Tr. J. of Mathematics
20 (1996) , 395 – 398.

ON STRONGLY REGULAR RINGS

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Abstract

Some characterizations of strongly regular rings will be given.

Let R be a ring and $I(\neq R)$ a right ideal of R. If, for each pair of right ideals A and B of R, $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$, then I is called a prime right ideal (or equivalently, if $aRb \not\subseteq I$ whenever a and b do not belong to I). I is strongly prime right ideal if, for each pair of a and b in R, $aIb \subseteq I$ and $ab \in I$ imply that either $a \in I$ or $b \in I$, and we call I a strongly semiprime right ideal whenever $aIa \subseteq I$ and $a^2 \in I$ imply that $a \in I$.

A strongly prime right ideal is trivially strongly semiprime, but the converse need not be true, as the following example shows:

Example Let

$$R = \begin{pmatrix} Z_2 & 0 & Z_2 \\ 0 & Z_2 & 0 \\ Z_2 & 0 & Z_2 \end{pmatrix} \text{ and } I = \begin{pmatrix} Z_2 & 0 & Z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where Z_2 is the ring of integers modulo 2. Then I is a strongly semiprime right ideal but not strongly prime.

Let $m(R)$ (resp. $sp(R), p(R)$) denote the set of all maximal (resp. strongly prime, prime) right ideals of R. Then it is known that

$$m(R) \subseteq sp(R) \subseteq p(R)$$

in s-unital rings [2].

In this note, we shall prove the following theorem.

Theorem . *The following are equivalent for any ring R:*

(1) R is strongly regular.

AMS subject classification (1991 revision): Primary: 16 E 50, Secondary: 16 D 15
Keywords: (Strongly, right weakly) regular ring,(strongly) prime right ideal. AC-ring.
The first author has been supported by TÜBİTAK, Ankara (TBAG-Ç2) and the second author by TÜBİTAK, (TBAG)

(2) R is regular and $p(R) \subseteq sp(R)$.

(3) R is right weakly regular and every element in $m(R)$ is a two-sided ideal.

(4) R is semiprime, $p(R) \subseteq sp(R)$, and R/P is regular for every completely prime ideal P of R.

(5) R is a right weakly regular, left s-unital AC-ring, and $p(R) \subseteq sp(R)$.

(6) Any right ideal (except R itself) is strongly semiprime.

Before proving the theorem, we need a lemma. We recall that a ring R is said to be right weakly regular if every right ideal of R is idempotent, and R is almost commutative (AC-ring) if for any prime ideal $P(\neq R)$ of R and $a \notin P$ there exists $x \in R$ such that ax is central but is not in P. For $I \in p(R)$, I^* will denote the largest ideal of R which is contained in I, and $r(a)$ will stand for the right annihilator of a for $a \in R$.

We note here that Proposition 2.1 in [4] remains true for rings without unity. So we can use it freely.

Lemma. *Let R be a ring satisfying $p(R) \subseteq sp(R)$ and let $I \in p(R)$.*

(i) If R is a right weakly regular ring, then R/I^* is a simple domain with unity.

(ii) If R is regular, then I is both a two-sided ideal and maximal one-sided ideal of R.

Proof. (i): In view of [4,Proposition 2.1], $I^* \in p(R)$, hence $I^* \in sp(R)$ by hypothesis, so the ring R/I^* has no nonzero divisor of zero since any ideal of a ring is strongly prime right ideal if and only if it is completely prime. On the other hand, $r \in rRrR$ for every $r \in R$ by [5,Proposition 1]. Thus, for every $\overline{0} \neq \overline{r} \in \overline{R} = R/I^*$ we have $\overline{r} = \overline{r}\ \overline{u}$, where $\overline{u}$ is in the ideal of $\overline{R}$ generated by $\overline{r}$. Now for every $\overline{x}$ in $\overline{R}$ we have $\overline{r}\ \overline{x} = \overline{r}\ \overline{u}\ \overline{x}$, which implies that $\overline{x} = \overline{u}\ \overline{x}$, that is, every nonzero ideal of $\overline{R}$ has a left unity of the ring $\overline{R}$. Therefore, $\overline{R}$ is a simple ring. As is well-known, every simple ring with a left unity has a unity.

(ii) We deduce from the regularity of R and from part (i) that $\overline{R}$ is a division ring, hence I^* is a maximal one-sided ideal of R. Thus $I^* = I$. □

A ring R is right (left) s-unital if $a \in aR (a \in Ra)$ for each $a \in R$, and is s-unital if it is both right and left s-unital.

We are now in a position to prove our theorem.

