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PIRINCIPLES OF RADIATION FOR HELMHOLTZ EQUATION IN N-DIMENSIONAL LAYER WITH IMPEDENCE BOUNDARY CONDITIONS.

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Abstract

The principle of limit absorption and A. G. Sveshnikov's partial conditions of radition for Helmholts equation in multile dimensinal layer with impedance boundary conditions were studied. The behavior of the solutions of corresponding initial-boundary value problem for non-stationary wave equations as $t \to +\infty$ was studied too.

Introduction

The study of diss emination of waves in homogeneous layer bounded by plane-parallel boundaries from two sides is very important for physics. The dissemination of the radio-wave at a great distance in the atmosphere, dissemination of sound in sea relate to phenomena like these. The enumerated physical phenomena lead to the boundary problem in the layer for Helmholtz equation. The principle of the limit absorption in two-dimensional layer was researched by L. M. Brekthovskikht in [1]. The principles of the limit absorption and the limit amplitude and A. G. Svesnikov's partial conditions of radiation were researched by A. G. Sveshnikov in [2].

The principles of radiation in n-dimensional layer for Helmholtz equation with the boundary conditions of Dirichlet and Neumann were studied in [3]. The same problems for cylindrical domin were studied in [4], [5]. The principles of radiation for Helmholtz equation in n-dimensional layer with impedance boundary conditions were proved in this paper.

1. Construction of Green's function and the principle of limit absorption.

Let

$$\prod = \{x : (x', x_{n+1}), \quad x'(x_1 x_2, \ldots, x_n), \\ -\infty < x_j < +\infty, -a < x_{n+1} < a, \ j = 1, 2, \ldots, n; \ a > 0\}$$

be the layer in $n+1$-dimensional Euclidean space R_{n+1}. We consider the next boundary value problem in $\prod$

$$(\Delta + \kappa^2)\, \mathcal{U}(\kappa, x) = f(x), \tag{1}$$

$$\left(\frac{\partial}{\partial x_{n+1}} + \alpha\kappa\right) \mathcal{U}(\kappa, x) \Big|_{x_{n+1}=\pm a} = 0, \tag{2}$$

where Δ is the Laplacian; $f(x)$-finitary infinitely derived function; κ, α-complex parameters.

Definition 1. *The function $\mathcal{U}(\kappa, x)$ satisfying equation (1) and boundary conditions (2) in the sense of the generalized functions [6] is called a solution of the problem (1)-(2).*
We consider the problem

$$(\Delta + \kappa^2)\, G(\kappa, x, y) = \delta(x, y), \tag{3}$$

$$\left(\frac{\partial}{\partial x_{n+1}} + \alpha\kappa\right) G(\kappa, x, y) \Big|_{x_{n+1}=\pm a} = 0, \tag{4}$$

together with the problem (1)-(2).

Definition 2. *Let Im $\kappa^2 \neq 0$. The decreasing solution of the problem (3)-(4) in $\prod$ when $x \to \infty$ is called Green's function of problem (1)-(2) where $\delta(x, y)$ is Dirac's function.*

Now we consider the construction of Green's function. By principle of the limit absorption the unique solution of boundary value problem (1)-(2) with real parameter κ^2 is searched as limit of solution of boundary value problem (1)-(2) with complex parameter $\kappa_\varepsilon^2 = \kappa^2 + i\varepsilon(\varepsilon > 0)$ when $\varepsilon \to 0$. So we shall solve the boundary value problem (3)-(4) with the complex parameter κ_ε^2. To solve this problem we apply Fourier transformation with respect to x'

$$\left[\frac{d^2}{dx_{n+1}^2} + \left(\kappa_\varepsilon^2 - \rho^2\right)\right] \tilde{G}\left(\xi', x_{n+1}, y_{n+1}, \kappa_\varepsilon\right) = \delta\left(x_{n+1}, y_{n+1}\right) e^{i\xi' y'}, \tag{5}$$

$$\left(\frac{d}{dx_{n+1}} + \alpha\kappa_\varepsilon\right) \tilde{G}\left(\xi', x_{n+1}, y_{n+1}, \kappa_\varepsilon\right) \Big|_{x_{n+1}=\pm a} = 0, \tag{6}$$

where $\tilde{G} = \mathcal{F}(G)$, $\mathcal{F}_{x' \to \xi'}$ *is Fourier transformation with respect to* $x', \rho = |\xi|$.

