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FINITE DIMENSIONAL ATTRACTORS FOR A CLASS OF SEMILINEAR WAVE EQUATIONS

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Abstract

In this paper we give a self-contained survey of results related with the global attractors for a class of nonlinear wave equations with damping or viscosity terms. In particular, we prove the existence of a finite dimensional attractor and estimate its fractal dimension by imbedding it in an exponential attractor. Some results on global stability, existence of finite dimensional attractors were already partially discussed in Kalantarov [44] and in Eden et. al. [25], however we simplify the framework by introducing a unified approach to both the existence of attractors through α-contractions and the construction of exponential attractors via some Lipschitzianity condition of the non-linear operator.

1. Introduction

We consider nonlinear damped or viscous wave equations that can be realized as abstract evolution equation of the form

$$Pu_{tt} + Qu_t + Au + F[u] = h,$$

with initial conditions

$$u_t(0) = u_1 \text{ and } u(0) = u_0,$$

where u_1 and u_0 come from an appropriate Hilbert space H. Under some natural conditions on the linear operators P, Q, and A and on the nonlinear term F, the existence of a dynamical system can be established. In particular, we show that by the uniqueness and existence result, Theorem 1, one can define a continuous semigroup of operators $S(t)$ from an appropriate Hilbert space X into itself, satisfying the semigroup condition $S(T+s) = S(t)S(s)$. Also in the first section we prove the existence of an absorbing set B in X. In the process we define both a Lyapunov like functional $L(t)$ and an Energy function $E(t)$, that is equivalent to the usual norm of the Hilbert space X that reveals the dissipative structure of the underlying evolution equation. The energy function will be used throughout the rest of the paper when we show the existence of a global attractor by the theory of α-contractions, i.e. Hale's approach [37]. Then the existence of an exponential attractor is established via the discrete squeezing property that was introduced in [32] and utilized in the construction of exponential attractor in [28], where we also utilize a recent result for the existence of exponential attractors for α-contractions (see [23]). Since the same energy function is used uniformly this approach simplifies a similar one introduced in [25] and in [26] for a more restricted class of equations. In the second section, in addition to the uniform approach that we bring to these problems, we also show the existence of exponential attractors for all of the examples given in the last section. In particular the existence of an exponential attractor for the sin-Gordon equation in any space dimension and for the modified von Karman equations are new results. In order to obtain these results we also use basic imbedding theorems and interpolation theory as applied to Sobolev and Besov spaces. Since the latter results are not all easily accesible, we also give the statements of the required results. In the final section, we discuss some equations for which the theory is applicable.

2. Existence and Uniqueness of Solutions: The Absorbing Set

Consider the following abstract Cauchy problem on a separable Hilbert space H:

$$Pu''(t) + Qu'(t) + Au(t) + F(u(t)) = h \qquad (u' = u_t) \tag{2.1}$$

$$u(0) = u_0, u'(0) = u_1. \tag{2.2}$$

Here A, P and Q are linear (not necessarily bounded) positive self-adjoint operators on H; the inverse of A is compact and the domains of these operators satisfy:

$$D(A) \subseteq D(Q) \subseteq D(P).$$

There exist positive constants α, β, γ such that

$$\alpha \parallel u \parallel \leq \parallel Q^{1/2}u \parallel \leq \beta \parallel A^{1/2}u \parallel \text{ for all } u \in D(A^{1/2}) \tag{2.3}$$

$$\parallel P^{1/2}u \parallel \leq \lambda \parallel Q^{1/2}u \parallel \text{ for all } u \in D(Q^{1/2}) \tag{2.4}$$

We set $D_1 = D(P^{1/2}), D_2 = D(Q^{1/2})$ and denote the dual spaces with respect to the inner product in H by D_{-1}, D_{-2} and D_{-3}, respectively. For notational ease we will use the bracket (u,v) both for the inner product in H and the duality pairing between the elements of D_i nad D_{-i} where $i = 1,2,3$. F is a nonlinear operator that is defined on D_3 and that maps D_3 into D_{-2} in a continuous and bounded fashion. We also assume that $F(.)$ is a gradient of some functional $G(.) : D_3 \rightarrow R$ and there exists $C \geq 0$ such that

$$(F(u),u) - G(u) \geq -C, G(u) \geq -C \text{ for all } u\ inD_3. \tag{2.5}$$

Finally, we assume that h is an element of D_{-2}. Let X denote the Hilbert space $D_3 \times D_1$. We define the inner product on X in the usual way by

$$((u,v),(w,z)) = (A^{1/2}u, A^{1/2}w) + (P^{1/2}v, P^{1/2}z).$$

In the third section we will also need an assumption on the relation between P, Q and $A : PA = AP; QA = AQ$ on $D(A)$.

A) Existence of Absorbing Set:

For the moment we will assume that the Cauchy problem admits a unique solution for initial data in X. Later on, we will utilize the apriori estimates obtained in the process to deduce the existence and uniqueness of solution.

Theorem 1. *Suppose that the operators A, P, and Q and the non-linear function $F(\cdot)$ satisfy all the properties cited at the beginning, especially those given in (2.3), (2.4) and (2.5), and further assume that the problem (2.1) and (2.2) with initial values in X has a unique weak solution (see definition 3 below) $u(t)$, $t > 0$ with the properties*

$$u(\cdot) \in C(R^+; D_3), u'(\cdot) \in C(R^+; D_1) \cap L_{2,loc}(R^+; D_2)$$

and for a.e. $t > 0$,

$$\begin{aligned} \frac{d}{dt}L(y(t)) \equiv & \frac{d}{dt}[\frac{1}{2} \parallel P^{1/2}u'(t) \parallel^2 + \frac{1}{2} \parallel A^{1/2}u(t) \parallel^2 + G(u(t)) - (h, u(t))] = \\ & - \parallel Q^{1/2}u'(t) \parallel^2, \end{aligned} \tag{2.6}$$

where $y(t) = (u(t), (u'(t)))$. Then the semigroup $S(t), t > 0$ (defined by $S(t)(u_0, u_1) = (u(t), u'(t))$) generated by the problem (2.1) and (2.2) is bounded dissipative.

Proof. Let us rewrite the equation (2.1) in an equivalent from as the system

$$u' = v \text{ and } Pv' = -Au - Qv - F(u) + h, \tag{2.7}$$

and consider the Lyapunov like functional defined on X by

$$\begin{aligned} \phi_\eta(w,z) = & \frac{1}{2} \parallel P^{1/2}z \parallel^2 + \frac{1}{2} \parallel A^{1/2}w \parallel^2 + G(w) - (h,w) \\ & + \eta(Pw, z) + \frac{\eta}{2} \parallel Q^{1/2}w \parallel^2, \end{aligned} \tag{2.8}$$

with $(w,z) \epsilon X$, where $\eta > 0$. Let $(u(t), v(t))$ be the solution of the system (2.7). Then

$$\begin{aligned}\frac{d}{dt}\phi_\eta(u(t),v(t)) &= (Pv'(t),v(t))+(Au(t),v(t))+(F(u(t)),v(t))-(h,v(t))\\&+\ \eta(Pv(t),v(t))+\eta(Pv'(t),u(t))+\eta(Qv(t),u(t))\\&= (Pv'(t)+F(u(t))-h,v(t))+\eta \parallel P^{1/2}v(t)\parallel^2\\&+\ \eta(Pv'(t)+Qv(t),u(t)).\end{aligned}$$

