Tr. J. of Mathematics
20 (1996) , 335 – 342.

OPERATIONS WITH THE PERIODIC DECIMAL EXPANSIONS

H. Ardahan

Abstract

In this paper, we prove the rules of direct addition and subtraction for the finite decimal expansions of fractions which are periodic. It has been shown that these rules are valid for the fractions which can be expanded as a periodic decimal with p figures in the period or have the mixed decimal part containing ν non-periodic and p periodic figures. Also, it has been given a rule of multiplication for these periodic decimals by $10^{\nu}, \nu \in \mathbb{N}$. Last of all, if a rational fraction has a period of length p, then it can be expressed by a decimal expansion, containing np-periodic figures, where n denotes the number of repetition of p-periodic figures.

Keywords: Periodic decimals, decimal expansions.

1. Introduction

Notations given in this paper are usual notations of Number Theory [1,2,3]. Method and techniques are elemntary and original.

These rules are for themselves worthy of arithmetical studies, and moreover they will lead to a new approach to the arithmetical operations.

2. Theorems and Proofs

Let $\mathbb{N} = Z^{+} \sqcup \{0\}$ and $\beta = \{0, 1, 2, \ldots 9\}$ be the base of the number system.

Lemma 1. If $r \in Q$ is a rational fraction and $t, \overline{a}$ is a decimal expansion of r, then for every integer n grater than 1, we express the fraction in the following way:

$$t, \bar{a} = t, \underset{\text{n-digits}}{\overline{aa \ldots a}}$$

Proof. Let A and B defined as $A = (99 \ldots 9)^{-1}(taa \ldots a - t)$ and $B = 9^{-1}(ta - t)$. To show that A is congruent to B, we enlarge the fraction B by $(11 \ldots 1)$ with finite n digits. Thus, we have

*AMS Subject Classification 46A40

$$B = \frac{(11\ldots1)(ta-t)}{99\ldots9} = \frac{(11\ldots1)(10t+a-t)}{99\ldots9} = \frac{(99\ldots9)t+aa\ldots a}{99\ldots9} = t,\overline{aa\ldots a} = A.$$

We can also use induction method to prove this Lemma. □

Lemma 2. For every $a,b,c,d \in \beta$ with $ab+cd > 99$ and $ab+cd = k\ell m$ $(k,\ell m \in \beta)$, we have

$$\frac{k\ell m}{99} = k + \frac{\ell m + k}{99} = k,\overline{\ell m + k}$$

Proof. According to the hypothesis, we have

$$\frac{k\ell m}{99} = \frac{100k+10\ell+m}{99} = \frac{99k+10\ell+m+k}{99} = k + \frac{\ell m + k}{99} = k,\overline{\ell m + k}.$$

In this case, it can always be shown that $k=1$. Indeed, if $ab+cd = k\ell m$ with three digits, then we have

$$\max(k\ell m) \leq \max(ab) + \max(cd) = 99+99 = 198,$$

and so we conclude that $k=1$. Therefore, we can express Lemman 2 in the following way:

$$(99)^{-1}k\ell m = 1,\overline{\ell m + 1}$$

□

Corollary 1. *If $A = t,\bar{a}$ and $B = p,\bar{b}$, then we have*
i) $A+B = t+p,\overline{a+b}$, *if* $a+b<9$, **ii)** $A+B = t+p+k,\overline{k+\ell}$, *if* $a+b \geq 9$ *and* $a+b = k\ell$ *for all* $t,p \in \mathbb{N}$ *and* $a,b,\in \beta$.

Corollary 2. *If $A = t,\overline{ab}$ and $B = p,\bar{c}$ then we have*
i) $A+B = t+p,\overline{ab+cc}$, *if* $ab+cc < 99$ **ii)** $A+B = t+p+k,\overline{\ell m+k},$, *if* $ab+cc = k\ell m \geq 99$, *where* $t,p \in \mathbb{N}$ *and* $a,b,c \in \beta$.

Theorem 3. *If* $A = t, \overline{a_1 a_2 \dots a_n}$, $B = p, \overline{b_1 b_2 \dots b_n}$ *and* $S = a_1 a_2 \dots a_n + b_1 b_2 \dots b_n$, *then for* $S = k_1 k_2 \dots k_n k_{n+1}$ *we have*

i) $A + B = t + p, \overline{S}$, *if* $S < 99 \dots 9$,

ii) $A + B = t + p + k_1, \overline{k_2 k_3 \dots k_n k_{n+1} + k_1}$, *if* $S \geq 99 \dots 9$,

where $t, p \in \mathbb{N}, a_i, b_i, k_i \in \beta$, $1 \leq i \leq n$ *and* $(99 \dots 9)$ *with finite n digits.*

Proof. i) If $S < 99 \dots 9$ then we obtain the required result from Lemman 1.

ii) If $S \geq 99 \dots 9$ then $S = k_1 k_2 \dots k_n k_{n+1}$ with finite $(n+1)$ digits and

$$\begin{aligned}\frac{S}{99\dots 9} &= \frac{(99\dots 9)k_1 + k_2 k_2 \dots k_n k_{n+1} + k_1}{99\dots 9} \\ &= k_1 + \frac{k_2 k_3 \dots k_n k_{n+1} + k_1}{99\dots 9} = k_1, \overline{k_2 k_3 \dots k_n k_{n+1} + k_1}.\end{aligned}$$

This gives the required result.