The solution of the boundary value problem (5)-(6) has the following representation:

$$G(\xi', x_{n+1}, y_{n+1}, \kappa_\varepsilon) = e^{i\xi' y'} \left[-\frac{\cos h \left[i\sqrt{\kappa_\varepsilon^2 - \rho^2} \left(2a - |x_{n+1} - y_{n+1}| \right) \right]}{2i\sqrt{\kappa_\varepsilon^2 - \rho^2} \sin h \left(2ia\sqrt{\kappa_\varepsilon^2 - \rho^2} \right)} + \right.$$

$$+ \frac{\left[\left(\alpha^2 - 1 \right) \kappa_\varepsilon^2 + \rho^2 \right] \cos h \left[i\sqrt{\kappa_\varepsilon^2 - \rho^2} \left(x_{n+1} - y_{n+1} \right) \right]}{2i\sqrt{\kappa_\varepsilon^2 - \rho^2} \left[\left(\alpha^2 + 1 \right) \kappa_\varepsilon^2 - \rho^2 \right] \sin h \left(2ia\sqrt{\kappa_\varepsilon^2 - \rho^2} \right)} -$$

$$\left. - \frac{\alpha \kappa_\varepsilon \sin h \left[i\sqrt{\kappa_\varepsilon^2 - \rho^2} \left(x_{n+1} + y_{n+1} \right) \right]}{\left[\left(\alpha^2 + 1 \right) \kappa_\varepsilon^2 - \rho^2 \right] \sin h \left(2ia\sqrt{\kappa_\varepsilon^2 - \rho^2} \right)} \right]. \tag{7}$$

from (7) it follows that $\tilde{G}$ *is an analytic function with respect to* ρ *and the points* $\eta_{1,2} = \pm\sqrt{\left(1 + \alpha^2\right) \kappa_\varepsilon^2}$, $\eta_\ell^\pm = \pm\sqrt{\kappa_\varepsilon^2 - \frac{\pi^2 \ell^2}{4a^2}}$, $\ell = 0, 1, 2 \ldots$ *are simple poles of* $\tilde{G}$.

If $x_{n+1} \neq y_{n+1}$, *so* $\tilde{G}$ *is absolutely integrable function with respect to* ξ'. *Applying Fourier inverse transformation we obtain*

$$G(\kappa_\varepsilon, x, y) = -\frac{i\tau^{1-n/2}}{4(2\pi)^{n/2-1}} \left\{ \left[\sqrt{\left(1 + \alpha^2\right) \kappa_\varepsilon^2} \right]^{n/2-1} g_\circ(\kappa_\varepsilon, x_{n+1}) \, g_\circ(\kappa_\varepsilon, y_{n+1}) \times \right.$$

$$\left. \times H^{(1)}_{n/2-1} \left(\sqrt{\left(1 + \alpha^2\right) \kappa_\varepsilon^2} \tau \right) + \sum_{\ell=1}^{\infty} \kappa_\ell^{n/2-1} g_\ell(\kappa_\varepsilon, x_{n+1}) \, g_\ell(\kappa_\varepsilon, y_{n+1}) \, H^{(1)}_{n/2-1}(\kappa_\ell \tau) \right\}, \tag{8}$$

where

$$g_\circ(\kappa_\varepsilon, x_{n+1}) = \left(\frac{\alpha \kappa_\varepsilon}{\sin h \left(2a\alpha\kappa_\varepsilon \right)} \right)^{1/2} e^{-\alpha \kappa_\varepsilon x_{n+1}},$$

$$g_\ell(\kappa_\varepsilon, x_{n+1}) = \frac{\frac{\pi \ell}{2a} \cos h \left(a - x_{n+1} \right) \frac{\pi \ell}{2a} + \alpha \kappa_\varepsilon \sin h \left(a - x_{n+1} \right) \frac{\pi \ell}{2a}}{\left[a \left(\alpha^2 \kappa_\varepsilon^2 + \frac{\pi^2 \ell^2}{4a^2} \right) \right]^{1/2}},$$

$H_{n/2-1}(Z)$ *is the Hankel function of the first kind,* $\tau = |x' - y'|, \kappa_\ell = \sqrt{\kappa_\varepsilon^2 - \frac{\pi^2 \ell^2}{4a^2}}$. *So we proved the theorem.*

Theorem 1. *Green's function of problem (1)-(2) (when* $n = 1, 2; \alpha \neq i$*) is analytic function with respect to* κ_ε *without branch points* $\kappa_\varepsilon = \frac{\pi \ell}{2a}$ *and simple pole-points* $\kappa_\varepsilon = i\frac{\pi \ell}{2a\alpha}$ $(\ell = 0, \pm 1, \pm 2, \ldots)$ *and the representation (8) is true.*

The solution of the boundary value problem (1)-(2) is defined by formula

$$\mathcal{U}(\kappa_\varepsilon, x) = \int_{\sqcap} \dots \int G(\kappa_\varepsilon, x' - \xi', x_{n+1}, \xi_{n+1}) f(\xi)\, d\xi. \tag{9}$$

Using the asymptotic representation of Hankel function $H^{(1)}_{n/2-1}(Z)$ *when* $Z \to \infty$ *we obtain that series (8) and its derivative converge uniformly by* ε *when* $\tau > 0$. *We can take the limit in (9) when* $\varepsilon \to 0$ *and get the following theorem.*

Theorem 2. *The principle of the limit absorption takes place for the problem (1)-(2) when* $\kappa \neq i\frac{\pi\ell}{2a\alpha}$ *(and for* $n = 1, 2$ *when* $\kappa \neq \frac{\pi\ell}{2a}$ *).*

