Tr. J. of Mathematics
20 (1996) , 323 – 334.
© TÜBİTAK

PRODUCTS AND QUOTIENTS OF (p, σ)-ABSOLUTELY CONTINUOUS OPERATOR IDEALS*

Enrique A. Sanchez Perez

Abstract

We obtain a generalization of the ideal $\mathcal{M}_{(q,p)}$ of (q,p)-mixing operators-in the sense of Pietsch- as a consequence of the study of the quotients of (p,σ)-absolutely continuous operator ideals $\mathcal{P}_{p,\sigma}$ -in the sense of Jarchow and Matter-. Inclusions $\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}_{(q,s)}(E,F)$ are also investigated, specially for the cases $E = \mathcal{C}(K)$ and $E = L_1$.

The ideal $\mathcal{P}_{p,\sigma}$ of (p,σ)- absolutely continous operators -where $1 \leq p < \infty$ and $0 \leq \sigma \leq 1-$ was defined by Matter [6] in order to give good characterizations of super-reflexivity and other properties of Banach spaces. It is closely related to the ideal of absolutely continuous operators defined by Niculescu [8], and it was introduced as an interpolated operator ideal between $\mathcal{P}_p$ -the ideal of p-absolutely summing operators- and $\mathcal{L}$ -the ideal of continuous operators-using an interpolative procedure ([4], [12]). This technique was motivated by the characterization of the uniform closure of the injective hull of an operator ideal proved by Jarchow and Pelczynski [3].

The ideal $\mathcal{P}_{p,\sigma}$ satisfies intermediate properties between $\mathcal{P}_p$ and $\mathcal{P}_{(\frac{p}{1-\sigma},p)}$ -the ideal of $(\frac{p}{1-\sigma},p)$-absolutely summing operators- and its description generalizes the case $\mathcal{P}_p$. The aim of the first section of our work is to study those operators -that we call (q,p,σ)-mixing operators and we denote $\mathcal{M}_{(q,p,\sigma)}$- that satisfy $\mathcal{P}_{q,\sigma} o \mathcal{M}_{(q,p,\sigma)} \subseteq \mathcal{P}_{q,\sigma}$. We obtain in this way a generalization of (q,p)-mixing operators. The second part of this paper is devoted to find inclusions between the ideals of (p,σ)-absolutely continuous operators and the ideals of (q,p)-mixing operators. Special attention is paid to operators from $\mathcal{C}(K)$-spaces and L_1 -spaces on arbitrary Banach spaces F. In this study we obtain some properties of operators belonging to $\mathcal{L}(L_1,F)$ that factorize through Lorentz function spaces and spaces of Schatten-Von Neumann classes, that are closely related to a theorem due to Carl and Defant (see [1] and [2]). In the third section we obtain some results about products of (p,σ)-absolutely continuous operators.

* This research has been supported by a grant of the Ministerio de Educacion y Ciencia.

0. Background and Notation

Throughout this paper we employ standard Banach space notation. We shall consider only operators on Banach spaces. E, F and G are Banach spaces and B_E is the unit ball of E. $W(B_{E'})$ is the set of all regular Borel probabilities on $B_{E'}$ in the weak* topology. If $(x_i) \in l_p(E)$, we denote

$$W_p((x_i)) := \sup_{x' \in B_{E'}} \Big(\sum_{i=1}^{\infty} | < x_i, x' > |^p\Big)^{1/p}, \quad 1_p((x_i)) := \Big(\sum_{i=1}^{\infty} \| x_i \|^p\Big)^{1/p}$$

$$\text{and} \qquad \delta_{p,\sigma}((x_i)) := \sup_{x' \in B_{E'}} \Big(\sum_{i=1}^{\infty} \Big(| < x_i, x' > |^{1-\sigma} \| x_i \|^{\sigma}\Big)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}.$$

The following definition is due to Matter.

Definition 0.1. *[6]. Let $\mathcal{U}$ be an operator ideal and let $0 \leq \sigma < 1$. An operator $T : E \to F$ belongs to $\mathcal{U}_\sigma$ if there exist a Banach space G and an operator $S \in \mathcal{U}(E, G)$ such that $\| Tx \| \leq \| x \| \; \| Sx \| \qquad \forall x \in E$. If $\mathcal{U}$ is a normed operator ideal and α is its norm, $\mathcal{U}_\sigma$ is a normed operator ideal with norm $inf \alpha(S)^{1-\sigma}$.*

For the particular case $\mathcal{U} = \mathcal{P}_p$, the following theorem holds.

Theorem 0.2 *[6]. For every operator $T : E \to F$, the following are equivalent:*

(i) $T \in \mathcal{P}_{p,\sigma}(E, F)$.

(ii) There is a constant $C > 0$ and a probability measure μ on $B_{E'}$ such that

$$\| Tx \| \leq C\Big(\int_{B_{E'}} \Big(| < x, x' > |^{1-\sigma} \| x \|^{\sigma}\Big)^{\frac{p}{1-\sigma}} d\mu(x')\Big)^{\frac{1-\sigma}{p}} \qquad \forall x \in E.$$

(iii) There exist a constant $C > 0$ such that for every finite sequence $x_1, \cdots, x_n$ in E $1_{\frac{p}{1-\sigma}}((Tx_i)) \leq C\delta_{p,\sigma}((x_i))$.

