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ON HOMOGENEOUS RIEMANN BOUNDARY VALUE PROBLEM

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Abstract

In this work we consider homogeneous Riemann boundary value problem (BVP) on general open rectifiable curves. We prove certain estimates for Cauchy type integrals in terms of local moduli of continuity and local maximum of modulus, which allow us to describe the solvability of Riemann BVP with unbounded oscillating coefficients on a wide class of non-smooth rectifiable Jordan curves.

1. Introduction

Let γ be an open rectifiable curve in $\mathbb{C}$ with ends a_1, a_2, oriented from a_1 to $a_2 (\gamma = \overset{\smile}{a_1 a_2})$.

It is well known (see [3], [7]) that the main tool in investigating Riemann boundary value problem (BVP; see (1) below) is the Cauchy type integral

$$F_\varphi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\tau)}{\tau - z} d\tau, z \notin \gamma,$$

In recent years great attention has been paid to the consideration of Riemann BVP on non-smooth curves (see [2], [4], [8], [11]) and even on non-rectifiable curves (see [5], [12], [13]) under various assumptions about coefficients G, g in (1). In particular, in [6], [9], where further references can be found, the case of oscillating coefficients has been considered.

In this paper we use techniques from [11] and [9] to get estimates for Cauchy type integral that are used to solve Riemann BVP on a wide class of non-smooth open rectifiable Jordan curves. Our results intersect with the ones in [9], but we impose less restrictive conditions on the curve γ.

The class of functions $\phi(z)$, defined and holomorphic off γ, continuously extendable to $\gamma\backslash\{a_1, a_2\}$ from both sides of γ, and such that in the vicinity of endpoints they satisfy the estimates $|\phi(z)| \leq C|z - a_k|^{-\nu_k}, \nu_k < 1, k = 1, 2$ and $\phi(\infty) = 0$, will be denoted by $K(\gamma)$ and the functions in $K(\gamma)$ will be called picewise-holomorphic functions (PHF) with jump line γ. $\phi^{\pm}(t)$ will stand for the boundary values of $\phi(z)$ at a point t from left and from right correspondingly. With $B(\gamma)$ we shall denote the class of functions with properties as above except that they are just bounded off γ.

Now consider the problem of finding a function $\phi \in K(\gamma)$ (or $\phi \in B(\gamma)$), satisfying the boundary condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t), t \in \hat{\gamma} \stackrel{def}{=} \gamma \backslash \{a_1, a_2\}, \tag{1}$$

where G, g are given coefficients. Usually this problem is called Riemann boundary value problem (BVP).

If $g \equiv 0$ in (1) the problem is called homogeneous.

Following [10] let us introduce the characteristics

$$\begin{aligned} \theta_t(\delta) &= meas\{\xi \in \gamma : |t - \xi| \leq \delta\} = meas\gamma_\delta(t), t \in \gamma, \delta \geq 0, \\ \theta(\delta) &= \sup_{t\in\gamma}\theta_t(\delta), \delta \geq 0. \end{aligned}$$

It is evident that $\theta_t(\delta), \theta(\delta)$ are non-negative, non-decreasing functions of δ and that $\lim\limits_{\delta\to 0}\theta_t(\delta) = \lim\limits_{\delta\to 0}\theta(\delta) = 0$.

For a function $f : \gamma\backslash\{a_k\} \to \mathbb{C}$, which is bounded for each ξ on the set $\gamma\backslash\gamma_\xi(a_k)$, consider the following characteristics (cf. [1])

$$\begin{aligned} \Omega_f^{a_k}(\xi) &= \sup_{t\in\gamma\backslash\gamma_\xi(a_k)} |f(t)|, \qquad \xi > 0, \\ \omega_f^{a_k}(\delta, \xi) &= \sup_{\substack{t,\tau\in\gamma\backslash\gamma_\xi(a_k) \\ |t-\tau|\leq\delta}} |f(t) - f(\tau)|, \qquad \delta \geq 0, \quad \xi > 0. \end{aligned}$$

Functions $\Omega_f^{a_k}(\xi)$, $\omega_f^{a_k}(\delta, \xi)$ are non-increasing in $\xi, \omega_f^{a_k}(\delta, \xi)$ is non-decreasing in $\delta\omega_f^{a_k}(\delta, \xi) \leq 2\Omega_f^{a_k}(\xi)$ and the fact that there exists $\lim\limits_{\delta\to 0}\omega_f^{a_k}(\delta, \xi) = 0$ is equivalent to the continuity of f on $\gamma\backslash\{a_k\}$.

Denote with $L(\gamma)$ the class of Lebesque integrable functions on γ.

We write $f = 0(g)$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$ in the domain of f and g, and write $f \asymp g$, if $f = 0(g)$ and $g = 0(f)$.