From this relation and (2.7) we deduce that

$$\begin{aligned}\frac{d}{dt}\phi_\eta(u(t),v(t)) &= -\parallel Q^{1/2}v(t)\parallel^2+\eta\parallel P^{1/2}v(t)\parallel^2-\eta\parallel A^{1/2}u(t)\parallel^2\\&-\ \eta(F(u(t)),u(t))+\eta(h,u(t)).\end{aligned} \tag{2.9}$$

Now from (2.9), for some $\delta, 0<\delta<\eta$, we have:

$$\begin{aligned}\frac{d}{dt}\phi_\eta(u(t),v(t))+\delta\phi_\eta(u(t),v(t)) &= -\parallel Q^{1/2}v(t)\parallel^2+(\eta+\frac{\delta}{2})\parallel P^{1/2}v(t)\parallel^2+\\&+\ (-\eta+\frac{\delta}{2})\parallel A^{1/2}u(t)\parallel^2-\eta(F(u(t)),u(t))\\&+\ (\eta-\delta)(h,u(t))+\delta G(u(t)+\eta\delta(Pu(t),v(t))\\&+\ \frac{\delta\eta}{2}\parallel Q^{1/2}u(t)\parallel^2.\end{aligned}$$

Due to conditions (2.3), (2.4) and (2.5), it follows from the last equality that

$$\begin{aligned}&\frac{d}{dt}\phi_\eta(u(t),v(t))+\delta\phi_\eta(u(t),v(t))\le-\parallel Q^{1/2}v(t)\parallel^2+(\eta+\frac{\delta}{2})\lambda^2\parallel Q^{1/2}v(t)\parallel^2\\&+\ (-\eta+\frac{\delta}{2})\parallel A^{1/2}u(t)\parallel^2+\delta[G(u(t))-(F(u(t)),u(t))]+(\delta-\eta)(F(u(t),u(t))\\&+\ (\eta-\delta)\parallel Q^{-1/2}h\parallel\parallel Q^{1/2}u(t)\parallel+\eta\delta\parallel P^{1/2}u(t)\parallel\parallel P^{1/2}v(t)\parallel+\frac{\delta\eta}{2}\parallel Q^{1/2}u(t)\parallel^2\\&\le-\parallel Q^{1/2}v(t)\parallel^2+(\eta+\frac{\delta}{2})\lambda^2\parallel Q^{1/2}v(t)\parallel^2+(-\eta+\frac{\delta}{2})\parallel A^{1/2}u(t)\parallel^2\\&+\ \eta C+\beta(\eta-\delta)\parallel Q^{-1/2}h\parallel\parallel A^{1/2}u(t)\parallel\\&+\ \eta\delta\lambda^2\beta\parallel A^{1/2}u(t)\parallel\parallel Q^{1/2}v(t)\parallel\\&+\ \frac{\delta\eta\beta^2}{2}\parallel A^{1/2}u(t)\parallel^2\le(-1+(\eta+\frac{\delta}{2})\lambda^2+\frac{\beta\eta^2\lambda^2}{2})\parallel Q^{1/2}v(t)\parallel^2\\&+\ (-\eta+\frac{\delta}{2}+\frac{\delta^2\lambda^2\beta}{2}+\frac{\delta\eta\beta^2}{2}+\frac{\eta}{2})\parallel A^{1/2}u(t)\parallel^2+\eta C\\&+\ \frac{\beta^2(\eta-\delta)^2}{2\eta}\parallel Q^{-1/2}h\parallel^2.\end{aligned} \tag{2.10}$$

Choosing the numbers δ and η sufficiently small we get that

$$\frac{d}{dt}\phi_\eta(u(t),v(t)) + \delta\phi_\eta(u(t),v(t)) \leq K \tag{2.11}$$

where K depends only on C, λ, β and $\| Q^{-1/2}h \|^2$. From (2.11), it follows from the standard arguments that there exists an absorbing ball B in X. Thus the semigroup $S(t), t > 0$ is bounded dissipative. □

B) Weak Solution. Existence and Uniqueness:

Definition 1. *$W^1(a,b)$ is the Hilbert space of vector-valued functions $\nu(t)$ such that $\nu \epsilon L^2(a,b;D_3), \nu' \epsilon L^2(a,b;D_1)$ furnished with the inner product $((,))$ defined by*

$$((u,\nu)) = \int_a^b \{(P^{1/2}u'(t), P^{1/2}\nu'(t)) + (A^{1/2}u(t), A^{1/2}\nu(t))\}dt$$

Definition 2 *$W^{1,0}(O,T)$ is the set of functions ν such that $\nu \epsilon W^1(O,T)$ and $\nu(T) = 0$.*

Definition 3. *If $u \epsilon W^1(O,T), u(O) = u_0$ and for every $\eta \epsilon W^{1,0}(O,T)$ the following equation is satisfied*

$$\begin{aligned}\int_0^T \{-(P^{1/2}u'(t), P^{1/2}\eta'(t)) &- (Q^{1/2}u(t), Q^{1/2}\eta'(t)) + (A^{1/2}u(t), A^{1/2}\eta(t)) \\ &+ (F(u(t)),\eta(t))\}dt = (h,\eta(t)) + (P^{1/2}u_1, P^{1/2}\eta(0)) \\ &+ (Q^{1/2}u_0, Q^{1/2}\eta(0)).\end{aligned} \tag{2.12}$$

then we will say that $u(t)$ is a weak solution of the problem (2.1) and (2.2).

Since we have assumed that the inverse of A is compact it follows that the spaces $D(A^\gamma)$ defined in the usual way are compactly imbedded in D_3, whenever $\gamma < 1/2$. Henceforth, we will use the following short notation for various $D(A^\gamma$ norms:

$$\| u \|_\gamma = D(A^\gamma) \text{ norm of } u.$$

Theorem 2. *Let the operators P, Q, A and F(.) satisfy the conditions (3), (4) and (5). Suppose also that*

$$\| Q^{-1/2}(F(u) - F(\nu)) \| \leq M(\| A^{1/2}u \|, \| A^{1/2}\nu \|) \| u - \nu \|_\gamma \text{ for all } u,\nu \in D_3, \tag{V}$$

where $M(\cdot,\cdot) : R^2 \to R^+$ is a continuous function, and $\gamma < 1/2$. Then the problem (2.1) and (2.2) with u_0 in D_3, u_1 in D_1 has a unique weak solution in the sense of definition 3.