2. The behavour of the solutions of initial-boundary value problem for the wave equation as $t \to +\infty$.

Now we consider the non-stationary problem corresponding to the problem (1)-(2)

$$\left(-\frac{\partial}{\partial t^2} + \Delta\right) \mathcal{U}(t, x) = f(x) e^{i\omega t}, \tag{10}$$

with initial conditions

$$\mathcal{U}(0, x) = 0, \quad \frac{\partial \mathcal{U}(0, x)}{\partial t} = 0, \tag{11}$$

and impedance boundary condition

$$\left(\frac{\partial}{\partial x_{n+1}} + \alpha \frac{\partial}{\partial t}\right) \mathcal{U}(t, x) \Big|_{x_{n+1} = \pm a} = 0, \tag{12}$$

where $f(x)$ *is finitary infinitely derived function with a supp*$\{f(x)\} = \Omega \subset \prod$, ω *is a real parameter. We can prove the following theorem.*

Theorem 3. *Let* $\omega \neq \frac{\pi\ell}{2a}, \omega \neq \frac{\pi\ell}{2a\alpha}, \ell = 0, \pm 1, \pm 2, \dots; \alpha -$ *a real numbber* ($|\alpha| \neq 1$ *when* $n = 1, 2$*). The asymptotic representation takes place for the solution of problem (10)-(12)*

$$e^{i\omega t} \mathcal{U}(t, x) = V(i\omega, x) + \Phi(t, x, \omega, \alpha) + \Psi(t, x, \omega, \alpha) t^{-\frac{n}{2}} + O\left(t^{-\frac{n}{2}-1}\right),$$

when $t \to +\infty$ *uniformly with respect to* x *in every compact from* $\sqcap$. *Here* $\nu(i\omega, x)$ *is the solution of corresponding stationary problem chosen by he principle of the limit absorption. The functions* $\Phi(t, x, \omega, \alpha)$, $\Psi(t, x, \omega\alpha)$ *are bounded on all changes and will be defined below.*

Proof. Implying $\mathcal{U}(t, x)$ as generalized function over the space $\mathcal{D}'$ (look [2]) we apply in (10)-(12) Laplace transformation with respect to t. So we have the following problem

$$\left(\Delta+\kappa^{2}\right)V\left(\kappa,x\right)=\frac{f(x)}{\kappa-i\omega}, \tag{13}$$

$$\left(\frac{\partial}{\partial x_{n+1}}+\alpha\kappa\right)V\left(\kappa,x\right)\Big|_{x_{n+1}=\pm a}=0, \tag{14}$$

where $V(\kappa,x)=\mathcal{L}\mathcal{U}(t,x),\cdot\mathcal{L}$ is Laplace transformation and $Re\kappa>0$. Let $\mathcal{J}_m(i\kappa)\neq 0$ and so Green's function of problem (3)-(4) exists and is unique. The solution of the boundary value problem (13)-(14) is defined by means of the formula (9) in which κ is replaced with $i\kappa,\alpha$ with $-i\alpha$ and the solution of (10)-(12) is defined with the solution of (13)- (14) with Laplace transformation in distribution.

$$\mathcal{U}\left(t,x\right)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}V\left(\kappa,\infty\right)e^{\kappa t}d\kappa,$$

where $\varepsilon>0$.

Series in expression $V(\kappa,x)$ converge uniformly with respect to κ. Therefore integrating this expression we obtain the following representation for the solution of the problem (10)-(12)

$$\begin{aligned}\mathcal{U}\left(t,x\right)=&-\frac{i}{4\left(2\pi\right)^{n/2-1}}\left\{\int\limits_{\Omega}\cdots\int\tau^{1-\frac{n}{2}}G_{\circ}\left(t,x,y\right)f(y)dy+\right.\\&+\frac{1}{a}\sum_{\ell-1}^{\infty}\left[\frac{\pi^{2}\ell^{2}}{4a^{2}}\,cosh\left(a-x_{n+1}\right)\frac{\pi\ell}{2a}\int\limits_{\Omega}\cdots\int\tau^{1-\frac{n}{2}}f(y)G_{\ell,\circ}\left(t,x',y'\right)\cos h\left(a-y_{n+1}\right)\frac{\pi\ell}{2a}dy\right.\\&+\frac{\pi\ell}{2a}\int\limits_{\Omega}\cdots\int\tau^{1-\frac{n}{2}}f(y)G_{\ell,1}\left(t,x',y'\right)\sin h\left[2a-\left(x_{n+1}-y_{n+1}\right)\right]\frac{\pi\ell}{2a}dy+\\&\left.\left.+\ \sin h\left(a-x_{n+1}\right)\frac{\pi\ell}{2a}\int\limits_{\Omega}\cdots\int\tau^{1-\frac{n}{2}}f(y)G_{\ell,2}\left(t,x',y'\right)\sin h\left(a-y_{n+1}\right)\frac{\pi\ell}{2a}dy\right]\right\},\end{aligned} \tag{15}$$

where

$$G_{\circ}\left(t,x,y\right)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\alpha\kappa e^{\kappa[t-\alpha(x_{n+1}-y_{n+1})]}}{\left(\kappa-i\omega\right)\sin h2a\alpha\kappa}\left(i\sqrt{(1-\alpha^{2})\kappa^{2}}\right)^{n/2-1}\times \tag{16}$$
$$H^{(1)}_{\frac{n}{2}-1}\left(i\tau\sqrt{(1-\alpha^{2})\kappa^{2}}\right)d\kappa,$$

$$G_{\ell,m}\left(t,x,y\right)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\left(\alpha\kappa\right)^{m}\left(i\kappa_{\ell}\right)^{n/2-1}}{\left(\kappa-i\omega\right)\left(\alpha^{2}\kappa^{2}+\frac{\pi^{2}\ell^{2}}{4a^{2}}\right)}H^{(1)}_{\frac{n}{2}-1}\left(i\kappa_{\ell}\tau\right)e^{\kappa t}d\kappa. \tag{17}$$

