Tr. J. of Mathematics
20 (1996) , 343 – 351.

A NOTE ON INTUITIONISTIC SETS AND INTUITIONISTIC POINTS

Doğan Çoker

Abstract

The purpose of this note is to define the so-called "intuitionistic sets" and "intuitionistic points", and obtain their fundamental properties.

Key words: Intuitionistic set; intuitionistic point.

1. Introduction

After the introduction of the concept of fuzzy set by Zadeh [8] several researches were conducted on the generalizations of the notion of fuzzy set. The idea of "intuitionistic fuzzy set" was first given by Krassimir T. Atanassov [1]. Here we shall present the classical version of this concept.

2. Preliminaries

Here we shall present the fundamental definitions. The following one is obviously inspired by K. T. Atanassov [1,2,3,4]:

Definition 2.1. *Let X be a nonempty fixed set. An intuitionistic set (IS for short) A is an object having the form*

$$A =< x, A_1, A_2 >$$

where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. The set A_1 is called the set of members of A, while A_2 is called the set of nonmembers of A.

Remark 2.2. An intuitionistic set $A =< x, A_1, A_2 >$ in X can be identified to an ordered pair in $2^X x 2^X$ or to an element in $(2x2)^X$.

AMS Subject Classification: 04A99

Example 2.3. Every set A on a nonempty set X is obviously an IS having the form $< x, A, A^{\subset} >$.

One can define several relations and operations between IS's as follows:

Definition 2.4. *(cf.[3.4.5]) Let X be a nonempty set, and the IS's A and B be in the form $A =< x, A_1, A_2 >$, $B =< x, B_1, B_2 >$, respectively. Then*

(a) $A \subseteq_{[\,]} B$ *iff* $A_1 \subseteq B_1$; (b) $A \subseteq_{\langle\rangle} B$ *iff* $A_2 \supseteq B_2$;
(c) $A \subseteq B$ *iff* $A \subseteq_{[\,]} B$ *and* $A \subseteq_{\langle\rangle} B$;
(d) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$;
(e) $A \sqsubset B$ *iff* $A_1^{\subset} \cap A_2^{\subset} \subseteq B_1^{\subset} \cap B_2^{\subset}$ *or, equivalently,* $A_1 \cup A_2 \supseteq B_1 \cup B_2$;
(f) $\bar{A} =< x, A_2, A_1 >$;
(g) $A \cap B =< x, A_1 \cap B_1, A_2 \cup B_2 >$; (h) $A \cup B =< x, A_1 \cup B_1, A_2 \cap B_2 >$;
(i) $A - B = A \cap \bar{B}$;
(j) $[\,]A =< x, A_1, A_1^{\subset} >$; (k) $\langle\rangle A =< x, A_2^{\subset}, A_2 >$.

Notice that ordinary sets can be interpreted as IS's. Now we can easily generalize the operations of intersection and union in Definition 2.4 to arbitrary family of IS's as follows:

Definition 2.5 *(cf.[5]) Let $\{A_1 : i \in J\}$ be an arbitrary family of IS's in X, where $A_i =< x, A_i^{(1)}, A_i^{(2)} >$. Then*

$$(a)\ \cup A_i =< x, \cup A_i^{(1)}, \cap A_i^{(2)} >, \quad (b)\ \cap A_i =< x, \cap A_i^{(1)}, \cup A_i^{(2)} >.$$

Definition 2.6. *(cf.[5])* $\underset{\sim}{\phi} =< x, \phi, X >$ *and* $\underset{\sim}{X} =< x, X, \phi >$.