Proof. Let $w_1, w_2, \cdots, w_n, \cdots$ be a complete system of linearly independent elements of D_3, e.g. the complete system of eigenvalues of A. We define an approximate solution of the problem (2.1), (2.2) in the form

$$\sum_{j=1}^{n} c_j(t)w_j = u_m(t)$$

as the solution of the following problem

$$(P^{1/2}u_m''(t), P^{1/2}w_j) + (Q^{1/2}u_m'(t), Q^{1/2}w_j) + (A^{1/2}u_m(t), A^{1/2}w_j) + (F(u_m(t), w_j) = (h, w_j), j = 1, 2, \cdots, m, \tag{2.13}$$

$$u_m(0) = u_{0m} \text{ and } u_m'(0) = u_{1m} \tag{2.14}$$

where u_{0m} (respectively u_{1m}) is the projection in D_3 of u_0 (respectively u_1) onto the n-dimensional space spanned by $w_1, w_2, \cdots, w_n$. According to the Peano's existence theorem for ordinary differential equations this problem is locally solvable. In order to obtain solutions for all finite time, we deduce an a priori estimate on the solutions by multiplying (2.13) by $c_j'(t)$ and summing up from $j = 1$ to $j = m$, we get

$$\frac{d}{dt}\{\frac{1}{2} \| P^{1/2}u_m'(t) \|^2 + \frac{1}{2} \| A^{1/2}u_m(t) \|^2 + G(u_m(t))\} + \| Q^{1/2}u_m'(t) \|^2 = (h, u_m'(t)) \leq \frac{1}{2} \| Q^{-1/2}h \|^2 + \frac{1}{2} \| Q^{1/2}u_m'(t) \|^2 . \tag{2.15}$$

It follows that

$$\| P^{1/2}u_m'(t) \|^2 + \| A^{1/2}u_m(t) \|^2 \leq C_T. \tag{2.16}$$

The last inequality implies that the sequence of approximate solutions $\{u_m(t)\}$ is bounded in the Hilbert space $W^1(0,T)$. Therefore there exists a weakly convergent subsequence which converges to an element u of $W^1(0,T)$ in the norm of $W^1(0,T)$. It is clear that this subsequence will also converge in $W^1(a,b)$ whenever $0 < a < b < T$. Without loss of generality using the same indices for the subsequence, for any a, b such that $0 < a < b < T$, we have

$$\int_a^b (\| P^{1/2}u'(t) \|^2 + \| A^{1/2}u(t) \|^2)dt \leq$$

$$\liminf_{k\to\infty}\int_a^b \{\| P^{1/2}u_m'(t) \|^2 + \| A^{1/2}u_m(t) \|^2\}dt \le C_T(b-a) \tag{2.17}$$

since a and b was arbitrary it follows that for almost all t, $0 < t < T$

$$\| P^{1/2}u'(t) \|^2 + \| A^{1/2}u(t) \|^2 \le C_T, \tag{2.18}$$

that is,

$$u' \epsilon L^\infty(0,T;D_1), u \epsilon L^\infty(0,T;D_3). \tag{2.19}$$

Now let us prove that for each $\eta \epsilon L^2(0,T;D_2)$,

$$\lim_{m\to\infty}\int_0^T (F(u_m(t)),\eta(t))dt = \int_0^T (F(u(t),\eta(t))dt. \tag{2.20}$$

Due to the condition (V) of the Theorem 2 and (2.16), (2.18) we have

$$\begin{aligned}
&\int_O^T (F(u_m(t)) - F(u(t)),\eta(t))dt \\
&\le \int_O^T \| Q^{-1/2}(F(u_m(t)) - F(u(t))) - F(u(t))) \| \| Q^{1/2}\eta(t) \| \, dt \\
&\le M_1\{\int_O^T \| u_m(t) - u(t) \|_\gamma^2 \, dt\}^{1/2}\{\int_O^T \| Q^{1/2}\eta(t) \|^2 \, dt\}^{1/2}
\end{aligned}$$

where the condition (V) on the nonlinearity is used. The space $W^1(0,T)$ is compactly imbedded into $L^2((0,T;D(A^\gamma))$ (see [51] and [52]), therefore

$$\int_O^T \| u_m(t) - u(t) \|_\gamma^2 \, dt \to 0, \text{ as } m \to \infty.$$

Hence (2.20) is valid. Now using the standard scheme we can easily prove that $u(t)$ is a weak solution of (2.1), (2.2). The condition (V) also implies the uniqueness of solutions as we will see in the next section. □

3. α-Contractions and Exponential Attractors

We impose another Lipschitz condition on the nonlinearity F in the form

$$\| Q^{-1/2}(F(u) - F(v)) \|_{\gamma} \leq M(\| A^{1/2}u \|, \| A^{1/2}v \|) \| A^{1/2}(u - v) \|$$
$$\text{for all } u, v \in D(A^{1/2}), \tag{W}$$

where $M(\cdot,\cdot)$ is a continuous function of its vatiables and γ is positive but very smal in the applications, see the last theorem of this section. This condition on the nonlinearity will be only needed on the absorbing set B, i.e. when u and v ranges over a bounded subset of either $D(A^{1/2})$ or, if needed, a more regular space, see the applications section. Let $u(t)$ and $v(t)$ be two solutions of (2.1) and (2.2) with respective initial values, we set

$$w(t) = u(t) - v(t) \text{ for } t > 0, \tag{3.1}$$

then w satisfies the evolution equation

$$Pw_{tt} + Qw_t + Aw + F[u] - F[v] = 0, \tag{3:2}$$

In H we take the inner product of (3.2) with the function $2(w_t + hw)$, after some simplification we obtain

$$\frac{d}{dt}E^2(w_t, w) + 2\eta \| A^{1/2}w \| + 2(\| Q^{1/2}w_t \|^2 - \eta \| P^{1/2}w_t \|^2) \leq 2(\Gamma, w_t + \eta w), \tag{3.3}$$

here the energy functional E is defined on X by

$$E^2(w_t, w) = \| P^{1/2}w_t \|^2 + \| A^{1/2}w \|^2 + \| Q^{1/2}w \|^2 + 2(Pw_t, w). \tag{3.4}$$

It follows from the assumptions (2.3) and (2.4) that if η is chosen so that

$$\eta\lambda\beta < 1/2, \tag{3.5}$$

then the energy functional defined by (3.4) is equivalent to the usual norm on X, more precisely,

$$(1/2) \| (w_t, w) \|^2 \leq E^2(w_t, w) \leq \mu\lambda \| (w_t, w) \| 2 \tag{3.6}$$

where $\mu = \max\{3/2, \beta\}$. The strategy is to further restrict the size of the parameter η to ensure the dissipativity of the energy and then by utilizing the equivalence in (3.6) establish the a-contractivity of the solution semigroup $S(t)$. The term in the parenthesis in (3.3) can be estimated from below by (2.3) since for

$$\eta\lambda^2 < 1/2, \tag{3.7}$$

it follows that

$$\| Q^{1/2}w_t \|^2 - \eta \| P^{1/2}w_t \|^2 > (1/2) \| Q^{1/2}w_t \|^2 \geq (1/2\lambda^2/ \| P^{1/2}w_t \|^2 > \eta \| P^{1/2}w_t \|^2 . \tag{3.8}$$

Hence the inequation in (3.3) results in the inequalities

$$\frac{d}{dt}E^2(w_t, w) + 2\eta \| A^{1/2}w \| + (1/2) \| Q^{1/2}w_t \| \leq 2|(\Gamma, w_t + \eta w)| \tag{3.9}$$

$$\frac{d}{dt}E^2(w_t, w) + 2\eta E^2(w_t, w) \leq 2|(\Gamma, w_t + \eta w)| \tag{3.10}$$

the differential inequality will be simplified using the assumption made on F. First, to establish the continuity of the solution semigroup we set

$$M = M(\| A^{1/2}u \|, \| A^{1/2}v \|) \tag{3.11}$$

and estimate the right hand side of the inequality (3.10), by

$$\begin{aligned} 2|(\Gamma, w_t + \eta w)| &\leq 2 \| Q^{-1/2}\Gamma \| \| Q^{1/2}(w_t + \eta w) \| \\ &\leq 2M\| |w \|_\gamma (\| Q^{1/2}w_t \| + \| Q^{1/2}\eta w \|) \\ &\leq 2M\| |w \|_\gamma (\| Q^{1/2}w_t \| + \eta\beta \| A^{1/2}w \|) \\ &\leq k(\| w \|_\gamma)^2 + (1/2) \| Q^{1/2}w_t \|^2 + \eta \| A^{1/2}w \|^2 \end{aligned} \tag{3.12}$$

where $k = 2M^2 + \eta\beta M^2$, with M as in (3.11), hence it follows that

$$\frac{d}{dt}E^2(w_t, w) + \eta \| A^{1/2}w \| + \| Q^{1/2}w_t \| \leq k(\| w \|_\gamma)^2 \tag{3.13}$$