The integrals in (16), (17) converges in the dictribution, but integral in (16) converges in usually sense if $|\alpha| > 1$, ($|\alpha| = 1 at\ n = 1, 2$). Consider now each integral in (16), (17) indivindually when $t \to +\infty$. The function under integral in (16),(17) in (16) has simple poles in points $\kappa = i\omega$ and $\kappa = i\frac{\pi\ell}{2a\alpha}$. Using Cauchy's theorem when $t \to +\infty$ we get

$$G_\circ(t, x, y) = \frac{\alpha i\omega e^{[i\omega(t-\alpha(x_{n+1} + y_{n+1})]}}{\sin h\,(2ai\alpha\omega)}\left(i|\omega|\sqrt{\alpha^2-1}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}\left(i|\omega|\tau\sqrt{\alpha^2-1}\right) +$$
$$+ \frac{\pi}{4\alpha^2}\sum_{\nu=1}^{\infty}\Theta_\nu(t, x_{n+1}, y_{n+1})\,\nu\left(i\frac{\pi\nu}{2a}\sqrt{1-\frac{1}{\alpha^2}}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}\left(i\frac{\pi\nu}{2a}\tau\sqrt{1-\frac{1}{\alpha^2}}\right), \tag{18}$$

where

$$\theta_\nu(t, x_{n+1}, y_{n+1}) = (-1)^\nu\left\{\frac{e^{i\frac{\pi\nu}{2a\alpha}[t-\alpha(x_{n+1} + y_{n+1})]}}{\frac{\pi\nu}{2a} - \alpha\omega} + \frac{e^{-i\frac{\pi\nu}{2a\alpha}[t-\alpha(x_{n+1} + y_{n+1})]}}{\frac{\pi\nu}{2a} + \alpha\omega}\right\} \tag{19}$$

Now we can consider the functions $G_{\ell,m}(t, x, y)$. The function under integral in expression for $G_{\ell,m}(t, x, y)$ has simple poles in $\kappa = i\omega$, $\kappa = i\frac{\pi\ell}{2a\alpha}$ and branch points in $\kappa = \pm i\frac{\pi\ell}{2a}$. We make cut on the plane and consider contour

$$\Gamma_\ell = \Gamma^+_\ell \cup C^{(1)}_\varepsilon \cup \Gamma^-_\ell \cup C^{(2)}_\varepsilon,$$

where $C^{(1)}_\varepsilon$ and $C^{(2)}_\varepsilon$ are circles with radius ε and centres in points $\kappa = \pm i\frac{\pi\ell}{2a}$ and $\Gamma^+_\ell, \Gamma^-_\ell$ banks of cut. We suppose the parts of cut near the points $\kappa = \pm i\frac{\pi\ell}{2a}$ to be parallel to the real line for comfort of applying the Laplace method.

Then

$$G_{\ell,m}(t, x', y') = \frac{(\alpha i\omega)^m e^{i\omega t}}{\frac{\pi^2\ell^2}{4a^2} - \alpha^2\omega^2}\left(i\sqrt{\frac{\pi^2\ell^2}{4a^2} - \omega^2}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}\left(i\tau\sqrt{\frac{\pi^2\ell^2}{4a^2} - \omega^2}\right) +$$
$$+ \left(i\frac{\pi\ell}{2a}\sqrt{1-\frac{1}{\alpha^2}}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}\left(i\frac{\pi\ell}{2a}\tau\sqrt{1-\frac{1}{\alpha^2}}\right)\Phi_{\ell,m}(\alpha, \omega, t) + Q_{\ell,m}(t, x, y) \tag{20}$$

where

$$\Phi_{\ell,m}(\alpha, \omega, t) = \frac{i\alpha}{2}\left(\frac{i\pi\ell}{2a}\right)^{m-1}\left[\frac{e^{i\frac{\pi\ell}{2a\alpha}t}}{\alpha\omega - \frac{\pi\ell}{2a}} - \frac{(-1)^m e^{-i\frac{\pi\ell}{2a\alpha}t}}{\alpha\omega + \frac{\pi\ell}{2a}}\right], \tag{21}$$
$$Q_{\ell,m}(t, x', y') = \frac{1}{2\pi i}\int_{\Gamma_\ell}\frac{(\alpha\kappa)^m (i\kappa_\ell)^{\frac{\pi}{2}-1} e^{\kappa t}}{(\kappa - i\omega)\left(\alpha^2\kappa^2 + \frac{\pi^2\ell^2}{4a^2}\right)} H^{(1)}_{\frac{n}{2}-1}(i\kappa_\ell\tau)\,d\kappa.$$