2. Cauchy Type Integral

To solve Riemann BVP we first will need the following

Lemma *Let $\gamma = \overset{\smile}{a_1 a_2}$ be an open rectifiable Jordan curve and let $f : \gamma\backslash\{a_k\} \to \mathbb{C}$ be bounded for every $\xi > 0$ on $\gamma\backslash\gamma_\xi(a_k)$ and $f \in L(\gamma)$.*

Then for Cauchy type integral

$$F_f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\tau)}{\tau - z}d\tau, \quad z \notin \gamma,$$

the following estimate holds ($\epsilon = |z - a_k|$)

$$\begin{aligned} &|\frac{1}{2\pi i}\int_\gamma \frac{f(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{\gamma\setminus\gamma_\epsilon(a_k)} \frac{f(\tau)}{\tau}dt| \le \\ &\le C(\frac{1}{\epsilon}\int_0^\epsilon \omega_f^{a_k}(2\epsilon, x)d\theta(x) + \epsilon\int_{\epsilon/2}^d \frac{\omega_f^{a_k}(x,\epsilon/2)}{x^2}d\theta(x) + \\ &+ \int_0^{3\epsilon} \frac{\omega_f^{a_k}(x,\epsilon/2)}{x}d\theta(x) + \Omega_f^{a_k}(\epsilon/2)(\frac{\theta(2\epsilon)}{\epsilon} + \epsilon\int_\epsilon^d \frac{d\theta(x)}{x^2}), \end{aligned} \tag{2}$$

where the constant $C > 0$ *does not depend on* z, *and* $d = diam\gamma$.

Note that estimate (2) is meaningful only if

$$\int_0^{3\epsilon} \frac{\omega_f^{a_k}(x,\epsilon/2)}{x}d\theta(x) < \infty.$$

Corollary *If in addition* $\Omega_f^{a_k}(\xi) = 0(\ln\frac{1}{\xi})$, $\omega_f^{a_k}(\delta,\xi) = 0(\delta/\xi)$ *for* $\delta \le \xi$, $k = 1,2$, *and* $\theta(\delta) \asymp \delta$, *then*

$$|\frac{1}{2\pi i}\int_\gamma \frac{f(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{\gamma\setminus\gamma_\epsilon(a_k)} \frac{f(\tau)}{\tau}d\tau| \le C\ln\frac{1}{\epsilon}, \epsilon = |z - a_k|, \tag{3}$$

where $C > 0$ *is independent of* z.

Proof of Lemma Without loss of generality we may assume that $a_1 = 0$ and $a_2 = 1$. Now consider two cases.

1) Let $\rho(z) \overset{def}{=} dist(z,\gamma) \ge \epsilon/8, \epsilon = |z|$. Denote with τ_z any of the points in $\gamma \cap \sum_\epsilon(0)$ ($\sum_r(a) = \{\xi \in \mathbb{C} : |\xi - a| = r\}$) and write the representation

$$\begin{aligned} A &= \frac{1}{2\pi i}\int_\gamma \frac{f(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{\gamma\setminus\gamma_\epsilon(0)} \frac{f(\tau)}{\tau}d\tau = \\ &= \frac{1}{2\pi i}\int_{\gamma_\epsilon(0)} \frac{f(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{\gamma\setminus\gamma_\epsilon(0)} \frac{f(\tau) - f(\tau_z)}{\tau(\tau - z)}d\tau + \\ &+ \frac{f(\tau_z)}{2\pi i}(\int_\gamma \frac{d\tau}{\tau - z} - \int_{\gamma\setminus\gamma_\epsilon(0)} \frac{d\tau}{\tau}) \overset{def}{=} A_1 + A_2 + A_3, \end{aligned}$$

and estimate the terms A_1, A_2 and A_3.

Since in A_1 we have $|f(\tau)-f(\tau_z)| \le \omega_f^0(|\tau-\tau_z|, \min\{|\tau|,|\tau_z|\}) \le \omega_f^0(|\tau-\tau_z|,|\tau|)$, $|\tau - z| \ge dist(z,\gamma) \ge \epsilon\backslash 8$ and $|\tau-\tau_z| \le 2\epsilon$, it follows

$$|A_1| \le \frac{4}{\pi\epsilon}\int_{\gamma_\epsilon(0)} \omega_f^0(|\tau-\tau_z|,|\tau|)|d\tau| \le \frac{4}{\pi\epsilon}\int_{\gamma_\epsilon(0)} \omega_f^0(2\epsilon,|\tau|)|d\tau|.$$

Applying now lemma from [10] we get

$$|A_1| \le \frac{4}{\pi\epsilon}\int_0^\epsilon \omega_f^0(2\epsilon, x)d\theta(x).$$