Now we choose the value 1/2 for γ and utilize the inequality (3.6) to deduce that

$$\frac{d}{dt}E^2(w_t, w) + \eta \| A^{1/2}w \| + \| Q^{1/2}w_t \| \leq 2kE^2(w_t, w) \tag{3.14}$$

it follows that

$$E^2(w_t, w) \leq \exp\{2kt\}E^2(w_1, w_0) \tag{3.15}$$

From the last inequality we can deduce various conclusions. First of all, it guarantees the uniqueness of solution as stated in Theorem 2, secondly, once the existence and uniqueness is established we can proceed with the solution semigroup as stated in the first section and prove the existence of the absorbing set B. Furthermore, we can use our conditions on the nonlinearity, given in (V), only on the absorbing set B in order to get an explicit estimate on k in (3.15), since the norm on X is equivalent to the one furnished by E we can then conclude that the solution semigroup is in fact Lipschitz continuous for every t, and the Lipschitz constant can be estimated by

$$Lip(S(t)) \leq (2\mu)^{1/2} \exp\{kt\} \tag{3.16}$$

Now we will use the stronger condition on the nonlinearity and assume that $\gamma < 1/2$, this would allow us to utilize the compact imbedding of

$$W(0,T) \to L^2(0,T,D(A^{1/2})). \tag{3.17}$$

Returning back to inequality (3.13) and using (3.6) once again we obtain that

$$\frac{d}{dt}E^2(w_t,w) + (\eta/\mu)E^2(w_t,w) \leq k \parallel w \parallel_\gamma^2 \tag{3.18}$$

The last inequality when integrated from 0 to t_0, implies that the solution semigroup is an α-contraction, hence guarantees the existence of a compact attractor. Let us remind that since the solution semigroup does not consist of compact maps, we do especially need the asymptotic smoothness property that is satisfied by α-contractions. Now let us recall the basic definitions and theorems related with α-contractions, we follow here the outline given in [26].

Since on a Hilbert space E, a set is compact if and only if it is closed and totally bounded, the only way a closed set fail to be compact is when it is not totally bounded. Kuratowski [46] introduced a way to measure the noncompactness of a set by means of a real valued function defined as

$$\alpha(A) = \inf\{d > 0 : \text{ there is a finite cover of A with open sets of diameter less than d}\} \tag{3.19}$$

Notice that if the Lipschitz constant of the solution semigroup is strictly less than 1, then all the solutions converge to the unique steady state. Hence the only way to have non-trivial dynamical behavior is when the solution semigroup fails to be a strict contraction. The notion of a α-contraction comes into the play at this point:

Definition 4. *(see [37]) A continuous map T from E to itself is called an α-contraction if there exists $q, 0 < g < 1$, such that $\alpha(TA) \leq q\alpha(A)$ for every subset A of E.*

Recall that a real valued function n on E is called a *pseudonorm* when all the properties except the one on vanishing norms, i.e. $n(x) = 0$ no longer implies that $x = 0$. We will call n to be *precompact* with respect to the usual norm of E if every bounded sequence in $(E, \| \ \|)$ has an n-convergent subsequence.

The concept of a precompact pseudonorm allows a useful characterization, see [37], namely,

Proposition *Let B be bounded subset of E and T is a continuous function on E leaving B invariant , i.e. TB is a subset of B, and satisfying, for some q, $0 < q < 1$,*

$$\| Tx - Ty \| \leq q \| x - y \| + n(x - y) \text{ for all } x \text{ and } y \text{ in } B \tag{3.20}$$

with n precompact pseudonorm on E, then T is an α-contraction.

If the solution semigroup is an α-contraction on B then the ω-limit set of B, denoted by A, under T is a compact subset of E, where

$$A = \omega(B) = \cap\{cl_E(T^n B) : \text{ a natural number }\}. \tag{3.21}$$

This set remains invariant under T and furthermore attracts all solutions in the sense that the distance of $T^n x$ to A goes to zero as n goes to infinity. A is called the *global attractor*. Hence in order to guarantee the existence of a compact attractor, it suffices to show that for a fixed time t_0 the map defined by $T = S(t_0)$ is an α-contraction. Going back to the inequality (3.18) and integrating it from 0 to t_0, we obtain that

$$E^2(w_t, w) \leq \exp\{-\eta t_0/\mu\} E^2(w_1, w_0) + \int_0^{t_0} k \| w(s) \|^2 \, ds \tag{3.22}$$

If we choose t_0 such that $\exp\{\eta t_0/\mu) = q < 1$, then $T = S(t_0)$ is an α-contraction when $H = D(A^\gamma)$, and the compactness of the imbedding given in (3.17) shows that the pseudonorm introduced by the integral terms is precompact with respect to the usual X norm. Consequently, we obtain a compact set A that attracts all solutions in the invariant set B uniformly. A similar argument results in a stronger property of the solution semigroup, the discrete spueezing property:

Definition 5. *A solution semigroup $\{S(t) : t > 0\}$ is said to satisfy the discrete spueezing property if there exists $t_0 > 0$, such that the map defined by $T = S(t_0)$ satisfies: there exists an orthogonal projection P of rank M such that for every x and y in X, if*

$$\| P(Tx - Ty) \| \leq \| (I - P)(Tx - Ty) \| \tag{3.23}$$

implies that

$$\| Tx - Ty \| \leq (1/8) \| x - y \| . \tag{3.24}$$

This property allows one to construct a finite dimensional set that attracts all solutions at a uniform exponential rate. Such a set is called an *exponential attractor* and is not unique. The argument that leads to this property is exactly parallel to the one given for α-contractions, the only difference lies in the fact that the estimates are done after projecting the equation by $I - P$ to the complement of the finite dimensional space PX. In the process we finally make use of the assumption that the operators P and Q commute with A on $D(A)$.

Definition 6. *A set M is called an exponential attractor for the solution semigroup $\{S(t) : t > 0\}$ on the set B if*

(i) $A \subseteq M \subseteq B$,

(ii) $S(t)M \subseteq M$,

(iii) M has finite fractal dimansion,

(iv) for every x in B, dist $(S(t)x, M) \leq c_1 \exp\{-c_2 t\}$ where c_1 and c_2 are universal constants.

Theorem 3. *([28], [25], [23]) If the solution semigroup $\{S(t) : t > 0\}$ satisfies the discrete squeezing property on B and if the map $T = S(t_0)$ is Lipschitz with Lipschitz constant L then there exists an exponential attractor M for the solution semigroup satisfying*

$$d_F(M) \leq M \max\{1, ln(16L + 1)/ln2\} \tag{3.25}$$

and for all $t > 0$,

$$dist(S(t)x, M) \leq c_1 \exp\{(-c_2/t_0)t\} \tag{3.26}$$

where c_1 and c_2 are universal constants independent of L,M and t_0.