In addition, the operator norm $\pi_{p,\sigma}(T)$ on $\mathcal{P}_{p,\sigma}(E, F)$ is the smallest number C for which (ii) and (iii) hold.

Let E be a Banach space and consider μ a probability defined on $B_{E'}$. We denote by J_p the map $E \to L_p(B_{E'}, \mu)$ given by $J_p(x) =< x, . >$, and by $N(J_p)$ the kernel of J_p. We write E_μ for the quotient space $E/N(J_p)$, and $\| \; \|_\mu$ for the quotient norm.

Consider an interpolation couple $(E_0, E_1)_{1-\sigma,1}$. The norm restricted to E_0 is equivalent to

$$\inf \left\{ \sum_{i=1}^{n} \| x_i \|_1^{1-\sigma} \| x_i \|_0^{\sigma} : \sum_{i=1}^{n} x_i = x, \; x_i \in E_0 \; \forall 1 \leq i \leq n \right\}$$

(see [7]). Throughout this paper we use this expression for the interpolation norm, since we only need its explicit formula for the elements $x \in E_0$.

1. (q,p,σ)-Mixing Operators

Definition 1.1. *Let T be an operator. We say that T is (q,p,σ)-mixing if it belongs to the quotient operator ideal $\mathcal{M}_{(q,p,\sigma)} := \mathcal{P}_{q,\sigma}^{-1} o \mathcal{P}_{p,\sigma}$. We denote by $M_{(q,p,\sigma)}$ the quotient ideal norm $\sup\{\pi_{p,\sigma}(SoT) : \pi_{q,\sigma}(S) \leq 1\}$.*

Obviously, this definition and the following characterization can be adapted to the case $\mathcal{P}_{p,\nu}$ and $\mathcal{P}_{q,\sigma}$ when $\sigma \neq \nu$. We restrict our attention to the case $\sigma = \nu$.

Theorem 1.2. *For every operator $T : E \to F$ the following are equivalent:*

(i) $T \in \mathcal{M}_{(q,p,\sigma)}(E,F)$.

(ii) There is a constant $C > 0$ such that for each probability measure μ on $B_{F'}$ there is a probability measure ν on $B_{F'}$ such that

$$\inf \left\{ \sum_{i=1}^{n} \Big(\int_{B_{F'}} \Big(| < y_i, y' > |^{1-\sigma} \parallel y_i \parallel^{\sigma} \Big)^{\frac{q}{1-\sigma}} d\mu(y') \Big)^{\frac{1-\sigma}{q}} : \sum_{i=1}^{n} y_i = Tx \right\} \leq$$
$$\leq C \inf \left\{ \sum_{i=1}^{n} \Big(\int_{B_{E'}} \Big(| < x_i, x' > |^{1-\sigma} \parallel x_i \parallel^{\sigma} \Big)^{\frac{p}{1-\sigma}} d\nu(x') \Big)^{\frac{1-\sigma}{p}} : \sum_{i=1}^{n} x_i = x \right\}$$
$\forall x \in E$.

(iii) There exist a constant $c > 0$ such that for every finite sequence $x_1, \cdots, X_n$ in E

$$\Big(\sum_{j=1}^{m} \inf \left\{ \sum_{i=1}^{s_j} \Big(\sum_{k=1}^{n} | < y_i^j, y_k' > |^q \parallel y_i^j \parallel^{\frac{\sigma q}{1-\sigma}} \Big)^{\frac{1-\sigma}{p}} : \sum_{i=1}^{n} y_j^i = Tx_j \right\}^{\frac{p}{1-\sigma}} \Big)^{\frac{1-\sigma}{p}}$$
$$\leq C \delta_{p,\sigma}((x_j)) l_q^{1-\sigma}((y_k'))$$

In this case, $M_{(q,p,\sigma)} = \inf C$, where the infimun is taken over all C satisfying (ii) or all satisfying (iii).

Proof. (i)$\to$(ii) If $T \in \mathcal{M}_{(q,p,\sigma)}(E,F)$ and μ is a probability measure on $B_{F'}$, the canonical embedding $I : F \to F_\mu \to (F_\mu, L_q(\mu))_{1-\sigma,1}$ is (p,σ)-absolutely continuous [7] and hence $I_o T \in \mathcal{P}_{p,\sigma}(E, (F_\mu, L_q(\mu))_{1-\sigma,1})$. By theorem 0.2 there exists a probability measure ν on $B_{E'}$ such that for all $x \in E$

$$\| IoTx \| \leq \pi_{p,\sigma}(IoT)\Big(\int_{B_{E'}} \Big(| < x, x' > |^{1-\sigma} \| x \|^{\sigma}\Big)^{\frac{p}{1-\sigma}} d\nu(x')\Big)^{\frac{1-\sigma}{p}} \tag{1}$$

where $\| IoTx \|$ is the norm of the element $IoTx$ of the interpolated space $(F_{\mu}, L_q(\mu))_{1-\sigma,1}$, i.e.

$$\| IoTx \| = \inf \left\{ \sum_{i=1}^{n} \Big(\int_{B_{F'}} \Big(| < y_i, y' > |^{1-\sigma} \| y_i \|_{\mu}^{\sigma}\Big)^{\frac{q}{1-\sigma}} d\mu(y')\Big)^{\frac{1-\sigma}{q}} : \sum_{i=1}^{n} y_i = Tx \right\}.$$