Now estimate A_2. We have $|f(\tau)-f(\tau_z)| \le \omega_f^0(|\tau-\tau_z|, \min\{|\tau|,|\tau_z|\}) \le \omega_f^0(|\tau| + |\tau_z|,\epsilon) \le \omega_f^0(2|\tau|,\epsilon)$ and since $\rho(z) \ge \epsilon/8$ it follows that $|\tau| \le 9|\tau - z|$. Applying once again lemma from [10] we get

$$\begin{aligned} |A_2| &= \frac{\epsilon}{2\pi}\left|\int_{\gamma\backslash\gamma_\epsilon(0)} \frac{f(\tau)-f(\tau_z)}{\tau(\tau-z)}d\tau\right| \le \\ &\le \frac{9\epsilon}{2\pi}\int_{\gamma\backslash\gamma_\epsilon(0)} \frac{\omega_f^0(2|\tau|,\epsilon)}{|\tau|^2}|d\tau| \le \frac{9\epsilon}{2\pi}\int_\epsilon^d \frac{\omega_f^0(x,\epsilon/2)}{x^2}d\theta(x). \end{aligned}$$

To estimate A_3 write $A_3 = \frac{f(\tau_z)}{2\pi i}\beta$, where

$$\beta = \int_{\gamma_\epsilon(0)} \frac{d\tau}{\tau - z} - \int_{\gamma\backslash\gamma_\epsilon(0)} \left(\frac{1}{\tau-z} - \frac{1}{\tau}\right)d\tau = \beta_1 + \beta_2.$$

As above, evidently

$$|\beta_2| \le \frac{\epsilon}{2\pi}\left|\int_{\gamma\backslash\gamma_\epsilon(0)} \frac{d\tau}{\tau(\tau-z)}\right| \le \frac{9\epsilon}{2\pi}\int_{\gamma\backslash\gamma_\epsilon(0)} \frac{|d\tau|}{|\tau|^2} \le \int_\epsilon^d \frac{d\theta(x)}{x^2}.$$

Now estimate β_1. Denote with $\breve{Ob}$ the component of $\gamma_\epsilon(0)$, containing $a_1 = 0$ and with $\overline{Ob}$ the segment joining O to b. Then we have decomposition $\gamma_\epsilon(0) = (\cup_{j=1}^p \gamma_j) \cup \breve{Ob}$, where $p \le \infty$, arcs γ_j have both endpoints on $\sum_\epsilon(0)$ and do not intersect with each other. Denote with Γ_j the arcs of $\sum_\epsilon(0)$ with the same endpoints as γ_j and such that z is not contained in domain D_j, bounded by $\gamma_j \cup \Gamma_j$. Then, since $\rho(z) \ge \epsilon/8$, we have meas $\Gamma_j \le B$ meas γ_j with absolute constant B (meas E stands for the arclength measure on $\sum_\epsilon(0)$ or on γ). Then applying Cauchy theorem we obtain

$$\begin{aligned} |\beta_1| &= \left|\sum_j\left(-\int_{\Gamma_j}\frac{d\tau}{\tau-z}\right) + \int_{\breve{Ob}}\frac{d\tau}{\tau-z}\right| \le \frac{8}{\epsilon}\sum_j meas\Gamma_j + \left|\int_{\overline{Ob}}\frac{d\tau}{\tau-z}\right| \le \\ &\le \frac{8}{\epsilon}\sum_j B meas\gamma_j + \frac{8}{\epsilon}meas\overline{Ob} \le \frac{8}{\epsilon}(\theta(\epsilon)+\epsilon) \le \frac{9}{\epsilon}\theta(\epsilon). \end{aligned}$$

The obtained estimates prove (2) in first case.

2) Let now $\rho(z) < \epsilon/8$. Denote with x_z any of the points for which $|z - x_z| = \rho(z)$ and write

$$A = \frac{1}{2\pi i}\int_{\gamma_\epsilon(0)} \frac{f(\tau) - f(x_z)}{\tau - z} d\tau + \frac{z}{2\pi i}\int_{\gamma\setminus\gamma_\epsilon(0)} \frac{f(\tau) - f(x_z)}{\tau(\tau - z)} d\tau +$$
$$+ \frac{f(x_z)}{2\pi i}\Big(\int_\gamma \frac{d\tau}{\tau - z} \int_{\gamma\setminus\gamma_\epsilon(0)} \frac{d\tau}{\tau}\Big) \stackrel{def}{=} B_1 + B_2 + B_3.$$

In B_1 we have $|f(\tau) - f(x_z)| \leq \omega_f^0(|\tau - x_z|, \min\{|\tau|, |x_z|\})$ and $|\tau - x_z| \leq |\tau - z| + |z - x_z| \leq 2|\tau - z|$, so that

$$|B_1| \leq \frac{1}{\pi}\int_{\gamma_\epsilon(0)} \frac{\omega_f^0(|\tau - x_z], \min\{|\tau|, |x_z|\})}{|\tau - x_z|} |d\tau| = B_1' + B_1'',$$

where integration in B_1' is over $\gamma_{\epsilon/2}(0)$ and in B_1'' over the remaining part $\gamma_\epsilon(0)\setminus\gamma_{\epsilon/2}(0)$.