We study no ω $Q_{\ell,m}(t,x'y')$ at $t\to\infty$. Let n is odd. Then using

$$H^{(1)}_{\frac{n}{2}-1}(Z)=\frac{J_{-(\frac{n}{2}-1)}(Z)+e^{i\pi(\frac{n}{2}-1)}J_{\frac{n}{2}-1}(Z)}{i\sin h\pi\left(\frac{n}{2}-1\right)}$$

Where $J_{\frac{n}{2}-1}(Z)$ is the Bessel function of ν order and that $Z^{\frac{n}{2}-1}J_{-(\frac{n}{2}-1)}(Z)$ is an entire function we get

$$Q_{\ell,m}(t,x',y')=\frac{1}{2\pi i}\int_{\Gamma_\ell}\frac{(\alpha\kappa)^m(i\kappa_\ell)^{\frac{n}{2}-1}e^{\kappa t}}{(\kappa-i\omega)\left(\alpha^2\kappa^2+\frac{\pi^2\ell^2}{4a^2}\right)}J_{\frac{n}{2}-1}(i\kappa_\ell\tau)\,d\kappa. \tag{22}$$

The integrals on contours $C_\varepsilon^{(j)}, j=1,2$ in (22) tent to zero uniformly with respect to τ at $\varepsilon\to 0$. Taking into consideration values of function $(i\kappa_\ell)^{\frac{n}{2}-1}J_{\frac{n}{2}-1}(i\kappa_\ell\tau)$ on shores of section and

$$Z^{\frac{n}{2}-1}J_{\frac{n}{2}-1}(-Z)=-Z^{\frac{n}{2}-1}J_{\frac{n}{2}-1}(Z)$$

we obtain

$$Q_{\ell,m}(t,x',y')=\frac{1}{\pi i}\int_{\Gamma_\ell}\frac{(\alpha\kappa)^m(i\kappa_\ell)^{\frac{n}{2}-1}e^{\kappa t}}{(\kappa-i\omega)\left(\alpha^2\kappa^2+\frac{\pi^2\ell^2}{4a^2}\right)}J_{\frac{n}{2}-1}(i\kappa_\ell\tau)\,d\kappa. \tag{23}$$

By using expansion $J_\nu(z)$ in series for non-integral ν in (23) and restricting ourselves only to the first members of the expansion (the other menbers of the expansion for large t will be smaller than first member) we obtain

$$Q_{\ell,m}(t,x',y')=\frac{\tau^{\frac{n}{2}-1}}{2^{\frac{n}{2}-1}\pi i\Gamma\left(\frac{n}{2}\right)}\int_{\Gamma_\ell^+}\frac{(\alpha\kappa)^m(i\kappa_\ell)^{n-2}e^{\kappa t}}{(\kappa-i\omega)\left(\alpha^2\kappa^2+\frac{\pi^2\ell^2}{4a^2}\right)}d\kappa. \tag{24}$$

Applaying the Laplace method to (24) we obtain

$$Q_{\ell,m}(t,x',y')=-\frac{(-\alpha)^m\tau^{\frac{n}{2}-1}i^n\left(i\frac{\pi\ell}{2a}\right)^{\frac{n}{2}+m-3}}{\pi(1-\alpha^2)}t^{-\frac{n}{2}}\Psi_\ell(t,\omega,m)+O\left(t^{-\frac{n}{2}-1}\right), \tag{25}$$

where

$$\Psi_\ell(t,\omega,m)=\frac{(-1)^{\frac{n}{2}+m-1}e^{i\frac{\pi\ell}{2a}t}}{\frac{\pi\ell}{2a}-\omega}+\frac{e^{-i\frac{\pi\ell}{2a}t}}{\frac{\pi\ell}{2a}+\omega}. \tag{26}$$

From (20), (25) it follows that

$$G_{\ell,m}(t,x',y') = \frac{(\alpha i\omega)^m e^{i\omega t}}{\frac{\pi^2\ell^2}{4a^2} - \alpha^2\omega^2}\left(i\sqrt{\frac{\pi^2\ell^2}{4a^2} - \omega^2}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}\left(i\tau\sqrt{\frac{\pi^2\ell^2}{4a^2} - \omega^2}\right) +$$
$$+ \left(i\frac{\pi\ell}{2a}\sqrt{1-\frac{1}{\alpha^2}}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}\left(i\frac{\pi\ell}{2a}\tau\sqrt{1-\frac{1}{\alpha^2}}\right)\Phi_{\ell,m}(\alpha,\omega,t) + O\left(t^{-\frac{n}{2}}\right).$$

Let n is an even number. Similary to the case above it can be shomwn that at $t \to +\infty$

$$Q_{\ell,m}(t,x',y') = \frac{4(-\alpha)^m \left(i\frac{\pi\ell}{2a}\right)^{\frac{n}{2}+m-3} i^n \tau^{\frac{n}{2}-1}}{\pi(1-\alpha^2)} t^{-\frac{n}{2}}\Psi_\ell(t,\omega,m) + O\left(t^{-\frac{n}{2}-1}\right),$$

Where $\Psi_\ell(t,\omega,m)$ is defined by (26).