If a rational fraction has a mixed decimal part, then can we reorder this in another way? Yo will find the answer in the following theorem. □

Theorem 4. *If* $s, t \in \mathbb{N}$ *and* $a_1, a_2 \in \beta$ *then we have*

$$t, s\overline{a_1 a_2} = t, s a_1 \overline{a_2 a_1}. \tag{1.1}$$

Proof.

$$\begin{aligned}A &= t, s\overline{a_1 a_2} = \frac{tsa_1a_2 - ts}{990} = \frac{10^3 t + 10^2 s + 10a_1 + a_2 - 10t - s}{990} \\ &= \frac{990t + 99s + 10a_1 + a_2}{990} = t + \frac{1}{10}[\frac{99s + 10a_1 + a_2}{99}], \\ B = t, sa_1\overline{a_2 a_1} &= \frac{tsa_1a_2a_1 - tsa_1}{9900} = \frac{(10^4 - 10^2)t + (10^3 - 10)s + 10^2 a_1 + 10a_2}{9900} \\ &= \frac{99.10^2 t + 99.10s + 10^2 a_1 + 10a_2}{9900} = t + \frac{1}{10}[\frac{99s + 10a_1 + a_2}{99}].\end{aligned}$$

We conclude that the mixed decimal A is congruent to B. □

Example 4.1 Let $A = 2, 5\overline{27}, B = 14, 34\overline{54}$. Then we find the value of A+B in the following way:

$$A + B = 2, 5\overline{27} + 14, 34\overline{54} = 2, 52\overline{72} + 14, 34\overline{54} = 16, 87\overline{27}, \tag{1.2}$$

$$A + B = 2, 5\overline{27} + 14, 34\overline{54} = 2, 527\overline{27} + 14, 345\overline{45} = 16, 872\overline{72}, \tag{1.3}$$

Let us now prove the eqwuivalency of (1.1) and (1.2). From the result of Theorem 4, it follows that

$$16,872\,\overline{72} = 16,8\,\overline{72} = 16,87\,\overline{27}.$$

Therefore, we get a rule of direct addition for the mixed decimals and multiplication by $10^n, n \in \mathbb{N}$ in any case.

3. Subtraction of Pure and Mixed Decimals

Theorem 5. Let $A = t,\overline{a_1a_2\ldots a_n}$ and $B = p,\overline{b_1b_2\ldots b_n}$ be two pure decimal expansions containing n-periodic figures, where $t,p,n \in \mathbb{N}$ and $a_ib_i \in \beta(1 \le i \le n)$. Then we have

i) $A\backslash B = t-p,\overline{a_1a_2\ldots a_n - b_1b_2\ldots b_n}$, if $a_1a_2\ldots a_n \ge b_1b_2\ldots b_n$,

ii) $A\backslash B = t-p-1,\overline{99\ldots 9 + a_1a_2\ldots a_n - b_1b_2\ldots b_n}$, if $a_1a_2\ldots a_n < b_1b_2\ldots b_n$.

Proof. i) From Lemma 1 and Theorem 3, we have

$A = t + (a_1a_2\ldots a_n)(99\ldots 9)^{-1}, \qquad B = p + (b_1b_2\ldots b_n)(99\ldots 9)^{-1}$,

and thus

$$A - B = t-p,\overline{a_1a_2\ldots a_n - b_1b_2\ldots b_n},$$

since $a_1a_2\ldots a_n - b_1b_2\ldots b_n < 99\ldots 9$.

ii) In the similar way, since $a_1a_2\ldots a_n - b_1b_2\ldots b_n < 0$ and $(99\ldots 9)+a_1a_2\ldots a_n - b_1b_2\ldots b_n < 99\ldots 9$, we have