In B_1' we have $|x_z| \geq |z| - |x_z - z| \geq \frac{7}{8}\epsilon > \epsilon/2$, $|\tau - x_z| \leq |\tau| - |x_z - z| + |z| < \epsilon/2 + \epsilon/8 + \epsilon < 2\epsilon$, $|\tau - x_z| \geq \epsilon/2 - \epsilon/8 > \epsilon/4$. Therefore

$$B_1' \leq \frac{4}{\pi\epsilon}\int_{\gamma_{\epsilon/2}(0)} \omega_f^0(2\epsilon, \min\{|\tau|, \epsilon/2\})|d\tau| \leq$$
$$\frac{4}{\pi\epsilon}\int_{\gamma_{\epsilon/2}} \omega_f^0(2\epsilon, |\tau|)|d\tau| \leq \frac{4}{\pi\epsilon}\int_0^\epsilon \omega_f^0(2\epsilon, x)d\theta(x).$$

In B_1'' we have $|x_z| \geq \epsilon/2$, $|\tau - x_z| \leq |\tau| + |x_z - z| + |z| < 3\epsilon$ and so $\gamma_\epsilon(0)\setminus\gamma_{\epsilon/2}(0) \subset \gamma_{3\epsilon}(x_z)$. Then

$$B_1'' \leq \frac{1}{2\pi}\int_{\gamma_{3\epsilon}(x_z)} \frac{\omega_f^0(|\tau - x_z|, \epsilon/2)}{|\tau - x_z|}|d\tau| \leq$$
$$\leq \frac{1}{2\pi}\int_0^{3\epsilon} \frac{\omega_f^0(y, \epsilon/2)}{y} d\theta_{x_z}(y) \leq \frac{1}{2\pi}\int_0^{3\epsilon} \frac{\omega_f^0(y, \epsilon/2)}{y} d\theta(y).$$

Further,

$$|B_2| \leq \frac{\epsilon}{2\pi}\int_{\gamma\setminus\gamma_\epsilon(0)} \frac{\omega_f^0(|\tau - x_z|, \min\{|\tau|, |x_z|\})}{|\tau||\tau - z|}|d\tau| \leq$$
$$\leq \frac{\epsilon}{2\pi}\int_{\gamma\setminus\gamma_\epsilon(0)} \frac{\omega_f^0(|\tau - x_z|, \frac{7}{8}\epsilon)}{|\tau||\tau - z|}|d\tau| = B_2' + B_2'',$$

where the integration in B_2' is over $\gamma\backslash\gamma_{2\epsilon}(0)$ and in B_2'' - over $\gamma_{2\epsilon}(0)\backslash\gamma_\epsilon(0)$.

In B_2' we have $|\tau - x_z| \leq |\tau| + \frac{9}{8}\epsilon < 3\epsilon, |\tau - x_z| \geq |\tau| - |x_z| \geq \frac{7}{8}\epsilon$, so

$$B_2' \leq \frac{3\epsilon}{2\pi}\int_{\gamma\backslash\gamma_{2\epsilon}(0)} \frac{\omega_f^0(|\tau - x_z|, \frac{7}{8}\epsilon)}{|\tau - x_z|^2}|d\tau| \leq \frac{3\epsilon}{2\pi}\int_{\gamma/\gamma_{7/8^\epsilon}(x_z)} \frac{\omega_f^0(|\tau - x_z|, \frac{7}{8}\epsilon)}{|\tau - x_z|^2}|d\tau| \leq$$

$$\leq \frac{3\epsilon}{2\pi}\int_{7/8\epsilon}^{d} \frac{\omega_f^0(y, \frac{7}{8}\epsilon)}{y^2}d\theta_{x_z}(y) \leq \frac{3\epsilon}{2\pi}\int_{\epsilon/2}^{d} \frac{\omega_f^0(y, \epsilon/2)}{y^2}d\theta(y).$$