Taking into account the resuits obtained for $G_{\ell,m}$ and $Q_{\ell,m}$ in (15) introduced denotements we have

$$e^{-i\omega t}\mathcal{U}(t,x) = V(i\omega,x) + \Phi(t,x,\alpha,\omega) + t^{-\frac{n}{2}}\Psi(t,x_{n+1},\alpha,\omega) + O\left(t^{-\frac{n}{2}-1}\right)$$

where

$$\Phi(t,x,\alpha,\omega) = -\frac{ie^{-i\omega t}}{2(2\pi)^{\frac{n}{2}-1}}\sum_{\ell=1}^{\infty}\left[\left(\frac{\pi\ell}{2a}\right)^2\Phi_{\ell,o}(\alpha,\omega,t)\, a_\ell(x')\cos h(a-x_{n+1})\frac{\pi\ell}{2a} +\right.$$
$$\left. + \ \Phi_{\ell,2}(\alpha,\omega,t)\, b_\ell(x')\sin h(a-x_{n+1})\frac{\pi\ell}{2a} + \frac{\pi\ell}{2a}\Phi_{\ell,1}(\alpha,\omega,t)\, C_\ell(x')\right]$$

$\Psi(t,x_{n+1},\alpha,\omega)$ is bounded function of the all variables, where

$$a_\ell(x') = \frac{1}{2a}\left(i\frac{\pi\ell}{2a}\right)^{\frac{n}{2}-1}\int\limits_{\Omega}\dots\int \tau^{1-\frac{n}{2}} f(y) H^{(1)}_{\frac{n}{2}-1}\left(i\frac{\pi\ell}{2a}\tau\sqrt{1-\frac{1}{\alpha^2}}\right)\cos h(a-y_{n+1})\frac{\pi\ell}{2a}dy,$$
$$b_\ell(x') = \frac{1}{2a}\left(i\frac{\pi\ell}{2a}\sqrt{1-\frac{1}{\alpha^2}}\right)^{\frac{n}{2}-1}\int\limits_{\Omega}\dots\int \tau^{1-\frac{n}{2}} f(y) H^{(1)}_{\frac{n}{2}-1}\left(i\frac{\pi\ell}{2a}\tau\sqrt{1-\frac{1}{\alpha^2}}\right)\sin h(a-y_{n+1})\frac{\pi\ell}{2a}dy,$$
$$c_\ell(x') = a_\ell(x')\sin h(a-x_{n+1})\frac{\pi\ell}{2a} + b_\ell(x')\cos h(a-x_{n+1})\frac{\pi\ell}{2a}.$$

The tehorem is proved □

Consequence. From theorem 3 it follows that the solution of non-stationary problem (10)-(12) consists of limit amplitute and counting number of normal wave corresponding to continuons and point spectrum of stationary problem (1)-(2) at $t \to +\infty$.

Theorem 3 is true when $n \geq 3$ in resonance case $\omega = \frac{\pi\ell}{2a}$, $\omega = \frac{\pi\ell}{2a\alpha}, \ell = 0, \pm1, \pm2, \ldots$ too. In that case when $n < 3$ the solution of the boundary value problem (10)-(12) grows. We can obtain it from the following theorem.

Theorem 4. *Let $\omega = \frac{\pi\ell}{2a}$, $n = 2$ and α is a real number so that $|\alpha| \neq 1$. Then for the solution of the problem (10)-(12) the following estimation takes place*

$$|\mathcal{U}\,(t,x)| \leq Ce^{\varepsilon t}$$

uniformly with respect to x in every compact subset from $\prod$, $\varepsilon > 0$ is sufficiently small number

Theorem 5. *Let $n = 2$, $\omega = \frac{\pi\ell}{2a\alpha}(\ell = 0, \pm1, \pm2, \ldots)$ and α is a real number so that $|\alpha| \neq 1$. Then the asymptotics takes place for the solution of problem (10)- (12) when $t \to +\infty$*

$$e^{-i\frac{\pi\ell}{2a\alpha}t}\mathcal{U}\,(t,x) = \frac{it}{4\alpha^2}\left(\frac{\pi\ell}{2a}\right)^2\left[a_\ell(x')\sin h(a-x_3)\frac{\pi\ell}{2a} + b_\ell(x')\cos h(a-x_3)\frac{\pi\ell}{2a} + \right.$$
$$\left. + \frac{1}{i}C_\ell(x')\right] + O(1)$$

uniformly with respect to x in every compact subset of $\prod$.

When $n = 1$ in the paper [7] it is shown that in resonance case when $\omega = \frac{\pi\ell}{2a}$ the solution of the problem (10)-(12) grows like $t^{1/2}$.