Just by using the triangle inequality, we find that the second part of (1) can be replaced by

$$\pi_{p,\sigma}(IoT) \inf \left\{ \sum_{i=1}^{n} \Big(\int_{B_{E'}} \Big(| < x_i, x' > |^{1-\sigma} \| x_i \|^{\sigma}\Big)^{\frac{p}{1-\sigma}} d\nu(x')\Big)^{\frac{1-\sigma}{p}} : \sum_{i=1}^{n} x_i = x \right\}.$$

Now we claim that $\| \, \|_{\mu}$ in (1) can also be replaced by $\| \, \|_F$; consider a representation $\sum_{i=1}^{n} y_i$ of Tx and suppose that $\| y_1 \|_{\mu} < \| y_1 \|$. For each $\epsilon > 0$ there is an $y_0 \in N(J_p)$ verifying $\| y_0 + y_1 \| < (1+\epsilon) \| y_1 \|_{\mu}$. Obviously,

$$\begin{aligned}
&\Big(\int_{B_{F'}} \Big(| < y_0 + y_1, y' > |^{1-\sigma} \| y_0 + y_1 \|^{\sigma}\Big)^{\frac{q}{1-\sigma}} d\mu(y')\Big)^{\frac{1-\sigma}{q}} + \\
&+\Big(\int_{B_{F'}} \Big(| < y_0, y' > |^{1-\sigma} \| y_0 \|^{\sigma}\Big)^{\frac{q}{1-\sigma}} d\mu(y')\Big)^{\frac{1-\sigma}{q}} \leq \\
&\leq (1+\epsilon)^{\sigma} \Big(\int_{B_{F'}} \Big(| < y_1, y' > |^{1-\sigma} \| y_1 \|_{\mu}^{\sigma}\Big)^{\frac{q}{1-\sigma}} d\mu(y')\Big)^{\frac{1-\sigma}{q}}
\end{aligned}$$

Thus, it is enough to consider the new representation $\sum_{i=2}^{n} y_i + (y_1 + y_o) - y_o = Tx$. The result is obtained by repeating the argument for all $2 \leq i \leq n$ and let $\epsilon \to 0$. Finally, since $\pi_{q,\sigma}(I) \leq 1, \pi_{p,\sigma}(IoT) \leq M_{(q,p,\sigma)}(T)$.

(ii)$\to$(iii) Let $(y'_k)_{k=1}^{n} \subset F'$ and consider the probability measure on $B_{F'}$ given by $\mu = (\sum_{k=1}^{n} \| y'_k \|^q \delta_k)(\sum_{i=1}^{n} \| y'_k \|^q)^{-1}$, where δ_k is the Dirac measure δ at the point $\frac{1}{\|y'_k\|} y'_k$. Then

$$\Big(\sum_{j=1}^{m} \inf \left\{ \sum_{i=1}^{s_j} \Big(\sum_{k=1}^{n} | < y_i^j, y'_k > |^q \| y_i^j \|^{\frac{\sigma q}{1-\sigma}}\Big)^{\frac{1-\sigma}{q}} : \sum_{i=1}^{n} y_j^i = Tx_j \right\}^{\frac{p}{1-\sigma}} \Big)^{\frac{1-\sigma}{p}} =$$

$$= 1_q^{1-\sigma}((y'_k))\Big(\sum_{j=1}^{m}\inf\left\{\sum_{i=1}^{s_j}\Big(\int_{B_{F'}} |< y_i^j, y' >|^q \parallel y_i^j \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(y')\Big)^{\frac{1-\sigma}{q}} : \right.$$

$$\left.\sum_{i=1}^{n} y_j^i = Tx_j\right\}^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}} \leq$$

$$\leq C1_q^{1-\sigma}((y'_k))\Big(\sum_{j=1}^{m}\Big(\int_{B_{E'}} |< x_j, x' >|^p \parallel x_j \parallel^{\frac{\sigma p}{1-\sigma}} d\nu(x')\Big)\Big)^{\frac{1-\sigma}{p}} \leq C\delta_{p,\sigma}((x_j))1_q^{1-\sigma}((y'_k)).$$

(iii)$\rightarrow$(i) Condition (iii) means that all discrete probability measures μ on $B_{F'}$ satisfy for all $x_1, \cdots, x_n \subset E$

$$\Big(\sum_{j=1}^{m}\inf\left\{\sum_{i=1}^{s_j}\Big(\int_{B_{F'}} |< y_i^j, y' >|^q \parallel y_i^j \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(y')\Big)^{\frac{1-\sigma}{q}} : \sum_{i=1}^{s_j} y_j^i = Tx_j\right\}^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}$$
$$\leq C\delta_{p,\sigma}((x_j)) \tag{2}$$