Analogously, taking into account that in B_2'' we have $|\tau| \geq \epsilon$ and $|\tau - x_z| < 3\epsilon$, we get

$$B_2'' \leq \frac{3\epsilon}{2\pi}\int_{\gamma_{2\epsilon}(0)\backslash\gamma_\epsilon(0)} \frac{\omega_f^0(|\tau - x_z|, \frac{7}{8}\epsilon)}{|\tau||\tau - x_z|}|d\tau| \leq \frac{1}{\pi}\int_{\gamma_{2\epsilon}(0)\backslash\gamma_\epsilon(0)} \frac{\omega_f^0(|\tau - x_z|, \frac{7}{8}\epsilon)}{|\tau - x_z|^2}|d\tau| \leq$$

$$\leq \frac{1}{\pi}\int_{\gamma_{3\epsilon}(x_z)} \frac{\omega_f^0(|\tau - x_z|, \epsilon/2)}{|\tau - x_z|}|d\tau| \leq \frac{1}{\pi}\int_0^{3\epsilon} \frac{\omega_f^0(y, \epsilon/2)}{y}d\theta(y).$$

To estimate B_3 write $B_3 = \frac{f(x_z)}{2\pi i}\sigma$, where

$$\sigma = \int_{\gamma_{2\epsilon}(0)} \frac{d\tau}{\tau - z} + z\int_{\gamma\backslash\gamma_{2\epsilon}(0)} \frac{d\tau}{\tau(\tau - z)} - \int_{\gamma_{2\epsilon}(0)\backslash\gamma_\epsilon(0)} \frac{d\tau}{\tau} = \sigma_1 + \sigma_2 + \sigma_3.$$

It is evident that

$$|\sigma_2| \leq 2\epsilon\int_{\gamma\backslash\gamma_{2\epsilon}(0)} \frac{|d\tau|}{|\tau|^2} \leq 2\epsilon\int_{2\epsilon}^{d} \frac{d\theta(x)}{x^2},$$

and

$$|\sigma_3| \leq \frac{1}{\epsilon}[\theta_0(2\epsilon) - \theta_0(\epsilon)] \leq \frac{\theta(2\epsilon)}{\epsilon}$$

To estimate σ_1 write

$$\sigma_1 = \int_{\gamma_{\epsilon/2}(x_z)} \frac{d\tau}{\tau - z} + \int_{\gamma_{2\epsilon}(0)\backslash\gamma_{\epsilon/2}(x_z)} \frac{d\tau}{\tau - z} = \sigma_1' + \sigma_1''.$$

In [11] it was proved that $|\sigma_1'| \leq C\frac{1}{\epsilon}meas\gamma_{\epsilon/2}(x_z)$ with absolute constant C, so $|\sigma_1'| \leq C\frac{1}{\epsilon}\theta(\epsilon/2)$.

In σ_1'' we have $|\tau - z| \leq |\tau| + |z| \leq 3\epsilon$, $|\tau - z| \geq |\tau - x_z| - |z - x_z| \geq \epsilon/2 - \epsilon/8 = \frac{3}{8}\epsilon$. Therefore

$$|\sigma_1''| \leq \frac{8}{3\epsilon}\int_{\gamma_{2\epsilon}(0)\setminus\gamma_{\epsilon/2}(x_z)} \leq \frac{8}{3\epsilon}\theta_0(2\epsilon) \leq \frac{8}{3\epsilon}\theta(2\epsilon).$$

Since $|f(x_z)| \leq \omega_f^{a_k}(|x_z|) \leq \Omega_f^{a_k}(\frac{7}{8}\epsilon) \leq \omega_f^{a_k}(\epsilon/2)$ the obtained estimates prove (2) in the second case as well and this completes the proof of lemma.

3. Homogeneous Riemann BVP

Now consider problem (1) with $g \equiv 0$ and $G(t) = \exp(2\pi i f(t))$, where $f \in L(\gamma)$. Introduce the quantities

$$\Delta_G^k = \lim_{\gamma \not\ni z \to a_k} \frac{1}{\ln|z-a_k|} Re(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau), \tag{4}$$

$$\Delta_f^k = \lim_{r\to 0}\frac{1}{\ln r} Re(\int_{\gamma\setminus\gamma_r(a_k)} \frac{f(\tau)}{\tau} d\tau), \tag{5}$$

$k = 1, 2$.

In case Δ_G^k is finite, let

$$æ_k = \begin{cases} \Delta_G^k, & \text{if } \Delta_G^k \in \mathbb{Z} \\ [\Delta_G^k] + 1, & \text{if } \Delta_G^k \notin \mathbb{Z}, k = 1, 2. \end{cases} \tag{6}$$

and

$$æ = æ_1 + æ_2.$$

Theorem. *Under the conditions of the above corollary the homogeneous Riemann BVP in $K(\gamma)$*

1) in case $\Delta_G^1 = -\infty$ or $\Delta_G^2 = -\infty$ has only trivial solution $\phi \equiv 0$;

2) in case when among the values Δ_G^k, k=1,2, there is $+\infty$ and there is not $-\infty$ has infinetely many linear independent solutions;