Proof of the Theorem. (1) $\Rightarrow$ (2) : R is a duo ring, hence if $I \in p(R)$, then I is a prime ideal of R, thus $I \in m(R) \subseteq sp(R)$.

(2) $\Rightarrow$ (3) : Clearly, R is right weakly regular, and every element in $m(R)$ is a two-sided ideal of R by the Lemma.

(3) $\Rightarrow$ (1) : We are going to show that $aR + r(a) = R$ for each $a \in R$. Suppose, to the contrary, that $bR + r(b) \underset{\neq}{\subset} R$ for some $b \in R$. Then, according to [7,Lemma 1], there exists a maximal right ideal I of R such that $bR + r(b) \subseteq I \underset{\neq}{\subset} R$. Also, since $b \in bRbR$

we can find $c \in R$ with $b = bc$, where $c \in RbR \subseteq I$. Let $d \in R \setminus I$. Since R is right s-unital, there exists $u \in R$ such that $bu = b$ and $du = d$ ([6,Theorem 1]). So $b(u - c) \doteq 0$ or $u - c \in r(b)$. Now $u = (u - c) + c$ implies that $u \in r(b) + I \subseteq I$, and since I is an ideal we get $d = du \in I$, a contradiction! Therefore $aR + r(a) = R$ for each $a \in R$. Finally, from the right s-unitality of R we can find $v = aa' + y$ in R with $y \in r(a)$ such that $a = av = a^2a'$.

(4) $\Rightarrow$ (1) : Let x be a nilpotent element with $x^n = 0 \in Q$ for any prime ideal Q of R. Then $x \in Q$ since Q is completely prime. Hence x is in the prime radical of R, which is zero. Thus R is reduced, and hence R is regular by [1, Corollary1.4].

(1) $\Rightarrow$ (5) : Reduced regular ring is an AC-ring and regular ring is a right weakly regular s-unital ring, and by the statement (2), $p(R) \subseteq sp(R)$.

(5) $\Rightarrow$ (3) : Let $I \in m(R)$. Then the s-unitality of R implies that $I \in p(R)$. By the Lemma, $\overline{R} = R/I^*$ is a simple domain with unity. Let $\overline{0} \neq \overline{a} \in \overline{R}$. Then $a \notin I^*$ and hence there exists $x \in R$ such that ax is central but is not in I^*. Thus, $\overline{a}\ \overline{x}\ \overline{R}$ is a nonzero two-sided ideal of $\overline{R}$, so $\overline{a}\ \overline{x}\ \overline{R} = \overline{R}$. But then $\overline{R} = \overline{a}\ \overline{x}\ \overline{R} \subseteq \overline{a}\ \overline{R} \subseteq \overline{R}$ so that $\overline{R} = \overline{a}\ \overline{R}$. Therefore, $\overline{R}$ is a division ring and I^* is a maximal right ideal of R, hence $I^* = I$.

(1) $\Rightarrow$ (6) : This follows from the fact that in a strongly regular ring a right ideal is strongly semiprime if and only if it is completely semi prime.

(6) $\Rightarrow$ (1) : For each $a \in R$, $a(a^2)_r a \subseteq (a^2)_r$ and $a^2 \in (a^2)_r$, where $(a^2)_r$ is the right ideal of R generated by a^2. By hypothesis $(a^2)_r$ is a strongly semiprime right ideal, so, for some integer m and b in R we have

$$a = ma^2 + a^2b = m(ma^3 + a^3b) + a^2b = a^2(m^2a + mab + b) = a^2x,$$

where $x = m^2a + mab + b$. Thus R is strongly regular.

The implication (2) $\Rightarrow$ (4) is trivial.

The following corollary will also provide a short proof of Theorem 1 in [3].

Corollary 1. *If a ring R satisfies one of the equivalent conditions of the Theorem, then R is a right V-ring.*

In [2] we proved that the regularity in reduced rings is equivalent to the condition that $sp(R) \subseteq m(R)$. In this respect, we have the following.

Corollary 2 . *Let R be a regular ring. Then R is reduced if and only if $p(R) \subseteq sp(R)$.*

Remark. The assumption in the third statement of the Theorem that R be right weakly regular cannot be weakened by taking the assumption that R be fully idempotent, for, there is an example of a fully idempotent ring that is not regular but in which every maximal right ideal is a two-sided ideal.(see [8]).

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KUVVETLİ REGÜLER HALKALAR ÜZERİNE

Özet

Kuvvetli regüler halkaların, kuvvetli asal sağ idealleri içeren bir karakterizasyonu verilmiştir.

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Received 16.2.1995