Here are the basic properties of inclusion and complementation:

Corollary 2.7. *Let A, B, C and A_i be IS's in X $(i \in J)$. Then*

(a) $A \subseteq B$ *and* $B \subseteq C \Longrightarrow A \subseteq C$
(b) $A_i \subseteq B$ *for each* $i \in J \Longrightarrow \cup A_i \subseteq B$
(c) $B \subseteq A_i$ *for each* $i \in J \Longrightarrow B \subseteq \cap A_i$
(d) $\overline{\cup A_i} = \cap \overline{A_i}$ (e) $\overline{\cap A_i} = \cup \overline{A_i}$
(f) $A \subseteq B \Longleftrightarrow \overline{B} \subseteq \overline{A}$ (g) $(\overline{A})^{-} = A$
(h) $\overline{\underset{\sim}{\phi}} = \underset{\sim}{X}$ (i) $\overline{\underset{\sim}{X}} = \underset{\sim}{\phi}$

Proof. Obvious. □

Let us now rephrase the relations between the operators

$$[\,]: 2^X x 2^X \to 2^X x 2^X, \quad \langle\rangle : 2^X x 2^X \to 2^X x 2^X,$$

other intuitionistic set operators and complementation:

Proposition 2.8. *Let A and A_i be IS's in X $(i \in J)$. Then*

(a) $[\,](\cup A_i) = \cup([\,]A_i)$ (b) $\langle\rangle(\cup A_i) = \cup(\langle\rangle A_i)$

(c) $[\,](\cap A_i) = \cap([\,]A_i)$ (d) $\langle\rangle(\cap A_i) = \cap(\langle\rangle A_i)$

(e) $[\,]([\,]A) = [\,]A$ (f) $\langle\rangle(\langle\rangle A) = \langle\rangle A$

(g) $\langle\rangle([\,]A) = [\,]A$ (h) $[\,](\langle\rangle A) = \langle\rangle A$

(i) $\overline{[\,]A} = \langle\rangle\overline{A}$ (j) $\overline{\langle\rangle A} = [\,]\overline{A}$

Proof. Straightforward. □

Proposition 2.9. *Let A, B, C be IS's in X. Then*

$$(a)\ A \sqsubset B \text{ and } B \sqsubset C \Longrightarrow A \sqsubset C, \quad (b)\ A \sqsubset B \text{ iff } \overline{A} \sqsubset \overline{B}.$$

Now we shall define the image and preimage of IS's. Let X and Y be two nonempty sets and $f : X \to Y$ a function.

Definition 2.10. *(cf.[5])*

(a) If $B =< y, B^{(1)}, B^{(2)} >$ is an IS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the IS in X defined by

$$f^{-1}(B) =< x, f^{-1}(B^{(1)}), f^{-1}(B^{(2)}) > .$$

(b) If $A =< x, A^{(1)}, A^{(2)} >$ is an IS in X, then the image of A under f, denoted by f(A), is the IS in Y defined by

$$f(A) =< y, f(A^{(1)}), f_-(A^{(2)}) >$$

where $f_-(A^{(2)}) = Y - (f(X - A^{(2)}))$.

Now we list the properties of images and preimages:

Corollary 2.11. *(cf.[5,7]) Let A, A_i $(i \in J)$ be IS's in X, B, B_j $(j \in K)$ IS's in Y and $f : X \to Y$ a function. Then*

(a) $A_1 \subseteq A_2 \Longrightarrow f(A_1) \subseteq f(A_2)$ *(b)* $B_1 \subseteq B_2 \Longrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(c) $A \subseteq f^{-1}(f(A))$ *and if f is 1-1, then* $A = f^{-1}(f(A))$.

(d) $f(f^{-1}(B)) \subseteq B$ *and if f is onto, then* $f(f^{-1}(B)) = B$.

(e) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$ *(f)* $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$

(g) $f(\cup A_i) = \cup f(A_i)$

(h) $f(\cap A_i) \subseteq \cap f(A_i)$ *and if f is 1-1, then* $f(\cap A_i) = \cap f(A_i)$.

(i) $f^{-1}(\underset{\sim}{Y}) = \underset{\sim}{X}$ *(j)* $f^{-1}(\underset{\sim}{\phi}) = \underset{\sim}{\phi}$

(k) $f(\underset{\sim}{X}) = \underset{\sim}{Y}$ *, if f is onto.* *(l)* $f(\underset{\sim}{\phi}) = \underset{\sim}{\phi}$

(m) If f is onto, then $\overline{f(A)} \subseteq f(\overline{A})$; *and if, furthermore, f is 1-1, we have* $\overline{f(A)} = f(\overline{A})$.