3) in case $\Delta_G^k \in \mathbb{R}$, k=1,2, has $\max\{æ, 0\}$ linear independent solutions. If $æ > 0$ one can take as such a system the system of functions $\chi(z), z\chi(z), \cdots, z^{æ-1}\chi(z)$, where

$$\chi(z), (z-a_1)^{-æ_1}(z-a_2)^{-æ_2}\exp(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau), z \in \mathbb{C}\setminus\gamma. \tag{7}$$

Proof. . For the sake of simplicity suppose now $a_1 = 0, a_2 = 1$ and $f(1) = 0$ (that is, $G(1) = 1$). The last assumption means that we can take into consideration only the behaviour at $a_1 = 0$, i.e. $\Delta_G^2 = 0$ and $æ_2 = 0$.

Now denote

$$Y(z) = \exp(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau), z \in \mathbb{C}\backslash\gamma.$$

It is evident that the assumptions of theorem imply that

$$\int_{\gamma\backslash\gamma_\epsilon(t)} \frac{f(\tau) - f(t)}{\tau - t} d\tau$$

converges as $\epsilon \to 0$ uniformly in t ranging over any compact $E \subset \gamma$, not containing endpoints a_1, a_2. Then by results in [14] the integral

$$\int_\gamma \frac{f(\tau)}{\tau - z} d\tau$$

is continuously extendable to $\hat{\gamma}$ from both sides of γ and the Plemelj-Sokhotsky formulae are true (see [3], [7]), which imply that Y satisfies boundary condition (1). It is obvious that $Y(z) \neq 0$ for all $z \in \mathbb{C}\backslash\gamma, Y^{\pm}(t) \neq 0$ for all $t \in \hat{\gamma}$.

Let $\phi(z)$ be one of the solutions to (1) in $K(\gamma)$. Then

$$\frac{\phi^+}{Y^+} = \frac{\phi^-}{Y^-} \text{ for all } t \in \hat{\gamma}.$$

Then the function $F(z) = \phi(z)/Y(z)$ can be extended to holomorphic in $\mathbb{C}\backslash\{a_1, a_2\}$ function and $F(\infty) = 0$.

Denote

$$\begin{aligned} A(z) &= Re(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau - \int_{\gamma\backslash\gamma_{|z|}(0)} \frac{f(\tau)}{\tau} d\tau), \\ B(z) &= Im(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau). \end{aligned}$$

Then one can write

$$\begin{aligned} F(z) &= \frac{\phi(z)}{Y(z)} = \phi(z) \exp(-Re \int_\gamma \frac{f(\tau)}{\tau - z} d\tau) \exp(iB(z)) = \\ &= \phi(z) \exp(iB(z)) \exp(-A(z)) \exp(-Re \int_{\gamma\backslash\gamma_{|z|}(0)} \frac{f(\tau)}{\tau} d\tau). \end{aligned}$$

It is not difficult to see that (cf. [9])

$$\Delta_f^k - const \leq \Delta_G^k \leq \Delta_f^k + const. \tag{8}$$

Consider following possible cases. Let $\Delta_f = \Delta_f^1, \Delta_G = \Delta_G^1$.
1) $\Delta_f = -\infty$ (or, equivalently, in view of (8), $\Delta_G = -\infty$). In this case write

$$\begin{aligned} F(z) &= \phi(z)\exp(iB(z))\exp(-A(z)) \times \\ &\times \exp(-\ln|z|(\frac{1}{\ln|z|} Re \int_{\gamma\setminus\gamma_{|z|}(0)} \frac{f(\tau)}{\tau} d\tau)). \end{aligned} \tag{9}$$

From the definition of Δ_f it follows that $\exists \{r_n\}_{n=1}^{\infty}\ \forall M > 0\ \exists n_M\ \forall n \geq n_M$ we have

$$\frac{1}{\ln r_n} Re(\int_{\gamma\setminus\gamma_{r_n}(0)} \frac{f(\tau)}{\tau} d\tau) \leq -M$$

and hence $\forall z, |z| = r_n$ we have

$$\begin{aligned} |F(z)| &\leq \frac{C}{|z|^{\nu}} \exp(C_A \ln \frac{1}{|z|}) \exp(\ln \frac{1}{|z|}(-M)) = \\ &= const \exp(-\nu \ln r_n + C_A \ln \frac{1}{r_n} - M \ln \frac{1}{r_n}) = \\ &= const \exp(\ln \frac{1}{r_n}(\nu + C_A - M)). \end{aligned} \tag{10}$$

Now put $M = \nu + C_A + 1$ in (10). Then $\forall z, |z| = r_n$ we have

$$|F(z)| \leq const \exp(-\ln \frac{1}{r_n})$$

and the righthand side tends to zero as $n \to \infty$. Then

$$\max_{|z|=r_n} |F(z)| \leq const \exp(-\ln \frac{1}{r_n}) \to 0, n \to \infty.$$