3. A. G. Sveshnikov's partial conditions of radiation.

Now we derive the conditions of infinitary for the solution of homogeneous boundary value problem corresponding to the problem (1)-(2) and these conditions provide trivial solution. We denote

$$\mathcal{U}_\ell\,(\kappa, x') = \int_{-a}^{a} \mathcal{U}\,(\kappa, x)\, g_\ell\,(\kappa, x_{n+1})\, dx_{n+1}$$

Consider A. G. Sveshnikov's partial conditions of radiation when $x' \to \infty$

$$\left(\frac{\partial}{\partial|x'|} - i\kappa_\ell)\mathcal{U}_\ell\,(\kappa, x') = 0\left(|x'|^{\frac{1-n}{2}}\right),\right.$$
$$\ell = 0, 1, \ldots, \nu \tag{27}$$

where $\nu = \left[\frac{2a|\kappa|}{\pi}\right]$

The following theorem is true.

Theorem 6. *The solution of homogeneous boundary value problem corresponding the problem (1)-(2) which satisfies the conditions (28) on infinity* $\kappa \neq i\frac{\pi\ell}{2a\alpha}$ *(when* $n = 1, 2, \kappa \neq \frac{\pi\ell}{2a}, \alpha \neq i$*),* $\ell = 0, \pm 1, \pm 2, \ldots$ *is the only trivial solution.*

Proof. let $\prod_r$ be the lateral surface of circular straight cylinder with basises on planes $x_{n+1} = \pm a$. Basises are circles with radius r and centres in points $(x', -a)$, (x', a).

Green's formula is true for the solution of homogeneous boundary value problem corresponding to the problem (1)-(2)

$$\mathcal{U}(\kappa, x) = \underset{\prod_r}{\int \ldots \int} \left(G_1 \frac{\partial u}{\partial \tau} - \mathcal{U} \frac{\partial G_1}{\partial \tau} \right) d \prod_r,$$

where

$$\begin{aligned} G_1(\kappa, x, y) &= G(\kappa, x, y) + \mathcal{H}(\kappa, x, y), \\ \mathcal{H}(\kappa, x, y) &= -\frac{i\tau^{1-\frac{n}{2}}}{2(2\pi)^{\frac{n}{2}-1}} \sum_{\ell=0}^{\nu} \gamma_\ell g_\ell(\kappa, x_{n+1}) g_\ell(\kappa, y_{n+1}) J_{\frac{n}{2}+1}(\kappa_\ell \tau). \end{aligned}$$

is regular solution in $\prod_r$ of the same problem and $\nu = \left[\frac{2a|\kappa|}{\pi}\right]$, $\kappa_\circ = \sqrt{(1+\alpha^2)\kappa^2}$, $\kappa_\ell = \sqrt{\kappa^2 - \frac{\pi^2\ell^2}{4a^2}}$.

We denote

$$W_\ell(\tau) = -\frac{i}{4}\left(\frac{\kappa_\ell}{2\pi\tau}\right)^{\frac{n}{2}-1} \left[(\gamma_\ell + 1) H^{(1)}_{\frac{n}{2}-1}(\kappa_\ell \tau) + \gamma_\ell H^{(2)}_{\frac{n}{2}-1}(\kappa_\ell \tau) \right],$$

where γ_ℓ is constant, $\gamma_\ell = 0$ when $\ell > \nu$.

Series in expressions for G and $\frac{\partial G}{\partial \tau}$ converge uniformly with respect to τ ($\tau \geq C_0 > 0$) and so

$$\mathcal{U}(\kappa, x) = \sum_{\ell=0}^{\infty} g_\ell(\kappa, x_{n+1}) \int_{S_r} \left[W_\ell(\tau) \frac{\partial \mathcal{U}_\ell(\kappa, y')}{\partial \tau} - \mathcal{U}_\ell(\kappa, y') \frac{\partial W_\ell(\tau)}{\partial \tau} \right] ds_r, \tag{28}$$

S_r is a sphere with radius r and centre in point $(x', 0)$ on $\prod_r$.

The functions $U_\circ(\kappa, x)$ and $\mathcal{U}_\ell(\kappa, x'), \ell = 1, 2, \ldots$ in R_n satisfy equations correspondly

$$[\Delta_n + (\alpha^2+1)]\,\mathcal{U}_\circ(\kappa, x') = 0, \qquad (29)$$
$$(\Delta_n + \kappa_\ell^2)\,\mathcal{U}_\ell(\kappa, x') = 0,$$

where Δ_n is the Laplacian of x'. We fix point $x = \dot{x} \in \prod$. Since $\mathcal{U}_\ell(\kappa, y')$ satisfies the equations (29) and conditions (27) taking into account the results of [4] we obtain that $\gamma_\ell = 0$ $(\ell = 0, 1, \ldots \nu)$ and $\mathcal{U}_\ell(\kappa, y')$ are functions of the first category. The following estimation is true

$$\mathcal{U}_\ell(\kappa, y') = e^{i\kappa_\ell |y'|}0\left(|x|^{\frac{1-n}{2}}\right). \qquad (30)$$

Taking into account expansion of the solution of the problem and othonormality of system $g_\ell(\kappa, x_{n+1})$ we obtain

$$\mathcal{U}_\ell(\kappa\dot{x}') = \int_{S_r}\left(W_\ell(\tau)\frac{\partial \mathcal{U}_\ell(\kappa, y')}{\partial \tau} - \mathcal{U}_\ell(\kappa, y')\frac{\partial W_\ell(\tau)}{\partial \tau}\right) dS_r. \qquad (31)$$