(n) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$

(o) $B_1 \sqsubset B_2 \Longrightarrow f^{-1}(B_1) \sqsubset f^{-1}(B_2)$

Proof. Let $B_j =< y, B_j^{(1)} B_j^{(2)} >$, $A_i =< x, A_i^{(1)}, A_i^{(2)} >$ $(i \in J, j \in K)$, $B =< x, B^{(1)}, B^{(2)} >$ and $A =< x, A^{(1)}, A^{(2)} >$.

(a) Let $A_1 \subseteq A_2$. Since $A_1^{(1)} \subseteq A_2^{(1)}$ and $A_1^{(2)} \supseteq A_2^{(2)}$, we obtain $f(A_1^{(1)}) \subseteq f(A_2^{(1)})$ and

$$X - A_1^{(2)} \subseteq X - A_2^{(2)} \Longrightarrow f(X - A_1^{(2)}) \subseteq f(X - A_2^{(2)})$$
$$\Longrightarrow Y - f(X - A_1^{(1)}) \supseteq Y - f(X - A_2^{(1)}) \Longrightarrow f_{\text{-}}(A_1) \supseteq f_{\text{-}}(A_2)$$

from which we easily obtain the result $f(A_1) \subseteq f(A_2)$.

(b) It is similar to (a).

$$\begin{aligned}(c) f^{-1}(f(A)) &= f^{-1}(f(< x, A^{(1)}, A^{(2)} >)) = f^{-1}(< y, f(A^{(1)}), f_{\text{-}}(A^{(2)}) >) \\ &= < x, f^{-1}(f(A^{(1)})), f^{-1}(f_{\text{-}}(A^{(2)})) \supseteq < x, A^{(1)}, A^{(2)} >= A.\end{aligned}$$

Notice that $f^{-1}(f(A^{(1)})) \supseteq A^{(1)}$ and

$$f^{-1}(f_{\text{-}}(A^{(2)})) = f^{-1}(Y - f(X - A^{(2)})) = X - f^{-1}(f(X - A^{(2)})) \subseteq$$
$$X - (X - A^{(2)}) = A^{(2)}.$$

$$\begin{aligned}(d) f(f^{-1}(B)) &= f(f^{-1}(< y, B^{(1)}, B^{(2)} >) = f(< x, f^{-1}(B^{(1)}), f^{-1}(B^{(2)}) >) \\ &= < y, f(f^{-1}(B^{(1)})), f_{\text{-}}(f^{-1}(B^{(2)})) > \subseteq < y, B^{(1)}, B^{(2)} >= B.\end{aligned}$$

Notice first that $f(f^{-1}(B^{(1)}) \subseteq B^{(1)}$. On the other hand, we also have

$$\begin{aligned}f_{\text{-}}(f^{-1}(B^{(2)})) &= Y - f(X - f^{-1}(B^{(2)})) = Y - f(f^{-1}(Y - B^{(2)})) \\ &\supseteq Y - (Y - B^{(2)}) = B^{(2)}, \text{ i.e } f_{\text{-}}(f^{-1}(B^{(2)})) \supseteq B^{(2)}.\end{aligned}$$

$$\begin{aligned}(e) f^{-1}(\cup B_j) &= f^{-1}(< y, \cup B_j^{(1)}, \cap B_j^{(2)} >) =< x, f^{-1}(\cup B_j^{(1)}), f^{-1}(\cap B_j^{(2)}) > \\ &= < x, \cup f^{-1}(B_j^{(1)}), \cap f^{-1}(B_j^{(2)}) >= \cup f^{-1}(B_j)\end{aligned}$$

(f) It is similar to (e).

