ such that $\mu o \theta = \alpha$. On the other hand, by [P] (see the introduction), there is a map $h : F \to Z^\star$ such that $g(a,b) = h(a+b) - h(a) - h(b) \in Z$ and $F_h Z \approx Z^\star$, furthermore the isomorhism δ between $F_h Z$ and $Z^\star$ may be chosen so that $\delta(0,n) = \theta(n)$ for all $n \in Z$. Note that $g(a,0) = 0$ so that $(a,x) = (a,0) + (0,x)$ for all $(a,x) \in F_h Z$. Let $\phi = \mu o \delta : F_h Z \to \hat{Z}$. Thus $\phi(0,n) = \mu o \delta(0,n) = \mu o \theta(n) = \alpha(n)$ if $n \in Z$. Let $\beta : F \to \hat{Z}$ be defined by $\beta(a) = \phi(a,0)$. Since ϕ is a homomorphism, we have

$$\phi(a,x) = \phi((a,0) + (0,x)) = \phi(a,0) + \phi(0,x) = \beta(a) + \alpha(x).$$

Thus

$$\phi((a,x) + (b,y) = \phi((a+b, x+y+g(a,b)) = \beta(a+b) + \alpha(x+y+g(a,b)).$$

On the other hand,

$$\phi((a,x) + (b,y)) = \phi(a,x) + \phi(b,y)) = \beta(a) + \beta(b) + \alpha(x) + \alpha(y).$$

Equating the last two equations, we get

$$\beta(a+b) - \beta(a) - \beta(b) = -\alpha(g(a,b)) \in Z.$$

Thus $F_\beta Z = F_h Z \approx Z^\star$. □

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Z'NİN NONSTANDARD MODELLERİNİN TOPLAMSAL GRUP YAPISI ÜZERİNE

Özet

Bu çalışmada, Z tamsayılar halkasının her nonstandard modelinin toplamsal grup yapısının, Q üzerine bir F vektör uzayı ve bir $\beta : F \to \hat{Z}$ dönüşümü için, $F_\beta Z$ grubuna eşyapısal olduğunu kanıtlıyoruz.

Ç. GENCER & M. TERZİLER
Ege University,
Department of Mathematics,
35100 Bornova, İzmir-TURKEY

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