For $\forall \epsilon > 0\ \exists N_\epsilon\ \forall n \geq N_\epsilon$ const $\exp(-\ln \frac{1}{r_n}) < \epsilon$ and then on the boundary of the annulus $K_{r_n, r_{n+1}} = \{z : r_{n+1} \leq |z| < r_n\}$ we have $|F(z)| < \epsilon$. Then in the annulus itself we have the same bound for $|F|$. As $B_{r_{N_\epsilon}}(0) = \{z : |z| < r_{N_\epsilon}\} = \cup_{n=N_\epsilon}^{\infty} K_{r_n, r_{n+1}}$, we get that $|F| < \epsilon$ in $B_{r_{N_\epsilon}(0)}$ and this implies the existence of $\lim_{z\to 0} F(z) = 0$. This in turn means that 0 is a removable singularity of F, i.e. can be extended to a function holomorphic in $\mathbb{C}$. Since $F(\infty) = 0$ it follows that $F \equiv 0$. Then $\phi(z) = F(z)Y(z) \equiv 0$.

Thus, in case $\Delta_f = -\infty$ Rieman BVP does not have solutions other than $\phi \equiv 0$ (in particular, it does not have linear independent solutions).

2) $\Delta_f = +\infty$(or $\Delta_G = +\infty$). In this case we have

$$\lim_{r\to 0}\frac{1}{\ln r}Re(\int_{\gamma\setminus\gamma_r(0)}\frac{f(\tau)}{\tau}d\tau) = +\infty.$$

Then $\forall M > 0\ \exists r_M\ \forall r < r_M$ we have

$$\frac{1}{\ln r}Re(\int_{\gamma\setminus\gamma_r(0)}\frac{f(\tau)}{\tau}d\tau) \geq M.$$

Now take $M = n + C_A + 1$, where $n \in \mathbb{N}$ is fixed, and then define the functions $\phi_n(z) = z^{-n}Y(z), z \in \mathbb{C}\setminus\gamma$.

We have the following estimate

$$\begin{aligned}
|\phi_n(z)| &= |z^{-n}Y(z)| = |\exp(-n\ln z)\exp(A(z))\exp(iB(z)) \times \\
&\times \exp(Re(\int_{\gamma\setminus\gamma_{|z|}(0)}\frac{f(\tau)}{\tau}d\tau)| \leq \\
&\leq \exp(C_A\ln\frac{1}{|z|} - n\ln|z|)\exp(\ln|z|(\frac{1}{\ln|z|}Re\int_{\gamma\setminus\gamma_{|z|}(0)}\frac{f(\tau)}{\tau}d\tau)) \leq \\
&\leq \exp(\ln\frac{1}{|z|}(C_A + n))\exp(M\ln|z|) = \exp(-\ln\frac{1}{|z|}),
\end{aligned}$$

and the righthand side tends to zero as $z \to 0$. Thus, ϕ_n is a solution of (1). Therefore the system $\{\phi_n\}_{n=1}^{\infty}$ is infinite system of evidently linear independent solutions.

3) $\Delta_f \in \mathbb{R}$(or $\Delta_G \in \mathbb{R}$). Let us show that in this case function F can only have a pole at 0. From the definition of Δ_f it follows that $\forall\epsilon > 0\ \exists r_\epsilon\ \forall r < r_\epsilon$ we have

$$\frac{1}{\ln r}Re(\int_{\gamma\setminus\gamma_r(0)}\frac{f(\tau)}{\tau}d\tau) \geq \Delta_f - \epsilon, \tag{11}$$

and $\exists\{r_n\}_{n=1}^{\infty}\exists n_\epsilon\ \forall n \geq n_\epsilon$ we have

$$\frac{1}{\ln r_n}Re(\int_{\gamma\setminus\gamma_{r_n}(0)}\frac{f(\tau)}{\tau}d\tau) \leq \Delta_f + \epsilon, \tag{12}$$

From the estimate

$$\begin{aligned}
|F(z)| &\leq \frac{C}{|z|^\nu}|\exp(-A(z)\exp(iB(z)\exp(-Re(\int_{\gamma\setminus\gamma_r(0)}\frac{f(\tau)}{\tau}d\tau))| \leq \\
&\leq C\exp((C_A + \nu)\ln\frac{1}{|z|}\exp(\ln\frac{1}{|z|}(\frac{1}{\ln|z|}Re\int_{\gamma\setminus\gamma_{|z|}(0)}\frac{f(\tau)}{\tau}d\tau))
\end{aligned}$$

for all $z \in \mathbb{C}\setminus\gamma$ and such that $|z| = r_n$ we get

$$|F(z)| \leq C \exp((C_A + \nu + \Delta_f + \epsilon) \ln \frac{1}{r_n}) = C \frac{1}{r_n^{C_A+\nu+\Delta_f+\epsilon}}. \tag{13}$$

From (13) it follows that $F(z) = z^m P(z)$ for some $m \in \mathbb{Z}$ and $P(z)$ is a polynomial such that $P(0) \neq 0$.