(g) $$\begin{aligned} f(\cup A_i) &= f(< x, \cup A_i^{(1)}, \cap A_i^{(2)} >) =< y, f(\cup A_i^{(1)}), f_{\text{-}}(\cap A_i^{(2)}) > \\ &= < y, \cup f(A_i^{(1)}), \cap f_{\text{-}}(A_i^{(2)}) >= \cup < y, f(A_i^{(1)}), f_{\text{-}}(A_i^{(2)}) >= \cup f(A_i) \end{aligned}$$

Notice that $f(\cup A_i^{(1)}) = \cup f(A_i^{(1)})$ and

$$\begin{aligned} f_{\text{-}}(\cap A_i^{(2)}) &= Y - f(X - \cap A_i^{(2)}) = Y - f(\cup(X - A_i^{(2)})) = Y - \cup f(X - A_i^{(2)}) \\ &= \cap(Y - f(X - A_i^{(2)})) = \cap f_{\text{-}}(A_i^{(2)}). \end{aligned}$$

(h) $$\begin{aligned} & f(\cap A_i) = f(< x, \cap A_i^{(1)}, \cup A_i^{(2)} >) =< y, f(\cap A_i^{(1)}), f_{\text{-}}(\cup A_i^{(2)}) > \\ & \subseteq < y, \cap f(A_i^{(1)}), \cup f_{\text{-}}(A_i^{(2)}) >= \cap f(A_i). \end{aligned}$$

Notice that $f(\cap A_i^{(1)}) \subseteq \cap f(A_i^{(1)})$ and

$$\begin{aligned} f_{\text{-}}(\cup A_i^{(2)}) &= Y - f(X - \cup A_i^{(2)}) = Y - f(\cap(X - A_i^{(2)})) \supseteq Y - \cap f(X - A_i^{(2)}) \\ &= \cup(Y - f(X - A_i^{(2)})) = \cup f_{\text{-}}(A_i^{(2)}). \end{aligned}$$

(i) $f^{-1}(\underset{\sim}{Y}) = f^{-1}(< y, Y, \phi >) =< x, f^{-1}(Y), f^{-1}(\phi) >=< x, X, \phi >= \underset{\sim}{X}$

(j), (k), (l) are similar to (i).

(m) Since

$$f(\overline{A}) = f(\overline{< x, A^{(1)}, A^{(2)} >}) = f(< x, A^{(2)}, A^{(1)} >) =< y, f(A^{(2)}), f_{\text{-}}(A^{(1)}) >$$

and

$$\overline{f(A)} = \overline{f(< x, A^{(1)}, A^{(2)} >)} = \overline{< y, f(A^{(1)} >), f_{\text{-}}(A^{(2)}) >} =< y, f_{\text{-}}(A^{(2)}), f(A^{(1)}) >,$$

we obtain the required result immediately from the fact that f is onto.

(n) Since

$$\begin{aligned} f^{-1}(\overline{B}) &= f^{-1}(\overline{< y, B^{(1)}, B^{(2)} >}) = f^{-1}(< y, B^{(2)}, B^{(1)} >) \\ &= < x, f^{-1}(B^{(2)}), f^{-1}(B^{(1)}) > \quad \text{and} \\ \overline{f^{-1}(B)} &= \overline{f^{-1}(< y, B^{(1)}, B^{(2)} >)} = \overline{< x, f^{-1}(B^{(1)}), f^{-1}(B^{(2)}) >} \\ &= < x, f^{-1}(B^{(2)}), f^{-1}(B^{(1)}) >, \end{aligned}$$

we obtain $f^{-1}(\overline{B}) = (f^{-1}(B))$.