Since the set of all discrete probabilities is dense in $W(B_{F'})$ with respect to the weak $\mathcal{C}(B_{F'})$-topology, we only need to verify that the function $f(\lambda)$ defined on $B_{F'}$ by

$$f(\lambda) := \Big(\sum_{j=1}^{m}\inf\left\{\sum_{i=1}^{s_j}\Big(\int_{B_{F'}} |< y_i^j, y' >|^q \parallel y_i^j \parallel^{\frac{\sigma q}{1-\sigma}} d\lambda(y')\Big)^{\frac{1-\sigma}{q}} : \sum_{i=1}^{s_j} y_j^i = Tx_j\right\}^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}$$

is continuous with respect to this topology to see that inequality (2) holds for every $\lambda \in W(B_{F'})$. But this holds since $\int_{B_{F'}} |< y, y' >|^q \parallel y \parallel^{\frac{\sigma q}{1-\sigma}} d\lambda(y')$ is continuous for each $y \in F$.
Hence if $S \in \mathcal{P}_{q,\sigma}(F,)$, theorem 0.2 gives

$$1_{\frac{p}{1-\sigma}}((SoTx_j))$$
$$\leq \pi_{q,\sigma}(S)\Big(\sum_{j=1}^{m}\inf\left\{\sum_{i=1}^{s_j}\Big(\int_{B_{F'}} |< y_i^j, y' >|^q \parallel y_i^j \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(y')\Big)^{\frac{1-\sigma}{q}} : \sum_{i=1}^{s_j} y_j^i = Tx_j\right\}^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}$$
$$\leq C\pi_{q,\sigma}(S)\delta_{p,\sigma}((x_j)).$$

This means that $\pi_{p,\sigma}(SoT) \leq C\pi_{q,\sigma}(T)$ which completes the proof. □

Definition 1.3. *Consider $1 \leq p \leq q \leq \infty$ such that $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$. For any finite collection of vectors $x_1, \cdots, x_n \in E$ we set*

$$m_{(q,p,\sigma)}((x_i)) := \inf\left\{1_{\frac{r}{1-\sigma}}((\tau_i))\delta_{q,\sigma}((x_i^0)) : \forall i, x_i = \tau_i x_i^0, x_i^0 \in E\right\}$$

We are going to use this expression to characterize when a Banach space operator belongs to $\mathcal{M}_{(q,p,\sigma)}$. We need the following lemma.

Lemma 1.4. *For every $(x_i)_{i=1}^n \subset E$,*

$$m_{(q,p,\sigma)}((x_i)) = \sup\left\{\Big(\sum_{i=1}^n\Big(\int_{B_{E'}} |< x_i, x' >|^q \parallel x_i \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')\Big)^{\frac{p}{q}}\Big)^{\frac{1-\sigma}{p}} : \mu \in W(B_{E'})\right\}.$$

Proof. For every set of factorizations $x_i = \tau_i x_i^0$, $1 \leq i \leq n$, and every $\mu \in W(B_{E'})$ the following inequalities hold, just by applying Hölder's inequality with indexes r/p and q/p.

$$\Big(\sum_{i=1}^n\Big(\int_{B_{E'}} |< x_i, x' >|^q \parallel x_i \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')\Big)^{\frac{p}{q}}\Big)^{\frac{1-\sigma}{p}} =$$
$$= \Big(\sum_{i=1}^n\Big(|\tau_i|\Big(\int_{B_{E'}} |< x_i^0, x' >|^q \parallel x_i^0 \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')\Big)^{\frac{1-\sigma}{q}}\Big)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}} \leq$$
$$\leq \Big(\sum_{i=1}^n |\tau_i|^{\frac{r}{1-\sigma}}\Big)^{\frac{1-\sigma}{r}} \Big(\sum_{i=1}^n \int_{B_{E'}} |< x_i^0, x' >|^q \parallel x_i^0 \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')\Big)^{\frac{1-\sigma}{q}} \leq 1_{\frac{r}{1-\sigma}}((\tau_i))\delta_{q,\sigma}((x_i^0))$$

The other inequality holds in the same way that on proposition [9]16.4.3. using the set $\mathcal{F}$ of continuous convex function as

$$\phi_{\mu,\epsilon}((\xi_i)) := \sum_{i=1}^n (\xi_i + \epsilon)^{-\frac{q}{p}} \int_{B_{E'}} |< x_i, x' >|^q \parallel x_i \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')$$

defined on

$$\mathcal{K} := \left\{(\xi_i) \in K^n : \sum_{i=1}^n \xi_i^{\frac{r}{p}} \leq \theta^{\frac{p}{1-\sigma}}, \xi \geq 0\right\}$$

$$\text{where} \quad \theta := \sup\Big(\sum_{i=1}^n\Big(\int_{B_{E'}} |< x_i, x' >|^q \parallel x_i \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')\Big)^{\frac{p}{q}}\Big)^{\frac{1-\sigma}{p}}$$

for some $\epsilon > 0$. Taking $\xi_i = \Big(\int_{B_{E'}} |< x_i, x' >|^q \parallel x_i \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')\Big)^{\frac{p}{r+q}}$ for each $1 \leq i \leq n$, we obtain $\phi_{\mu,\epsilon}((\xi_i)) \leq \theta^{\frac{p}{1-\sigma}}$ and $\sum_{i=1}^n \xi_i^{\frac{r}{p}} \leq \theta^{\frac{p}{1-\sigma}}$, since $\frac{r+q}{r} = \frac{q}{p}$ and