$A - B = t-p-1+[(99\ldots 9+) + a_1a_2\ldots a_n - b_1b_2\ldots b_n](99\ldots 9)^{-1}$

or

$$A - B = t-p-1,\overline{99\ldots 9 + a_1a_2\ldots a_n - b_1b_2\ldots b_n}.$$

This completes the proof of Theorem 5. □

Theorem 6. *Let $A = t, s_1s_2\ldots s_\mu\overline{a_1a_2}$ and $B = p, k_1k_2\ldots k_v\overline{b_1b_2}$ be mixed decimal expansions containing maximum v non-periodic and two periodic figures. For al $S_j, k_j \in \mathbb{N}, (1 \le j \le v)$ and $a_i, b_i \in \beta(1 \le i \le 2)$ we have*

i) $A - B = 10^{-v}(T-P),\overline{a_1a_2 - b_1b_2}$, if $\mu \le v$ and v is even and $a_1a_2 \ge b_1b_2$

$A - B = 10^{-v}(H-P),\overline{a_2a_1 - b_1b_2}$, if $\mu \le v$ and v is odd and $a_2a_1 \ge b_1b_2$

ii) $A\backslash B = 10^{-v}(T-P-1),\overline{99 + a_1a_2 - b_1b_2}$, if v is even and $a_1a_2 < b_1b_2$

$A\backslash B = 10^{-v}(H-P-1),\overline{99 + a_2a_1 - b_1b_2}$, if v is odd and $a_2a_1 < b_1b_2$

where $T = ts_1s_2\ldots s_\mu a_1a_2\ldots a_1a_2, a_1a_2; P = pk_1k_2\ldots k_v,\overline{b_1b_2}$;

$$H = t\,s_1s_2\ldots s_\mu a_1a_2a_1\ldots a_2a_1,\overline{a_2a_1}$$

Proof. **i)** From Theorem 4, we have

$$10^v A = t\, \underbrace{s_1 s_2 \ldots s_\mu a_1 a_2 \ldots a_1 a_2}_{(v+1)\text{digits}}, \overline{a_1 a_2} \text{ and } 10^v B = \underbrace{p k_1 k_2 \ldots k_v}_{(v+1)\text{digits}}, \overline{b_1 b_2}.$$

Hence

$$10^v(A\backslash B) = T - P, \overline{a_1 a_2 - b_1 b_2}, \text{ if } 0 \leq a_1 a_2 - b_1 b_2 < 99 \text{ and } v \text{ is an even number}$$
$$10^v A = t\, s_1 s_2 \ldots s_\mu a_1 a_2 a_1 \ldots a_2 a_1, \overline{a_2 a_1} = H, \overline{a_2 a_1}, \text{ if } \mu \leq v \text{ and } v \text{ is an odd number}$$

and so

$$10^v(A\backslash B) = H - P, \overline{a_2 a_1 - b_1 b_2}, \text{ since } 0 \leq a_2 a_1 - b_1 b_2 < 99.$$

ii) Similarly, we deduce that
$10^v(A\backslash B) = T - P - 1, \overline{99 + a_1 a_2 - b_1 b_2}$, if v is an even number,
since $a_1 b_2 - b_1 b_2 < 0$ implies $99 + a_1 a_2 - b_1 b_2 < 99$ and
$10^v(A\backslash B) = H - P - 1, \overline{99 + a_2 a_1 - b_1 b_2}$, if v is an odd number. □

Example 6.1. Let $A = 2,23\overline{45}$ and $B = 5,467\overline{34}$ be mixed decimals. Let us find the value of A-B:

Using Theorem 6, we write

$$10^3 A = 2234, \overline{54} \text{ and } 10^3 B = 5467, \overline{34}.$$

Hence

$$10^3(A - B) = -3233, \overline{54 - 34} \text{ or } A - B = -3,233\overline{20} = -(3,232\overline{79}.$$

Example 6.2. If $A = 4,57\overline{63}$ and $B = 28,478\overline{57}$, then $A - B = -(23,902\overline{21})$. Indeed,

$$\begin{aligned} 10^3(A - B) &= 4576, \overline{36} - 28478, \overline{57}(36 < 57) \\ &= -23902 - 1, \overline{99 + 36 - 57}, \text{ (by Theorem 6, ii)} \\ &= -23903, \overline{78} \\ &= -(23902, \overline{21}) \end{aligned}$$

and this impliles the required result.

Theorem 7. (Period Theorem). *If $r \in Q$ has pure periodic decimal expansion containing p-periodic figures then the decimal can be expressed as purely periodic decimal having a period of length np, where $n \in \mathbb{N}$ denotes the repeating number of p-periodic figures.*

Proof. Let t be any integer and $r_p = t, \overline{a_1 a_2 \dots a_p}$ be a pure periodic decimal expansion of r. From Lemma 1, we write

$$r_p = [(10^{-p} - 1)t + a_1 a_2 \dots a_p)](99\dots9)^{-1} = t + (a_1 a_2 \dots a_p)(\underbrace{99\dots9}_{\text{p-digits}})^{-1} \tag{1.4}$$

and

$$r_{np} = t + \overline{(a_1 a_2 \dots a_p) \dots (a_1 a_2 \dots a_p)} = t + (a_1 a_2 \dots a_p \dots a_1 a_2 \dots a_p)(\underbrace{99\dots9\dots99\dots9}_{\text{np digits}})^{-1} \tag{1.5}$$