(o) Let $B_1 \sqsubset B_2$. Hence

$$\begin{aligned} B_1^{(1)} \cup B_1^{(2)} \supseteq B_2^{(1)} \cup B_2^{(2)} &\implies f^{-1}(B_1^{(1)} \cup B_1^{(2)}) \supseteq f^{-1}(B_2^{(1)} \cup B_2^{(2)}) \\ &\implies f^{-1}(B_1^{(1)}) \cup f^{-1}(B_1^{(2)}) \supseteq f^{-1}(B_2^{(1)}) \cup f^{-1}(B_2^{(2)}) \\ &\implies f^{-1}(B_1) \sqsubset f^{-1}(B_2) \quad \text{follows.} \end{aligned}$$

□

3. Intuitionistic Points

One can easily define a natural IS in X, called "an intuitionistic point" in X, corresponding to an element X:

Definition 3.1 *(cf.[6]) Let X be a nonempty set and $p \in X$ a fixed element in X. Then the IS $\underset{\sim}{p}$ defined by $\underset{\sim}{p} =< x, \{p\}, \{p\}^{\complement} >$ is called an intuitionistic point (IP for short) in X.*

IP's in X can sometimes be inconvenient when expres an IS in X in terms of IP's. This situation will occur if $A =< x, A_1, A_2 >$ and $p \notin A_1$. Therefore we shall define "vanishing IP's" as follows:

Definition 3.2. *(cf.[6]) Let X be a nonempty set and $p \in X$ a fixed element in X. Then the IS*

$$\underset{\approx}{p} =< x, \phi, \{p\}^{\complement} >$$

is called a vanishing intuitionistic point (VIP for short) in X.

Now we shall present some types of inclusion of an IP to an IS:

Definition 3.3. *(cf.[6]) (a) Let $\underset{\sim}{p}$ be an IP in X and $A =< x, A_1, A_2 >$ an IS in X. $\underset{\sim}{p}$ is said to be contained in A($\underset{\sim}{p} \in A$ for short) iff $p \in A_1$.*

(b) Let $\underset{\approx}{p}$ be a VIP in X and $A =< x, A_1, A_2 >$ an IS in X. $\underset{\approx}{p}$ is said to be contained in A ($\underset{\approx}{p} \in A$ for short) iff $p \notin A_2$.

Proposition 3.4 *(cf.[6]) Let $\{A_i : i \in J\}$ be a family of IS's in X. Then*

(a_1) $\underset{\sim}{p} \in \bigcap_{i \in J} A_i$ *iff* $\underset{\sim}{p} \in A_i$ *for each* $i \in J$

(a_2) $\underset{\approx}{p} \in \bigcap_{i \in J} A_i$ *iff* $\underset{\approx}{p} \in A_i$ *for each* $i \in J$.

(b_1) $\underset{\sim}{p} \in \bigcup_{i \in J} A_i$ *iff* $\exists i \in J$ *such that* $\underset{\sim}{p} \in A_i$.

(b_2) $\underset{\approx}{p} \in \bigcap_{i \in J} A_i$ *iff* $\exists i \in J$ *such that* $\underset{\approx}{p} \in A_i$.

Proof. Straightforward. □

Proposition 3.5. *(cf.[6]) Let A and B be two IS's in X. Then*

(a) $A \subseteq B$ *iff for each* $\underset{\sim}{p}$ *we have* $\underset{\sim}{p} \in A \Longrightarrow \underset{\sim}{p} \in B$ *and*

for each $\underset{\approx}{p}$ *we have* $\underset{\sim}{p} \in A \Longrightarrow \underset{\approx}{p} \in B$.

(b) $A = B$ *iff for each* $\underset{\sim}{p}$ *we have* $\underset{\sim}{p} \in A \Longleftrightarrow \underset{\sim}{p} \in B$ *and*

for each $\underset{\approx}{p}$ *we have* $\underset{\approx}{p} \in A \Longleftrightarrow \underset{\approx}{p} \in B$.