$(\frac{p}{r+q})^{\frac{r}{p}} = \frac{p}{q}$. Since the set $\mathcal{F}$ is concave and for each function $\phi_{\mu,\epsilon}$ there is an element $(\xi_i) \in \mathcal{K}$ such that $\phi_{\mu,\epsilon}((\xi_i)) \leq \theta^{\frac{p}{1-\sigma}}$ we can apply Ky Fan's lemma (see for example E.4.[9]) in order to obtain an element $(\xi_i^0) \in \mathcal{K}$ verifying $\phi_{\mu,\epsilon}((\xi_i^0)) \leq \theta^{\frac{p}{1-\sigma}}$ for all $\phi_{\mu,\epsilon}$ simultaneously. Now, if we define $\tau_i = |\xi_i^0|^{\frac{1-\sigma}{p}}$ and $x_i^0 = \tau_i^{-1} x_i$ the inequality $1_{\frac{r}{1-\sigma}}((\tau_i))\delta_{q,\sigma}((x_i^0)) \leq \theta$ holds, using the fact that

$$\Big(\sum_{i=1}^{n} |< x_i^0, x' >|^q \parallel x_i^0 \parallel^{\frac{\sigma q}{1-\sigma}} d\mu(x')\Big)^{\frac{1-\sigma}{q}} = \lim_{\epsilon \to 0} \Big(\sum_{i=1}^{n} (\xi_i + \epsilon)^{-\frac{q}{p}} |< x_i, x' >|^q \parallel x_i \parallel^{\frac{\sigma q}{1-\sigma}}\Big)^{\frac{1-\sigma}{q}}$$
$$\leq \theta^{q\frac{1-\sigma}{p}}$$

for each $x' \in E'$ verifying $\parallel x' \parallel \leq 1$ as can be deduced from $\phi_{\delta_{x'},\epsilon}((\xi_i^0)) \leq \theta^{\frac{p}{1-\sigma}}$, where $\delta_{x'}$ is the Dirac measure at the point x'. This proves the lemma. □

Proposition 1.5. *Let $T \in \mathcal{L}(E, F)$. The following two statements are equivalent:*

(i) There is a $C_1 > 0$ such that for every $(x_i)_{i=1}^n \subset E$,

$$m_{(q,p,\sigma)}((Tx_i)) \leq C_1 \delta_{p,\sigma}((x_i)).$$

(ii) There is a $C_2 > 0$ such that for every $(x_i)_{i=1}^n \subset E$ and $(y_k')_{k=1}^m \subset F'$ the following inequality holds

$$\Big(\sum_{i=1}^{n}\Big(\sum_{k=1}^{m} |< Tx_i, y_k' >|^q \parallel Tx_i \parallel^{\frac{\sigma q}{1-\sigma}}\Big)^{\frac{p}{q}}\Big)^{\frac{1-\sigma}{p}} \leq C_2 \delta_{p,\sigma}((x_i)) 1_q^{1-\sigma}((y_k')).$$

Moreover, if T verifies these conditions, $\inf C_1 = \inf C_2$.

Proof. $(i) \to (ii)$ Let $T \in \mathcal{L}(E, F)$. Given $y_1', \cdots, y_k' \in F'$ we define the discrete probability μ as in theorem 1.2($(ii) \to (iii)$). We obtain in this way an integral expression of

$$\Big(\sum_{i=1}^{n}\Big(\sum_{k=1}^{m} |< Tx_i, y_k' >|^q \parallel Tx_i \parallel^{\frac{\sigma q}{1-\sigma}}\Big)^{\frac{p}{q}}\Big)^{\frac{1-\sigma}{p}}$$

for every $(x_i)_{i=1}^n$. Using the previous lemma the result holds. $(ii) \to (i)$ Take $(x_i)_{i=1}^n \subset E$. As in the case $(iii) \to (i)$ of theorem 1.2, $\Big(\sum_{i=1}^{n}\Big(\int_{B_{F'}} |< Tx_i, y_k' >|^q \parallel Tx_i \parallel^{\frac{\sigma q}{1-\sigma}}$

$d\nu\Big)^{\frac{p}{q}}\Big)^{\frac{1-\sigma}{p}} \leq C_2\delta_{p,\sigma}((x_i))$ holds for each discrete probability on $W(B_{F'})$. The fact that these probabilities are dense in $W(B_{F'})$ and lemma 1.4 complete the proof. □

Corollary 1.6 *Let $T \in \mathcal{L}(E,F)$. If T verifies (i) (and (ii)) of proposition 1.5, then $T \in \mathcal{M}_{(q,p,\sigma)}$.*

As inmediate consequences of the characterization given in theorem 1.2, the following corollaries hold.

Corollary 1.7. *Let $1 \leq p \leq q \leq r \leq \infty$ and $0 \leq \sigma < 1$. Then $\mathcal{M}_{(r,q,\sigma)} o \mathcal{M}_{(q,p,\sigma)} \subset \mathcal{M}_{(r,p,\sigma)}$.*

Corollary 1.8. *Let $1 \leq p_1 \leq p_2 \leq q_2 \leq q_1 \leq \infty$ and $0 \leq \sigma < 1$. Then $\mathcal{M}_{(q_1,p_1,\sigma)} \subset \mathcal{M}_{(q_2,p_2,\sigma)}$.*