We summarize the results discussed above in the following theorem:

Theorem 4. *Suppose that the linear operators P,Q, and A satisfy the properties assumed in Theorem 2, furthermore, we assume that they all commute with the orthogonal projection P_N onto the space spanned by the vectors $w_1, \cdots, w_N$, where the vectors $w_1, \cdots, w_N, \cdots$ form a basis of H. Assume further that the nonlinear term $F(u)$ satisfy the condition (ν) and for each u,v in D_3*

$$\| Q^{-1/2}(F(u) - F(v)) \|_\gamma \leq M_2(\| u \|_{1/2}, \| \nu \|_{1/2}) \| u - \nu \|_{1/2},$$

where $M_2(\cdot, \cdot)$ is a continuous function of its variables. Then the semigroup $S(t) : X \to X, t > 0$, generated by the problem (2.1) and (2.2) has an exponential attractor.

4. Imbedding Theorems and Interpolation Results

Typically, the operator that appears in (2.1) will either be the negative of the Laplace operator, i.e. $-\Delta$, or the biharmonic operator, i.e. Δ^2 or a linear combination of these operators with the identity operator, furnished with the appropriate boundary conditions. In this paper we will only deal with the homogeneous Drichlet boundary conditions. Therefore, either

$$A = -\Delta, u = 0 \text{ on the boundary of } \Omega, D(A) = H^2(\Omega) \cap H_0^1(\Omega) \tag{4.1}$$

$$\text{or } A = \Delta^2, u = \frac{\partial u}{\partial n} = 0 \text{ on the boundary of } \Omega, D(A) = H^4(\Omega) \cap H_0^2(\Omega) \tag{4.2}$$

It follows that the interpolation spaces that appear in the conditions (V) and (W) are either:

$$D(A^{s/2}) \subseteq H^s(\Omega) \text{ and the corresponding norms coincide,} \tag{4.3}$$

$$\text{or } D(A^{s/2}) \subseteq H^{2s}(\Omega) \text{ and the corresponding norms coincide.} \tag{4.4}$$

Hence in the conditions (V) and (W) one is required to estimate the growth of the non-linear term on the fractional order Sobolev spaces $H^s(\Omega)$, where $0 < s < 1/2$ (or $0 < s < 1$). In addition to the standard Sobolev space theory for fractional orders, (see for example Adam's book [1], and Bergh and Lofstrom's book [11]) we will be needing the theory of Besov spaces which can be obtained from Sobolev spaces by interpolation. Here we use the definition given in Bennett and Sharpley [10]:

Definition 7. (Besov Spaces) *Let us denote the r^{th}-order difference operators by*

$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x) for x, h \in \mathcal{R}^n$$

$$\Delta_h^{r+1} f(x) = D_h^1(\Delta_h^r f(x)), r = 1, 2, \cdots$$

and denote the modulus of continuity of these differences by

$$\omega_r(f,t)_p = \sup_{|h| \le t} \| \Delta_h^r f \|_p, \quad for \quad f \in L^p(\mathcal{R}^n), t > 0, 1 \le p \le \infty.$$

For $\alpha > 0, 1 \le p, q \le \infty, r > \alpha$ and r natural number, the Besov space is defined by

$$B_{p,q}^{\alpha} = \left\{ f \in L^p(\mathcal{R}^n) : \| f \|_{B_{p,q}^{\alpha}} < +\infty \right\} \tag{4.5}$$

where

$$\| f \|_{B_{p,q}^{\alpha}} = \| f \|_p + \| t^{-\alpha} \omega_r(f,t)_p \|_{L^q(0,\infty;\frac{dt}{t})} . \tag{4.6}$$

Theorem 4. *(see [11] and [10])* ***(Imbedding Results)***

Let ω be a open bounded subset of $\mathcal{R}^n$ with smooth boundary then the following inclusions are continuous

(i) $B_{p,q}^s(\Omega) \subseteq B_{p_1,q_1}^{s_1}(\Omega)$ *for* $s - \frac{n}{p} = s_1 - \frac{n}{p_1}, (1 \le p \le p_1 \le \infty, 1 \le q, q_1 \le \infty)$

(ii) for $n > 2s$, and $q < 2n/(n-2s), H^s(\Omega) \to L^q(\Omega)$ is compact.

(iii) $B_{2,1}^{n/2}(\Omega) \subseteq L_\infty(\Omega)$.

Definition 8. (Interpolation of Banach Spaces by K-method): *Let A_0 and A_1 be two Banach spaces that are included in another Banach space A. For any element a in $A_0 + A_1$ define a function*

$$K(t,a) = \inf\{\| a_0 \|_{A_0} + t \| a_1 \|_{A_1} : a_0 \in A_0, a_1 \in A_1 \quad and \quad a = a_0 + a_1\}$$

for $0 < \theta < 1$ and $1 \leq q < \infty$, set

$$(A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 : t^{-\theta} K(t,a) \in L^p(0,\infty, dt/t)\} \tag{4.7}$$

and furnish this space with the norm

$$\| a \|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t} \right)^{\frac{1}{q}} \tag{4.8}$$

The Besov spaces as defined in definition 7 can also be captured by interpolation of Sobolev spaces:

Theorem 5. *(see [11]) $(H^{s_0}(\Omega), H^{s_1}(\Omega))_{\theta,q} = B^s_{2,q}(\Omega)$ where $s = (1-\theta)s_0 + Qs_1$.*

The Besov spaces which interpolate between the Sobolev spaces have better imbedding properties than the Sobolev spaces that they interpolate. see for example Theorem 4 (iii). Due partly to this better imbedding property they play a crucial role even in the estimates involving only Sobolev norms. The following results is inspired by a similar result stated in Chueshov [15, 17] for $n = 2$:

Proposition 2. *Let $s > 0, q > 1$ and set $\alpha = s + (n/2(1 - 1/q)$ and $\beta = n/(2q)$*

$$\| u\nu \|_{H^s(\Omega)} \leq C \| u \|_{H^\alpha(\Omega)} \| \nu \|_{H^\beta(\Omega)} \quad whenever \quad u \in H^\alpha(\Omega) \quad and \quad \nu \in H^\beta(\Omega). \tag{4.9}$$

Proof. We use the fact that $H^s(\Omega) = B^s_{2,2}(\Omega)$ and the Besov-norm on the Sobolev space $H^s(\Omega)$. Let us denote the H^s-norm of u by $\| u \|_s$, then for p and q Holder conjugate, i.e. $1/p + 1/q = 1$, we have

$$\| u\nu \|_s^2 \leq \| u\nu \|^2 + 2 \int_0^\infty t^{-s} \left\{ \sup_{|t|\leq h} \| \Delta_h u \|_{L^{2p}} \| \nu(\cdot + h \|_{L^{2q}} + \| \Delta_h \nu \|_{L^{2q}} \| u \|_{L^{2p}} \right\}$$
$$\frac{dt}{t} \leq C \| u \|^2_{B^s_{2p,2}} \| \nu \|^2_{B^s_{2q,2}}$$

Then we use the imbedding of Sobolev spaces into Besov spaces, namely,

$$B^s_{2q,2} \supseteq H^\alpha = B^\alpha_{2,2}, \alpha - n/2 = s - n/2,$$

and similarly, for α replaced by β and p replaced by q, and obtain the desired result. □

Remark 1. We will be making some standard choices for the parameters that appear in the spaces above. When $n = 2$, our estimate gives

$$\| u\nu \|_{H^s(\Omega)} \leq C \| u \|_{H^{1-t}(\Omega)} \cdot \| \nu \|_{H^{t+2s}(\Omega)}, \quad with \quad \frac{1}{q} - s \equiv t > 0, \tag{4.10}$$

and when $n = 3$, taking $s = 1/q$, we obtain that

$$\| u\nu \|_{H^s(\Omega)} \leq C \| u \|_{H^{(3-s)/2}(\Omega)} \| \nu \|_{H^{5s/2}(\Omega)} . \tag{4.11}$$