Proof. Obvious. □

Proposition 3.6. *Let A be an IS in X. Then*

$$A = (\cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}) \cup (\cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}).$$

Proof. It is sufficient to show the following equalities:

$$A_1 = (\cup\{\{p\} : \underset{\sim}{p} \in A\}) \cup (\cup\{\phi : \underset{\approx}{p} \in A\}) \text{ and}$$

$$A_2 = (\cap\{\{p\}^C : \underset{\sim}{p} \in A\}) \cap (\cap\{\{p\}^C : \underset{\approx}{p} \in A\}),$$

which are fairly obvious. □

Proposition 3.6. states that any IS A in X can be written in the form $A = \underset{\sim}{A} \cup \underset{\approx}{A}$, where $\underset{\sim}{A} = \cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}$ and $\underset{\approx}{A} = \cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}$. It is easy to show that, if $A =< x, A_1, A_2 >$, then

$$\underset{\sim}{A} =< x, A_1, A_1^{\complement} > \text{ and } \underset{\approx}{A} =< x, \phi, A_2 > .$$

Definition 3.7. *Let $f : X \to Y$ be a function.*

(a) Let $\underset{\sim}{p}$ be an IP in X. Then the image of $\underset{\sim}{p}$ under f, denoted by $f\ \underset{\sim}{(p)}$, is defined by $f\ \underset{\sim}{(p)} =< y, \{q\}, \{q\}^{\complement} >$, where $q = f(p)$.

(b) Let $\underset{\approx}{p}$ be a VIP in X. Then the image of $\underset{\approx}{p}$ under f, denoted by $f\ \underset{\approx}{(p)}$, is defined by $f\ \underset{\approx}{(p)} =< y, \phi, \{q\}^{\complement} >$, where $q = f(p)$.

It is easy to see that $f\ \underset{\sim}{(p)}$ is indeed an IP in Y, namely $f\ \underset{\sim}{(p)} = \underset{\sim}{q}$ where $q = f(p)$, and it has exactly the same meaning of the image of an IS under the function f. $f\ \underset{\approx}{(p)}$ is also a VIP in Y, namely $f\ \underset{\approx}{(p)}\ ) = \underset{\approx}{p}$, where $q = f(p)$.

Proposition 3.8. *Let $f : X \to Y$ be a function and A an IS in X. Then we have $f(A) = f\ \underset{\sim}{(A)} \cup f\ \underset{\approx}{(A)}$.*

Proof. This is obvious from $A = \underset{\sim}{A} \cup \underset{\approx}{A}$. □

4. Conclusions

This note consists of an introduction to intuitionistic sets and other related concepts. One of the future developments in this direction may be the use the topological concepts defined by means of intuitionistic sets and intuitionistic points. A preliminary report [7] on this subject is prepared and some introductory results are obtained.

References

[1] K. Atanassov: "Intuitionistic fuzzy sets", *VII ITKR's Session,* Sofia (September, 1983) (in Bulgarian).

[2] K. Atanassov and S. Stoeva: "Intuitionistic fuzzy sets", *Polish Symp. on Interval & Fuzzy Mathematics,* Poznan (August, 1983), Proceedings: 23-26.

[3] K. Atanassov: "Intuitionistic fuzzy sets", *Fuzzy Sets and Systems* 20, 87-96 (1986).

[4] K. Atanassov: "Review and new results on intuitionistic fuzzy sets," *Preprint IM-MFAIS-*1-88, Sofia, pp. 1-8.

[5] D.Çoker: "An introduction to intuitionistic fuzzy topological spaces", submitted to *Fuzzy Sets and Systems.*

[6] D. Çoker: "An introduction to intuitionistic fuzzy points", submitted to *Notes on IFS.*

[7] D. Çoker: "An introduction to intuitionistic topological spaces", *Preliminary Report,* Ankara, 1995.

[8] L.A. Zadeh: "Fuzzy sets", *Information and Control* **8,** 338-353 (1965).

SEZGİSEL KÜMELER VE SEZGİSEL NOKTALAR ÜZERİNE BİR NOT

Özet

Bu notun amacı "sezgisel kümeler" ve "sezgisel noktalar" adlarını alan kavramları tanımlamak ve temel özelliklerini elde etmektir.

Doğan ÇOKER
Matematik Eğitimi Anabilim Dalı,
Hacettepe Üniversitesi,
Beytepe, Ankara-TURKEY

Received 18.1.1995