Remark 1.9. Let $1 \leq p \leq q \leq r \leq \infty$ and $\mathcal{U}$ an operator ideal such that $\mathcal{P}_q o \mathcal{U} \subset \mathcal{P}_p$. Consider an operator $T \in \mathcal{U}(E,F)$, $0 \leq \sigma < 1$ and $S \in \mathcal{P}_{q,\sigma}(F,G)$. By definition 0.1, there exists an $S_0 \in \mathcal{P}_q$ satisfying $\| Sy \| \leq \| y \|^{\sigma} \| S_0 y \|^{1-\sigma}$ for every $y \in F$. Thus

$$\| STx \| \leq \| Tx \|^{\sigma} \| S_0Tx \|^{1-\sigma} \leq \| T \|^{\sigma} \| x \|^{\sigma} \| S_0Tx \|^{1-\sigma}$$

for every $x \in E$ and $S_0T \in \mathcal{P}_p$. This means that $ST \in \mathcal{P}_{p,\sigma}$. Hence

$$\mathcal{U} \subset \mathcal{M}_{(q,p)} \quad \text{implies} \quad \mathcal{U} \subset \mathcal{M}(q,p,\sigma)$$

As an inmediate consequence, if r verifies $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, then

$$\mathcal{P}_r \subset \mathcal{M}_{(q,p)} \subset \mathcal{M}_{(q,p,\sigma)}$$

2. The Inclusion $\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}_{(q,p,\sigma)}$.

The purpose of this section is to study sufficient conditions to assure $\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}_{(q,p)}(E,F)$. We obtain special results in this direction in the case $E = \mathcal{C}(K)$ and $E = L_1$. The following assertion gives the best q verifying $\mathcal{P}_{p,\sigma} \subset \mathcal{M}_{(q-\epsilon,p)}$ for every $\epsilon > 0$ and a fixed σ.

Proposition 2.1. *Let $p \geq 1$, $0 \leq \sigma < 1$ and $\epsilon > 0$. Then $\mathcal{P}_{p,\sigma} \subset \mathcal{M}_{(p/(\sigma(1+\epsilon)),p)}$ for each $\epsilon > 0$.*

$$\textit{Moreover,} \qquad \frac{p}{\sigma} = \sup\left\{q : \mathcal{P}_{p,\sigma} \subset \mathcal{M}_{(q,p)}\right\}.$$

Proof. By [6] the minimum q satisfying $\mathcal{P}_{p,\sigma} \subset \mathcal{P}_{(q,p)}$ is $\frac{p}{1-\sigma}$. On the other hand, if $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ then $\mathcal{P}_{(q,p)} \subset \mathcal{M}_{(s-\epsilon,p)} \forall \epsilon > 0$. Thus $\mathcal{P}_{p,\sigma} \subset \mathcal{M}_{(p/(\sigma(1+\epsilon)),p)}$ for each $\epsilon > 0$. Since $\mathcal{M}_{(s,p)} \subset \mathcal{P}_{(q.p)}$ if $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$, if we take a $s > p/\sigma$ then $\mathcal{P}_{p,\sigma} \subset \mathcal{P}_{(\frac{p}{1-\sigma}-\epsilon,p)}$, a contradiction. □

There are many examples of Banach spaces F such that this inclusion is satisfied for $\epsilon = 0$. For example, if either $F \in$ space $(\mathcal{M}_{(p/\sigma,p)})$ or $E \in$ space $(\mathcal{M}_{(p/\sigma,p)})$ (see [9] for notation) then $\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}_{(p/\sigma,p)}(E,F)$ is easily verified.

Even equality is available in certain cases:
Let $1 \leq p,q \leq \infty$ with $\frac{1}{q} = |\frac{1}{2} - \frac{1}{p}|$. Carl and Defant proved that $\mathcal{L}(l_1, l_p) = \mathcal{M}_{(q,1)}(l_1, l_p)$ for these p and q. This also means that $\mathcal{L}(l_1, L_p) = \mathcal{M}_{(q,2)}(L_1, L_p)$, since $\mathcal{M}_{(q,1)} \subset \mathcal{M}_{(q,2)}$ [9]. on the other hand, Matter obtained the equality $\mathcal{L}(L_1, L_p) = \mathcal{P}_{2,\sigma}(L_1, L_p)$ for $\frac{2}{1+\sigma} \leq p \leq \frac{2}{1-\sigma}$ (Theorem 9.1.(i) [6]). In particular, this means that $\mathcal{P}_{2,\sigma}(L_1, L_p) = \mathcal{M}_{(2/\sigma,2)}(L_1, L_p)$.

An application of proposition 2.1 to another results of Matter allows us to obtain the following corollary about operators on L_1 factoring through spaces of Schatten-Von Neumann classes $S_{p,q}$ and Lorentz spaces $L_{p,q}$. It holds just by applying 2.1 to theorem 9.2 of Matter [6] and Grothendieck's theorem.

Corollary 2.2. *Let F a Banach space, $p,q \leq 1$ and $q \leq \sigma < 1$ such that $\frac{2}{1+\sigma} < p,q < \frac{2}{1-\sigma}$. Suppose that $T \in \mathcal{L}(L_1, F)$ admites a factorization $T = T_1 o T_0$ through $B = L_{p,q}$ or $S_{p,q}$ verifying $T_1 \in \mathcal{P}_{\frac{2}{\sigma}}(B,F)$. Then $T \in \mathcal{P}_1(L_1, F)$.*

However, the inclusion $\mathcal{P}_{p,\sigma}(\mathcal{C}(K), F) \subset \mathcal{M}_{(p/\sigma,p)}$ does not hold for the general case. The following proposition characterizes those Banach those Banach spaces F such that the inclusion

$$\mathcal{P}_{1,\sigma}(\mathcal{C}(K), F) \subset (\mathcal{M}_{1/\sigma,1)}(\mathcal{C}(K), F) \quad \text{holds.}$$

Using this result we find a Banach space F such that inclusion is not true.