It is evident that $\chi(z)$ [7] satisfies homogeneous boundary condition (1) and $\chi(z) \neq 0$ for $z \in \mathbb{C} \backslash \gamma, \chi^{\pm}(t) \neq 0$ for $t \in \hat{\gamma}$.Using the definition of Δ_G for any $\epsilon > 0$ and z close enough to 0 and off γ we have

$$\frac{1}{\ln |z|} Re(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau) \geq \Delta_G - \epsilon$$

and then for small enough ϵ

$$\begin{aligned} |\chi(z)| &= |\exp(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau) z^{-æ}| = \\ &= |\exp(-\ln \frac{1}{|z|}(\frac{1}{\ln |z|} Re(\int_\gamma \frac{f(\tau)}{\tau - z} d\tau)) z^{-æ}| \leq \exp(\ln \frac{1}{|z|}(æ - \Delta_G + \epsilon)) \leq \\ &\leq \exp((\mu + \epsilon) \ln \frac{1}{|z|}) = \frac{1}{|z|^{\mu+\epsilon}}, \end{aligned}$$

where

$$\mu = \begin{cases} 0, \text{ if } \Delta_G \in \mathbb{Z}, \\ 1 - \{\Delta_G\}, \text{ if } \Delta_G \notin \mathbb{Z}, \end{cases}$$

and we may assume that $\mu + \epsilon < 1$.

Let us show now that any solution of homogeneous BVP has the form

$$\phi(z) = \chi(z) P_{æ-1}, z \in \mathbb{C} \backslash \gamma, \tag{14}$$

and vice versa, where $P_{æ-1}$ is a polynomial of degree at most $æ - 1$ (the polynomials of negative degree are considered to be $\equiv 0$).

The inverse statement is evident. Let us prove the direct one. From the definition of Δ_G it follows that for some sequence $\{z_n\}_{n=1}^{\infty}$, tending to 0, for any $\epsilon > 0$ and large enough n we have

$$\frac{1}{\ln |z_n|} Re(\int_\gamma \frac{f(\tau)}{\tau - z_n} d\tau) \leq \Delta_G + \epsilon.$$

Since, as it was shown above, any solution has the form $\phi(z) = Y(z) z^m P(z), m \in \mathbb{Z}, P(0) \neq 0$, it follows that for large enough n

$$\frac{C}{|z_n|^\nu} \geq |\phi(z_n)| \geq C_p \exp(-\ln \frac{1}{|z_n|}(\frac{1}{\ln|z|}Re(\int_\gamma \frac{f(\tau)}{\tau - z_n}d\tau))|z_n|^m \geq$$
$$\geq C_p \exp(-\ln \frac{1}{|z_n|}(\Delta_G + \epsilon + m)) = \frac{C_p}{|z_n|^{-\Delta_G - \epsilon - m}}.$$

Comparing the degrees on both sides we get that $\Delta_G + \epsilon + m + \nu > 0$ or $m > -\Delta_G - \epsilon - \nu$ which implies $m \geq -\Delta_G - \nu > -\Delta_G$. From the definition of æ it follows that $\Delta_G \leq$ æ. Thus, $m >$ −æ. Then

$$\phi(z) = Y(z)z^m P(z) = Y(z)z^{-æ}(z^{æ+m}P(z)) = \chi(z)Q(z),$$

where $Q(z)$ is a polynomial. As $\phi(\infty) = 0$ and $|\chi(z)Q(z)| \sim |z|^{-æ+k}$, where k is the degree of Q, it follows that $k \leq$ æ − 1. This proves (14) and completes the proof of theorem.

Analogous theorem is true also for the class $B(\gamma)$. □

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HOMOJEN RİEMANN SINIR-DEĞER PROBLEMİ HAKKINDA

Özet

Bu çalışmada genel açık düzlendirilebilen eğriler üzerinde homojen Riemann sınır-değer problemini araştırıyoruz. Düzgün olmayan Jordan eğrilerinin geniş bir sınıfında, sınırsız salınım yapan katsayı fonksiyonu olduğu durumda Riemann sınır-değer probleminin çözülebilirliğine imkan veren lokal süreklilik modülü ve lokal maksimum modülleri cinsinden Cauchy tipli integraller için bazı tahminler ispat ediyoruz.

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