Since $\gamma_\ell = 0,\ \ell = 0, 1, 2, \ldots$ then

$$W_\ell(\tau) = -\frac{i}{4}\left(\frac{\kappa_\ell}{2\pi\tau}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}(\kappa_\ell \tau). \qquad (32)$$

From asymptotics of the Hankel functions $H^{(1)}_{\frac{n}{2}-1}(Z)$ and its derivative for $Z \to \infty$ we obtain

$$\left(\frac{\partial}{\partial \tau} - i\kappa_\ell\right) W_\ell(\tau) = e^{i\kappa_\ell \tau} O\left(\tau^{-\frac{n+1}{2}}\right). \qquad (33)$$

Since for large

$$\frac{\partial \mathcal{U}_\ell(\kappa, y')}{\partial \tau} = \frac{\partial \mathcal{U}_\ell(\kappa, y')}{\partial |y'|}\left(1 + O\left(\frac{1}{|y'|}\right)\right)$$

in view of (31)-(33) we get

$$\mathcal{U}_\ell(\kappa, \dot{x}') = \int_{S_r}\left(W_\ell(\tau)\frac{\partial \mathcal{U}_\ell(\kappa, y')}{\partial |y'|} - \mathcal{U}_\ell(\kappa, y')\frac{\partial W_\ell(\tau)}{\partial \tau}\right) dS_r +$$
$$+ \int_{S_r} W_\ell(\tau)\frac{\partial \mathcal{U}_\ell(\kappa, y')}{\partial |y'|} O\left(\frac{1}{|y'|}\right) dS_r \equiv \mathcal{J}_1(\kappa, \dot{x}') + \mathcal{J}_2(\kappa, \dot{x}'). \qquad (34)$$

From (27), (30), (32), (33) it follows that

$$\begin{aligned}\mathcal{J}_1(\kappa, \dot{x}') &= \int_{S_r}\left[W_\ell(\tau)e^{i\kappa_\ell|y'|}O\left(|y'|^{\frac{1-n}{2}}\right) - \mathcal{U}_\ell(\kappa, y')e^{i\kappa_\ell\tau}O\left(\tau^{-\frac{n+1}{2}}\right)\right]dS_r = \\ &= \left[\kappa_\ell^{\frac{n-3}{2}}O(1) + O(\tau^{-1})\right]e^{2i\kappa_\ell\tau} \end{aligned} \tag{35}$$

Similarly from (27), (30), (32) we obtain

$$\begin{aligned}&\mathcal{J}_2(\kappa, \dot{x}') = \kappa_\ell^{\frac{n-3}{2}}\int_{S_r}\tau^{-\frac{n-1}{2}}e^{i\kappa_\ell\tau}O\left(\frac{1}{|y'|}\right)\left[O\left(|y'|^{\frac{1-n}{2}}\right) + e^{i\kappa_\ell\tau}O\left(|y'|^{\frac{1-n}{2}}\right)\right]dS_r = \\ &= \kappa_\ell^{\frac{n-3}{2}}\int_{S_r}\tau^{-\frac{n+1}{2}}\left[O\left(|y'|^{\frac{1-n}{2}}\right) + O\left(|y'|^{\frac{1-n}{2}}\right)\right]dS_r = O(\tau^{-1}).\end{aligned} \tag{36}$$

At $r \to \infty$ from (34)-(36) it follows that

$$\mathcal{U}_\ell(\kappa, \dot{x}') = 0, \quad \ell = 0, 1, \ldots, \nu. \tag{37}$$

Now consider $\mathcal{U}_\ell(\kappa, x')$ for $\ell = \nu + 1, \nu + 2, \ldots$. In this case $\lambda_\ell > \kappa^2$. Considering $\mathcal{U}_\ell(\kappa, x')$ as a distribution with respect to x' in space $C_0^\infty(R_n)$ and applaying Fourier transformation to equation (29) we obtain

$$\left[-|\xi'|^2 - \lambda_\ell + \kappa^2\right]\tilde{\mathcal{U}}_\ell(\kappa, \xi') = 0.$$

Since $-|\xi'|^2 - \lambda_\ell + \kappa^2 < 0$, we obtain $\tilde{\mathcal{U}}_\ell(\kappa, \xi') = 0$. Therefore

$$\mathcal{U}_\ell(\kappa, x') = 0, \quad \ell = \nu + 1, \nu + 2, \ldots \tag{38}$$

From (28), (31), (37), (38) it follows that

$$\mathcal{U}(\kappa, \dot{x}) = 0.$$

Since the point $\dot{x}$ is arbitrary the theorem is proved. □

Remark From (8) and the asymptotics of the Hankel functions it follows that the solution of the problem (1)-(2) chosen by the principle of the limit absorption satisfies the partial conditions of radiation (27).

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N BOYUTLU KATMANDE İMPEDANS SINIR KOŞULLARINDA HELMHOLTZ DENKLEMİ IŞİNLANMA PRENSİPLERİ

Özet

Makalede n boyutlu katmande impedans sınır koşullarında Helmholtz denklemi için ışınlanma prensipleri ispatlanmış ve rezonans fenomeni incelenmiştir.

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