Proposition 2.3 *Let F be a Banach space and K compact set. The following assertions are equivalent for $0 < \sigma < 1$.*

(i) $\mathcal{P}_{1,\sigma}(\mathcal{C}(K), F) \subset \mathcal{M}_{(1/\sigma,1)}(\mathcal{C}(K), F)$.

(ii) $\mathcal{P}_{(\frac{1}{1-\sigma},1)}(\mathcal{C}(K), F) = \mathcal{P}_{\frac{1}{1-\sigma}}(\mathcal{C}(K), F)$.

(iii) For every Banach space G and every $T \in \mathcal{L}(\mathcal{C}(K), F)$, if T verifies that there is a probability measure λ on K such that there exist a factorization $T = T_1 o \bar{T} o I$ where I is the canonical injection $\mathcal{C}(K) \to L_{\frac{1}{1-\sigma},1}(\lambda)$, $\bar{T} \in \mathcal{L}(L_{\frac{1}{1-\sigma},1}(\lambda), F)$ and $T_1 \in \mathcal{P}_{\frac{1}{\sigma}}(F, G)$, then $T \in \mathcal{P}_1(\mathcal{C}(K), G)$.

Proof. $(i) \rightarrow (ii)$ For every Banach space F, $\mathcal{M}_{(1/\sigma,1)}(\mathcal{C}(K),F) = \mathcal{P}_{\frac{1}{1-\sigma}}(\mathcal{C}(K),F)$ holds (see [2] ex. 32.3). If $\mathcal{P}_{1,\sigma}(\mathcal{C}(K),F) \subset \mathcal{M}_{(1/\sigma,1)}(\mathcal{C}(K),F)$ then (ii) holds, since for $\mathcal{C}(K)$-spaces $\mathcal{P}_{(\frac{1}{1-\sigma},1)}(\mathcal{C}(K),F) = \mathcal{P}_{1,\sigma}(\mathcal{C}(K),F)$ is satisfied (see [7] and [11]). $(ii) \rightarrow (i)$ The inclusion $\mathcal{P}_{\frac{1}{1-\sigma}}(\mathcal{C}(K),F) \subset \mathcal{P}_{(\frac{1}{1-\sigma},1)}(\mathcal{C}(K),F)$ and (ii) imply (i). $(iii) \leftrightarrow (i)$ By (iv) of theorem 2.4 of [11], $\tilde{T}oI \in \mathcal{P}_{1,\sigma}(\mathcal{C}(K),F)$, and every $R \in \mathcal{P}_{1,\sigma}(C(K),F)$ can be factored in this way. On the other hand (iii) means that $R = \tilde{T}oI$ also belongs to $\mathcal{M}_{(1/\sigma,1)}(\mathcal{C}(K),F)$. □

Observe that $(ii) \rightarrow (i)$ holds for every space E, and not only for $E = \mathcal{C}(K)$.

Counterexaple 2.4 The equality (ii) is not valid for all F. Let $F = L_p([0,1])$ for $p = \frac{1}{1-\sigma} > 2$ and $E = L_\infty([0,1])$. Then by theorem 7 of [5], $\mathcal{L}(E,F) \neq \mathcal{P}_{\frac{1}{1-\sigma}}(E,F)$. However, by a theorem of Orlicz, $\mathcal{L}(E,F) = \mathcal{P}_{(\frac{1}{1-\sigma},1)}(E,F)$ (see e.g. 22.6.2 [9]).

Remark 2.5. Consider the canonical map $J_{p,\sigma} : E_\mu \rightarrow (E_\mu, L_p(B_{E'},\mu))_{1-\sigma,1}$ for a given probability μ defined on $B_{E'}$. The canonical inclusion $J_p : E_\mu \rightarrow L_p(B_{E'}\mu)$ is p-absolutely summing and thus $J_p \in \mathcal{M}_{(\infty,p)}$. Obviously the identity map of E_μ is continuous and thus belongs to $\mathcal{L}(E_\mu,E_\mu) = \mathcal{M}_{(p,p)}(E_\mu,E_\mu)$. Taking $s = p/\sigma$, $s_o = p$ and $s_1 = \infty$, and applying 20.1.13 [9] we obtain that

$$J_{p,\sigma} \in \mathcal{M}_{(p/\sigma,p)}(E_\mu, L_p(B_{E'},\mu))_{1-\sigma,1}).$$

This implies that for every couple of Banach spaces (E,F), $\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}_{(p/\sigma,p)}(E,F)$, a contradiction. These arguments show that there is a flow in [9] 20.1.13. However, there is no problem with the application of proposition 20.1.13 in the proof of theorem 20.1.15 [9], since 20.1.13 is only used there for the case $F_0 = F_1 = F$.

3. Products of (p,σ)-Absolutely Continuous Operators.

The following proposition extends in a certain sense the classical Pietsch result about products of p-absolutely summing operators [9]: $\mathcal{P}_q o \mathcal{P}_p \subset \mathcal{P}_r$ if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $p,q,r \geq 1$.

Proposition 3.1. *Let $0 \leq \sigma < 1$ and $1 \leq r,p,q \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then*

$$\mathcal{P}_{q,\sigma} o \mathcal{P}_{\frac{p}{\sigma(1+\epsilon)}} o \mathcal{P}_{r,\sigma} \quad \textit{for each} \quad \epsilon > 0.$$

Moreover, this inclusion is also valid for $\epsilon = 0$ for couples (E,F) of Banach spaces which satisfy $\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}_{(p/\sigma,p)}(E,F)$.