Remark 2. It is possible to improve the estimate for $n = 2$, following Chueshov [17]: using the fact that $H^{1-\delta}(\Omega)$ is continuously imbedded in $L^{2/\delta}(\Omega)$, for $0 < \delta < 1$, we estimate the difference operator in the following way

$$\| u\nu \|^2_s \leq \| u\nu \|^2 + 2 \int_0^\infty t^{-s} \left\{ \sup_{|t| \leq h} \| \Delta_h u \|_{L^{2p}} \| \nu(\cdot + h) \|_{L^{2q}} + \| \Delta_h \nu \|_{L^{2p'}} \| u \|_{L^{2q'}} \right\}$$
$$\frac{dt}{t} \leq C(\| u \|^2_{B^s_{2p,2}} \| \nu \|^2_{L^{2q}} + \| \nu \|^2_{B^s_{2p',2}} \| u \|^2_{L^{2q'}} 0$$

where $1/p + 1/q = 1 = 1/p' + 1/q'$, then the idea is to choose the parameters p, p', q, and q' in an optimal way and combine it with the imbedding theorem of a Sobolev space into the appropriate Besov space, namely

$$H^{1+s-n/2q} \subseteq B^s_{2p,2} \text{ for } q > 1.$$

Now one sets, $1/p = \beta = 1 - 1/q$ and $1/p' = 1/q - s$ and obtains that

$$\| u\nu \|_{H^s} \leq C(\| \nu \|_{H^{1-\frac{1}{p}}} \| u \|_{H^{1+s-\frac{1}{q}}} + \| u \|_{H^{1-\frac{1}{p'}}} \| \nu \|_{H^{1+s-\frac{1}{q'}}}) \leq 2C \| \nu \|_{H^{1-\beta}} \| u \|_{H^{s+\beta}} \tag{4.12}$$

We will also need an interpolation result on Lipschitz operators due to Peetre [60, 61] and Tartar [67]:

Theorem 9. *Let T be a Lipschitz operator from A_i to B_i, for $i = 0, 1$ with Lipschitz constants less than M, and let $A = (A_0, A_1)_{\theta,q}$ and $B = (B_0, B_1)_{\theta,q}$ then T is also Lipschitz from A to B with Lipschitz constant less than M.*

Remark 3. This result will be used for the particular nonlinear function sinu, although it is a Lipschitz function from L^p to L^p for $1 < p < \infty$, it is not Lipschitz from $H^1(\Omega)$ to itself for $n > 2$. Hence it will be necessary to seek Lipschitzianity between different order Sobolev spaces.

5. Applications and Examples

The framework that is developed in the second and third sections of the article applies to a variety of equations. However, the conditions that we impose on the non-linearity sometimes forces us to work with stronger damping terms than necessary. Therefore, although we are able to obtain new results in the cases of sin-Gordon equation, von Karman equations, dissipative Boussinesque equation etc., we are not able to capture the best results in the three dimensional Klein-Gordon equation and various beam equations. We refer the reader to specific articles where better results are obtained. Unless otherwise mentioned all the equations will be complemented with homogeneous Drichlet boundary conditions on the boundary of an open bounded domain in n-dimension with smooth boundary, so that all the imbedding theorems apply. Other boundary conditions can be treated similarly. We will also deal systematically with the homogeneous versions of the equations, however, the analysis given below easily extends to non-homogeneous cases where the forcing term $h(x)$ is time independent and comes from the appropriate space. We will give the details only for the selected applications. In the other applications one must perform parallel computations. The general references on these equations are the following books: Lions Ladyzhenskaya, Haraux, Babin and Vishik [6], Eden et al., Hale, Temam [68], Gajewski et al [33].

5.1. Nonlinear Klein-Gordon Equation:

$$u_{tt} - \Delta u + \gamma u_t + f(u) = 0, \text{ where } f(u) = -\lambda u + |u|^\rho u, \text{ with } \rho > 0, \tag{5.1}$$

In space dimension 3, these equations arise in relativistic quantum mechanics, with $\rho = 2$. The global stability of this equation has been studied by Strauss [66, Theorem 3, p. 150] where it was established that all solutions converge to the unique steady state $u \equiv 0$ when $\lambda = 0$.

Muszynksi and Makoveckaja [54], Ball [9], Webb [69]; Babin and Vishik [4,5] have obtained preliminary results on the existence of attractors, later Haraux [39], Ghidaglia and Temam [35] Ladyzhanskaya [50] have proven the existence of attractors, partial results on exponential attractors are obtained in Eden et al [2], and in Eden and Milani [20]. Although Webb in [69] has considered only the one space dimensional case, $n = 1$, he showed that the trajectories are precompact and his results have paved the way for further advances in the field. Whereas the asymptotic stability results established in Makoveckaja and Muszynski [54] did not influence the developments on these equations.

Another attempt on showing the precompactness of the trajectories was done by Babin and Vishik [4,5], they were only able to get some weak results in this direction. The first conclusive result was obtained by Haraux in. Haraux was able to prove the existence of the global attractor for the range $\rho < 2$, in three space dimension. Later, Ladyzhenskaya in was able to resolve the case of the critical exponent $\rho. = 2, n = 3$ for smooth solutions. The existence of the finite dimensional atractor and estimate of its dimension was obtained in Ghidaglia and Temam [35], Ladyzhenskaya [50] not including the critical case $\rho = 2$. Babin and Vishik [6] have proved the existence of the finite dimensional attractor in the critical case. The theory developed in Ladyzhenskaya [48], [50] was used by Kalantarov [38, 39, 40] and Chueshov [14, 15, 16] to prove the existence of a finite dimensional attractor for a variety of equations with stronger dissipative terms. Clearly, depending on the dimension of the space the value of the ρ that one is allowed to take changes. For example, in one and two space dimensions there is no restriction on it. However, in three dimensions $\rho = 2$, in contrast due to the strong nature of the conditions (V) and (W) of our specific framework, we are only able to allow $\rho = 1$. However, if we assume the regularity of the attractor then we can relax this restriction, see Babin and Vishik [6] for the H^2 regularity of the attractor. Using the H^2-regularity of the attractor, one can show as in Eden et al [25] that the cubic nonlinearity induces a Lipschitz map both from L^2 into itself and from H^1 into itself, on a bounded invariant set that contains the attractor, hence the conditions on the nonlinearity will be satisfied automatically. Here for illustrative purposes, we consider the two dimensional case with ρ arbitrary, for computational convenience we will take $\rho = 2p + 1$, with p a positive integer. One only needs to check the conditions (V) and (W) for the term u^{2p+1}, again using the imbedding of $H^{1-\delta}$ into $L^{2/\delta}$, for $0 < \delta < 2$, we see that on the absorbing set, which is bounded in $H^1(\Omega)$, the H^1-norms are uniformly bounded. Consequently,

$$\| u^{2p+1} - \nu^{2p+1} \| \leq \| u - \nu \|_{L^4} \sum_{\alpha+\beta=2p} \| u^\alpha \|_{L^{4q}} \| \nu^\beta \|_{L^{4q}}$$

for $1/q + 1/q' = 1$, where we have utilized the imbedding of $H^{1/2}$ into L^4 and a Holder estimate. The terms in the summation sign are all bounded by H^1-norm, which implies that on the absorbing set u^{2p+1} is Lipschitz from $H^{1/2}$ into L^2. To verify the condition (W) we proceed to estimate the H^s-norm of the difference using the estimate of Chueshov on fractal order estimates on the products, for $n = 2$, see Remark 2. If $s + (2p+1)_\gamma < 1$, then by induction on $\alpha + \beta = p$, one can prove that

$$\| (u - \nu)(u^{2p} + \cdots + \nu^{2p}) \|_{H^s} \leq C \| u - \nu \|_{H^{1-\gamma}} \sum \| u^\alpha \nu^\beta \|_{H^{s+\gamma}} \leq C_1 \| u - \nu \|_{H^{1-\gamma}}$$

here C_1 is a function of p and M, where M is the bound in H^1-norm for the absorbing set.