Now let us prove the equivalency of (1.4) and (1.5), i.e.,

$$(a_1 a_2 \dots a_p)(99\dots9)^{-1} = (a_1 a_2 \dots a_p \dots a_1 a_2 \dots a_p), (99\dots9\dots99\dots9)^{-1}.$$

□

Remark If a number has p digits and 10^{p-1} th digit is u and another number has d digits and v is any number in d digits and $u.v \le 9$, then the product of these numbers have p+(d-1) digits. If $u.v > 9$, then the product of these numbers have (p+d) digits.

So, we write

$$\begin{aligned} a_1 a_2 \dots a_p \dots a_1 a_2 \dots a_p &= (10^{np-1} + 10^{np-1-p} + 10^{np-1-2p} + \dots + 10^{p-1})a_1 + \\ &+ (10^{np-2} + 10^{np-2-p} + 10^{np-2-2p} + \dots + 10^{p-2})a_2 \\ &+ \dots \\ &+ (10^{np-p} + 10^{np-2p} + 10^{np-3p} + \dots + 10^{p} + 10^{p-p})a_p. \end{aligned}$$

Rearranging the above form, we have

$$\begin{aligned} a_1 a_2 \dots a_p \dots a_1 a_2 \dots a_p = &\ (\underbrace{100\dots0}_{\text{np-1 zeros}} + \underbrace{100\dots0}_{\text{np-1-p zeros}} + \underbrace{100\dots0}_{\text{np-1-2p zeros}} + \dots + \underbrace{100\dots0}_{\text{p-1 zeros}})a_1 \\ &+ (\underbrace{100\dots0}_{\text{np-2 zeros}} + \underbrace{100\dots0}_{\text{np-2-p zeros}} + \dots + \underbrace{100\dots0}_{\text{p-2 zeros}})a_2 + \end{aligned}$$

$$
\begin{aligned}
&+ \dots \\
&+ (100\dots0 + 100\dots0 + \dots + 1)a_p \\
&= (\underbrace{100\dots0}_{\text{p-1 zeros}}1\underbrace{00\dots0}_{\text{p-1 zeros}}1\,00\dots0\,1\underbrace{00\dots0}_{\text{p-1 zeros}}1)(a_1a_2\dots a_p). \quad (1.6)
\end{aligned}
$$

In this form the repeating number of the block $(100\dots0)$ containing p digits is $(n-1)$ by the Remark. If $a_1 = a_2 = \dots = a_p = 9$, then the result (1.6) is also correct. Hence

$$\frac{(a_1a_2\dots a_p\dots a_1a_2\dots a_p}{99\dots9\dots99\dots9} = \frac{100\dots0100\dots0\dots100\dots01)(a_1a_2\dots a_p)}{(100\dots0100\dots0\dots100\dots01)(99\dots9)}$$

As a result, if a number has $p(n-1)+1$ digits and another number has p digits, then the product of these numbers has $p(n-1)+1+p-1 = np$ digits by the Remark. This is the required result given in Theorem 7.

Finally, I thank to Doç. Dr Mehmet H. Oryan for his valuable guidance in preparing this paper.

References

[1] Dickson, L. E.,: History of theTheory of Numbers, Vol.1, Chelsea Pub. Comp., New York, 1952.

[2] Hardy, G. H. and Wright, E. M.,: an Introduction to the Theory of Numbers, Oxford University Press, London, 1954.

[3] Le Veque, W. J.,: Elementary Theory of Numbers, Addison-Wesley Publishing Company, New York, 1962.

PERİYODİK ONDALIK AÇILIMLARLA İŞLEMLER

Özet

Bu makalede kesirlerin periyodik olan sonlu ondalık açılımları için doğrudan toplama ve çıkarma kuralları ispatlandı. Bu kuralların, periyodunda p rakam olan ondalık açılım şeklinde ifade edilen kesirler veya ondalık kısmında p periyodik rakam ve v periyodik olmayan rakam içeren ondalık açılımlara sahip kesirler için de geçerli olduğu gösterildi. ayrıca, ondalık açılımların $10^v, v \in \mathbb{N}$ ile çarpım kuralı verildi. Son olarak, bir rasyonel kesirin p uzunluğunda bir periyoda sahip olması durumunda np-periyodik rakama sahip bir ondalık açılım ile ifade edilebileceği gösterildi. (Burada n,p-periyotlu rakamın tekrar sayısını göstermektedir).

Halil Ardahan
Selçuk University, Faculty of Education,
Department of Mathematics
42090 Konya-TURKEY

Received 16.1.1995