Proof. If $\sigma = 0$ then $\mathcal{P}_{\frac{p}{\sigma(1+\epsilon)}} = \mathcal{L}$ and nothing is to prove. If $\sigma > 0$, proposition 2.1 and remark 1.9 give the result.

Finally, we give some results for the case $E = \mathcal{C}(K)$. □

Proposition 3.2. *Let* $1 \le r, q \le \infty$ *such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\epsilon > 0$ *and* $\frac{1}{\frac{p}{1-\sigma}} + \frac{1}{(\frac{p}{1-\sigma})'} = 1$. *The following inclusions hold for all compact* K *and for all couple of Banach spaces* (F, G).

i) $\mathcal{P}_{\frac{q}{1-\sigma},\sigma}(F,G) o \mathcal{P}_{p,\sigma}(\mathcal{C}(K),F) \subset \mathcal{P}_{\frac{r}{1-\sigma}+\epsilon,\sigma}(\mathcal{C}(K),G)$

ii) $\mathcal{P}_{(\frac{p}{1-\sigma})'-\epsilon,\sigma}(F,G) o \mathcal{P}_{p,\sigma}(\mathcal{C}(K),F) \subset \mathcal{P}_{1,\sigma}(\mathcal{C}(K),G)$.

Proof. $\mathcal{P}_{\frac{p}{1-\sigma}}(\mathcal{C}(K),F) \subset \mathcal{P}_{p,\sigma}(\mathcal{C}(K),F) \subset \bigcap_{\epsilon>0} \mathcal{P}_{\frac{r}{1-\sigma}+\epsilon}(\mathcal{C}(K),F)$ are satisfied (see [6] 5.2). This means that the inclusion

$$\mathcal{P}_{\frac{q}{1-\sigma}}(F,G) o \mathcal{P}_{p,\sigma}(\mathcal{C}(K),F) \subset \mathcal{P}_{\frac{r}{1-\sigma}+\epsilon}(\mathcal{C}(K),G)$$

holds, and by remark 1.9, i) holds.
On the hand, since for each $\epsilon > 0$

$$\mathcal{P}_{p,\sigma}(\mathcal{C}(K),G) = \mathcal{P}_{(\frac{p}{1-\sigma},1)}(\mathcal{C}(K),G) \subset \mathcal{M}_{((\frac{p}{1-\sigma})'-\epsilon,1)}(\mathcal{C}(K),G)$$

(see [10], [11], [6] or [7], and [9]), we also have the inclusion ii) just by an application of remark 1.9. □

References

[1] Carl, B.; Defant, A. Tensor products and Grothendieck type inequalities of operators in L_p-spaces. Trans. Am. Math. Soc. 331, 1 (1992) 55-76.

[2] Defant, A.; Floret, K. *Tensor norms and Operator Ideals.* North-Holland Mathematics Studies 176. Amsterdam-London-New York-Tokyo 1993.

[3] Jarchow, H. *Locally Convex Spaces.* B.G. Teubner, Stuttgart 1981.

[4] Jarchow, H; Matter, U. Interpolative constuctions for operator ideals. Note di Matematica Vol VIII, 1, (1988) 45-56.

[5] Kwapien', S. On a theorem of L. Schwartz and its applications to absolutely summing operators. Studia Math. XXXVIII (1970) 193-201.

[6] Matter, U. Absolutely Continuous operators and Super-Reflexivity. Math. Nachr. 130 (1987)193-216.

[7] Matter, U. Factoring through interpolation spaces and super-reflexive Banach spaces. Rew. Roumane Math. Pures Appl. 34, (1989) 147-156.

[8] Niculescu, C. Absolute continuity in Banch space theory. Rew. Roumane Math. Pures Appl. 24, (1979) 413-422.

[9] Pietsch, A. *Operator Ideals.* North-Holland Publishing Company, Amsterdam-New York-Oxford 1980.

[10] Pisier, G. Factorisation des opérateurs (q,p)-sommants sur les C^*-algébres. C.R. Acad. Sc. Paris. Série I, 8 (1985) 403-405.

[11] Pisier, G. Factorization of Operators Through $L_{p,\infty}$ or $L_{p,1}$ and Non-Commutative Generalizations. Math. Ann. 276 (1986) 105-136.

[12] Räbiger, F. *Absolutstetigkeit und Ordnungsabsolutstetigkeit von Operatoren.* Reports of the Heidelberg Academy of Science. Section for Mathematics and Natural Sciences, 91-1. Springer-Verlag, Berlin 1991.

(p,σ) MUTLAK SÜREKLİ OPERATÖR İDEALLERİN ÇARPIMLARI VE BÖLÜMLERİ ÜZERİNE

Özet

Bu makalede A. Pietsch in tanımladığı $M_{(q,p)}$ operatör ideallerin bir genelleştirilmesi tanımlanım, bu kavramın çeşitli uygulamaları ele alınmıştır.

Enrique A. Sánchez PÉREZ,
Departamento de Matematica Aplicada.
E.T.S. Ingenieros Agronomos.
Universitat Politecnica de Valencia.
E-46071 Valencia SPAIN

Received 3.1.1995