5.2. Sin-Gordon Equation:

Most of the references given for the first equation also applies to this one, the systems given below are taken from Temam [68].

$$u_{tt} - \Delta u + \gamma u_t + \sin u = 0, \tag{5.2}$$

systems of sin-Gordon equations:

$$u_{tt} + u_t - \Delta u + \sin u + (u - v) = f_1 \text{ and } v_{tt} + v_t - \Delta v + \sin v + (v - u) = f_2; \tag{5.3}$$

$$u_{tt} + u_t - \Delta u + \sin(u + v) = f_1 \text{ and } v_{tt} + v_t - \Delta v + \sin(v - u) = f_2; \tag{5.4}$$

Here we only discuss the sin-Gordon equation, similar arguments will apply to systems given above. Since the non-linearity is mildly behaved we are able to prove the existence of an exponential attractor in any space dimension, this result improves the one obtained in Eden et al [25] where the space dimension was restricted to n 3, and regularity results for the attractor was used. Clearly, $f(u) = \sin u$ is Lipschitz from $L^2(\Omega)$ into itself with Lipschitz constant less than or equal to one. To estimate the difference $\sin u - \sin v$ in higher Sobolev norms, we will use the fact that $f(u)$ is also Lipschitz from $L(\Omega)$ into itself. Since Ω is a subset of $\mathcal{R}^n$, the Sobolev space $H^{(n+1)/2}(\Omega)$ is imbedded into $L(\Omega)$ only for $\epsilon > 0$. Therefore, we have the following estimate

$$\begin{aligned} \| \sin u - \sin \nu \|_{H^1} &\leq \| \nabla(u - \nu) \cos \| + \| (\cos u - \cos \nu) \nabla u \| \\ &\leq \| u - \nu \|_{H^1} + \| u - \nu \|_{L^\infty} \| u \|_{H^1} \end{aligned}$$

since $H^{(n+\epsilon)/2}$ is imbedded into L, the estimate above allows us to bound the Lipschitz constant uniformly on the absorbing set, which is a bounded subset of $H^1(\Omega)$. Next we utilize Peetre's Theorem on the interpolation of Lipschitz operators in conjuction with the characterization of the interpolation spaces of Sobolev type by the K-method and deduce that sin is a Lipschitz function from $(L^2(\Omega), H^{(n+\epsilon)/2}(\Omega))_{\theta,2} = H^{\theta(n+\epsilon)/2}(\Omega)$ into $(L^2(\Omega), H^1(\Omega))_{\theta,2} = H^\theta(\Omega)$. Choosing $\theta = 1/n$, it follows that sin is Lipschitz on the absorbing set from $H^1(\Omega)$ into $H^{1/n}(\Omega)$. Consequently, the condition (W) is also satisfied for the nonlinearity in any space dimension. To reveal the exact nature of the role of the space dimension in the estimate of the fractal dimension of the attractor requires finer analysis that we refrain from doing here.

5.3. Wave Equation With Strong Dissipative Term:

$$u_{tt} - \Delta u - \alpha \Delta u_t + f(u) = 0, \text{ where } f(u) \text{ is as above.} \tag{5.5}$$

The global stability of this equation was studied by Webb [70], Fitzgibbon [31], and Massatt [57] mainly for mild non-linearities f. Webb utilized this mild form of the non-linearity and the strong dissipation to establish the existence of a Lyapunov functional and proved the convergence to steady states. Whereas Fitzgibbon studied the asymptotic behavior of a similar equation where the function f is assumed to be Holder continuous but depends only on the gradient of u. Later on, Massat discussed a more general situation but he assumed the existence and uniqueness results without proof. The existence of an absorbing set was discussed in Kalantarov [43] under milder conditions on the non-linearity f, the existence of an attractor and the estimate of its finite dimensionality is established in [44] and is mentioned in Hale [37], and later on in more detail in Ghidaglia and Marzochhi [24]. The existence of an exponential attractor follows from the framework developed here. The values for ρ that are allowed ranges between 0 and 4, for $n = 3$, and is arbitrary for $n = 1$ or 2.

5.4. The Vibrating Beam Equation:

$$u_{tt} + 2(\alpha\Delta^2 + \delta)u_t + \Delta^2 u = f(u), \tag{5.6}$$

where $f(u) = \Gamma \cdot \Delta u + \rho \cdot \nabla u + \kappa < \nabla u, \nabla u > \Delta u + \sigma < \nabla u, \nabla u_t > \Delta u$ and ρ and Γ are constant vectors in $\mathcal{R}^n$. The global stability of the solutions and the convergence of solutions to steady states are discussed in Ball [7,8], Hale [37], refers to a preprint of Lopes and Ceron [53] for the proof on the existence of the attractor, results in similar direction are also obtained by Chueshov [14], and Eden and Milani [26]. When σ is not 0, our treatment does not apply automatically, since f is also a function of u_t, the argument that will be given for the generalized beam equation will also apply to the case above when $\sigma = 0$. The existence of an exponential attractor was established in [26] under some regularity conditions on the attractor. In our framework we are able to remove this extra condition on the regularity of the attractor.

$$(1 - \alpha\Delta)u_{tt} - (\delta - \epsilon\Delta)u_t + \Delta^2 u = -[v + \theta, u] \text{ and } \Delta^2 v = [u, u] \tag{5.7}$$

where $[u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}, (x, y)$ is from a domain Ω in the plane, $\theta \epsilon H^4(\Omega)$ the constants α and ϵ are assumed to be positive. This model was developed by Morozov in [58] as a dynamical system its properties were discussed in Aviles and Sandefur [3] the existence of a finite dimensional compact attractor was first proved in Chueshov [17], the analysis given in [14] is sufficient to verify that the linear and nonlinear terms satisfy the necessary conditions. Here we take $P = 1 - \alpha\Delta; Q = \delta - \epsilon\Delta$ and $A = \Delta^2$, the Drichlet boundary conditions on u and its normal derivatives at the boundary of Ω are set, hence

$D(A)$ is given as in (4.2). Instead of the condition (2.5) one can directly show the existence of an energy functional that is decreasing on trajectories. The same functional allows one to obtain an absorbing set. Here we will only check the two conditions (V) and (W) on the non-linear term. Let us recall the basic Sobolev estimates on the bracket [,] that are given in Chueshov [17]:

$$\| [u, \nu] \|_{-l} \leq C \| u \|_{2-\beta} \| \nu \|_{3-l+\beta} \quad where\ l = 1, 2, 0 < \beta < 1,$$
$$\| [u, \nu] \|_{-l-\theta} \leq C \| u \|_{2-\theta+\beta} \| \nu \|_{3-l-\beta} \quad where\ l = 0, 1, 0 < \beta \leq \theta < 1,$$

these estimates result in the following estimates:

$$\begin{aligned} \| Q^{-1/2}(F(u) - F(\bar{u})) \| &\leq \| [\bar{\nu} + \theta, \bar{u}] - [\nu + \theta, u] \|_{-1} \\ &\leq C \| \nu + \theta \|_{2+\beta} \| u - \bar{u} \|_{2-\beta} + C \| \nu - \bar{\nu} \|_{2+\beta} \| \bar{u} \|_{2-\beta} \\ &\leq C(M + \| [u, u] \|_{-2+\beta}) \| u - \bar{u} \|_{2-\beta} \\ &+ CM \| [u, u] - [\bar{u}, \bar{u}] \|_{-2+\beta} \end{aligned}$$

where we have used the facts that $(i) D(Q^{-1/2})$ has the same norm as H^{-1}; (ii) the first of the inequalities mentioned above with $l = 1$, twice; $(iii) \theta$ is a function in H^4, and the constant M depends on the H^2-norms of u. There remain two terms in the last line of the above inequality that need to be estimated.

$$\| [\bar{u}, \bar{u}] \|_{-2+\beta} = \| [\bar{u}, \bar{u}] \|_{-1-\theta} \leq C \| \bar{u} \|_2 \| \bar{u} \|_{2-\beta} \leq CM^2$$

and

$$\| [\bar{u}, \bar{u}] - [u, u] \|_{-2+\beta} \leq 2CM \| u - \bar{u} \|_{2-\beta}$$

where we have utilized the second inequality above first with $\theta = 1 - \beta$, then taking $l = 1$ and $\theta = \beta$ as parameters. The condition (V) for the nonlinear term now follows from the last inequalities. As for the (W) condition, we again use the second estimate on the Karman bracket with $\theta = \beta$ and $l = 0$:

$$\begin{aligned} \| [\bar{\nu} + \theta, \bar{u}] - [\nu + \theta, u] \|_{-\beta} &\leq C \| u - \bar{u} \|_2 \| \bar{\nu} + \theta \|_{3-\beta} + CM \| [u, u] - [\bar{u}, \bar{u}] \|_{-1-\beta} \\ &\leq C(M + \| [\bar{u}, \bar{u}] \|_{-1-\beta}) \| u - \bar{u} \|_2 + CM \| [u - \bar{u}, u] \|_{-1-\beta} \\ &+ CM \| [u - \bar{u}, \bar{u}] \|_{-1-\beta} \end{aligned}$$

by arguments similar to the previous ones the Lipschitzianity of the function into H^2 follows.

5.6. Generalized Beam Equations:

$$u''(t) + \alpha A^2 u(t) + (\delta + \gamma A)u'(t) + M(\| A^{1/2}u(t) \|^2)Au(t) = 0, \tag{5.8}$$

These equations arise naturally as an operator theoretic version of the similar equation considered by Ball [8], they were introduced by de Brito [13], where the global stability and convergence to steady states were established. de Brito assumed that A is a positive linear self-adjoint operator with compact inverse, and the continuous nonlinearity M, which is non-local type, is assumed to satisfy $M(s) > \beta + ks$, and $M'(s) > 0$, for $s > 0$, with $\beta, k > 0$. Here we only assume that $k > 0$, however for our conditions, we need to impose $\gamma > 0$, whereas de Brito and Ball deals with $\gamma = 0$. The latter case was treated in Eden and Milani [26] under some smallness conditions on the parameters that are involved. Later on, the restriction on the parameters were dropped in Eden et al. [23]. To apply the theory we just need to take $I, \delta + \gamma A$, αA^2, as P, Q, and A respectively. The energy functional approach allows us to obtain an absorbing set in $D(A)$, with A as in the equation. The estimates on the nonlinear term are of the form:

$$\begin{aligned} \| Q^{-1/2}(f(u)Au - f(v)Av) \| &\le |f(u) - f(v)| \, \| Q^{-1/2}Av \| + |f(u)| \, \| Q^{-1/2}(Au - Av) \| \\ &\le C \, \| A^{1/2}(u - v) \| \end{aligned}$$

where $f(u) = M(\| A^{1/2}u \|^2)$, using the conditions on the function $M(\cdot)$ this verifies (V), to verify (W) similar arguments results in

$$\| A^{1/2}Q^{-1/2}(f(u)Au - f(v)Av) \| \le C \, \| A(u - v) \| \, .$$

5.7 Dissipative Boussinesque equation:

$$u_{tt} + \Delta^2 u + \alpha \Delta^2 u_t - \Delta f(u) = 0, \tag{5.9}$$

with $\alpha > 0$. The global stability of these equation was discussed in Kalantarov [49], where the existence of an absorbing set was also established.

5.8. Dissipative Wave Equation With Strong and Weak Damping:

$$u_{tt} + u_{xxxx} - \alpha u_{xxt} + \gamma u_t - f(u_x)_x = 0, \tag{5.10}$$

where $f(u)$ satisfies an asymptotic monotonicity condition of the form $f(v)v - F(v) \ge -c$. This equation appears both in the theory of viscolelasticity and in gas dynamics, with different type of non-linearity $f(u)$. Interesting dynamics is observed when the function is not necessarily convex. The asymptotic stability of the steady states was discussed in Andrews and Ball [2], in Kalantarov [44], where the existence of an absorbing set is also established. In Nicolaenko [59] the existence of an inertial manifold was claimed for smooth solutions, later in Eden et al [2,7], the existence of an exponential attractor is established for less regular solutions.

5.9. Longtitudinal Vibrations in a Homogeneous Bar With Viscous Effects:

$$u_{tt} - \epsilon\Delta u_{tt} - \Delta u - \alpha\Delta u_t + f(u) = 0, \tag{5.11}$$

Kalantarov [44] established the existence of an absorbing set for this equation, later Iskenderov [42] has studied the global attractor and related problems. Here the main difference is on $P(= I - \epsilon\Delta), A = -\Delta$ and $Q = \alpha A$, the condition on the nonlinearity is as in the third equation.

5.10. An Abstract Parabolic Equation:

$$u_{tt} + A^2 u + Au_t + f(u) = 0, \tag{5.12}$$

Although hyperbolic in the first appearence, this equation can be factored to show its parabolic nature. The global stability results were discussed by Biler [12], the existence of an absorbing set was established in Kalantarov [44], the attractor was studied in Humbatov [41].

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BİR YARILİNEER DALGA DENKLEMİ SINIFI İÇİN SONLU BOYUTLU ÇEKENLER

Özet

Bu makalede lineer olmayan bir dalga denklemi sınıfının global çekenleri ile ilgili, kendi içinde bütün bir derleme veriyoruz. Özel olarak, bir üstel çekene gömerek, sonlu boyutlu bir çekenin varlığını ve onun fraktal boyutu hakkında bir kestirimi ispatlıyoruz. Global kararlılık ve sonlu boyutlu çekenlerin varlığı hakkında bazı sonuçlar kısmen Kalantarov [44] ve Eden ve diğerleri [25]'de tartışılmıştır. Ancak birleştirilmiş bir yaklaşım kullanarak hem α-daraltmalar yardımıyla çekenlerin varlığını ve hem de lineer olmayan operatör için Lipschitz olma koşulu aracılığı ile üstel çekenlerin inşaasını kolaylaştırmış oluyoruz.

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