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ON SURJECTIVITY OF THE ARENS HOMOMORPHISM*

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Abstract

We define approximate separation and extension properties of a Banach lattice and study the relation between these and topological fullness of the ideal centre. Banach lattices with topologically full ideal centre are characterized as those for which the Arens homomorphism $m : Z(E)'' \to Z(E')$ is surjective. Among the positive operators $T : E \to F$ between Banach lattices E, F we distinguish nearly $Z(E)$ and $Z(F)$ extremal ones and study their properties.

Introduction

Riesz spaces considered in this note are assumed to have separating order duals. If E is a Riesz space, we denote by $L_b(E)$ the space of all order bounded operators on E. $E^\sim$ will denote the order dual of E. $Z(E)$ will denote the ideal centre, $OrthE$ will denote the orthomorphisms of E. $E_n^\sim$ will denote the order continuous members of $E^\sim$. When $T : E \to F$ is order bounded, the adjoint of $T : F^\sim \to E^\sim$ is denoted by $T^\sim$. The dual of a normed space E is denoted by E'. In all undefined terminology we will adhere to the definitions in [1] and [6].

Let us recall that for any associative algebra A a multiplication called Arens multiplication can be introduced in the second algebraic dual A^{**} of A. In particular, for any Archimedean f-algebra A, the space $(A^\sim)_n^\sim$ is an Archimedean f-algebra with respect to the Arens multiplication [3]. $Z(E)$ is an unital f-algebra with unit. Thus $Z(E)''$ is an AM-space and with the Arens product, it is a partially ordered Banach algebra(an Archimedean f-algebra) with unit where the order unit and algebra unit coincide.

A Riesz space E is said to have topologically full centre if for each $x \in E_+$, the $\sigma(E, E^\sim)$ closure of $Z(E)x$ contains the ideal generated by x. Banach lattices with topologically full centre were initiated in [11]. The class of Riesz spaces and the class of Banach lattices that have topologically full centre are quite large. In a σ-Dedekind complete Riesz space E each positive element generates a projection band. Therefore,

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for each $x \in E_+, Z(E)x$ is an ideal and $Z(E)$ is topologically full [11]. However, not all Riesz spaces have topologically full centre [10].

Example. Let E be the Riesz space of piecewise affine continuous functions on [0,1]. Since E is cofinal in $C[0,1]$, we have $E^\sim = C[-,1]'$. But $Z(E)$ is trivial, that is, it contains only the scalar multiplies of the identity. On the other hand $E^\sim$, being Dedekind complete, has the projection property and the centre $Z(E^\sim)$ of $E^\sim$ contains all band projections and has to be large. Similarly for any infinite dimensional Riesz space E for which $Z(E)$ is finite dimensional $Z(E^\sim)$ cannot be a quotient of $Z(E) = Z(E)''$. In Proposition 3, Riesz spaces for which $Z(E^\sim)$ is a quotent of $Z(E)''$ are characterized as those with topologically full centre.

The example of an AM-space that does not have topologically full center was given in [2].

If E is a Banach lattice with a topological orthogonal system then $Z(E)$ is topologically full [11]. However there are Banach lattices with topologically full centre which are neither σ-Dedekind complete nor have a topological orthogonal system.

Example. Let βR denote the Stone-Čech compactification of R and $p \in \beta R \backslash R$. Let $C_0(\beta R)$ denote the continuous real functions vanishing at p. Then $C_0(\beta R)$ does not have a topological orthogonal system and it is not σ-Dedekind complete $\beta R \backslash \{p\}$ is connected). However $C_0(\beta R)$ is an order ideal of the Banach lattice $C(\beta R)$. Since $C(\beta R)$ has topologically full center, $C_0(\beta R)$ also has topologically full center.

Results

A Riesz space E is said to have separating orthomorphisms if, whenever $x \wedge y = 0$ in E, there exists $\pi \in OrthE$ such that $\pi(x) = x$ and $\pi(y) = 0$ (equivalently, for all $x \in E$ there exists $\pi \in OrthE$ such that $\pi(x^+)$ and $\pi(x^-) = 0$). Clearly, any Riesz space with the principal projection property has separating orthomorphisms. Note that if $x \wedge y = 0$ and $\pi \in OrthE$ satisfies $\pi(x) = x$ and $\pi(y) = 0$, then the orthomorphism $\pi_1 = |\pi| \wedge I$ satisfies $\pi_1(x) = x$ and $\pi_1(y) = 0$. Hence, we may assume that $0 \leq \pi \leq I$.

An order ideal I in E has the $Z(E)$-extension property if every $\pi_0 \in Z(I)$ has an extension $\pi \in Z(E)$. Obviously, any projection band has the $Z(E)$-extension property. These properties were defined in [8]. Among other things it was proved that a uniformly complete Riesz space E has separating orthomorphisms if and only if every principal ideal has the $Z(E)$-extension property. In particular, therefore, any σ-Dedekind complete Riesz space has both of these properties. On the other hand every order ideal in a uniformly complete Riesz space E has the $Z(E)$-extension property if and only if E is Dedekind complete [12].

We now define approximate extension and separation properties for a Banach lattice.

Definition. *(i) A Banach lattice E is said to have approximate separation property if, whenever $x \wedge y = 0$ in E, there exists (π_n) in $Z(E)$ with $\pi_n(x) \to x$ and $\pi_n(y) \to 0$.*

(ii) An ideal I in a Banach lattice E is said to have the approximate extension property if for each π_0 in $Z(I)$, there exists (π_n) in $Z(E)$ such that $\pi_n(x) \to \pi(x)$ for every $x \in I$.

A similar concept was introduced in [5]. A set $\mathcal{B}$ of operators on a topological vector lattice separates x from y if there exists a net (π_α) in $\mathcal{B}$ with $\pi_\alpha(x) \to x$ and $\pi_\alpha(y) \to 0$. In the same paper it is proved that if $\mathcal{B}$ is a Boolean ring of order projections on E and I_e is dense in E then E is the smallest closed subspace containing $\{\pi e : \pi \in \mathcal{B}\}$ iff $\mathcal{B}$ separates every pair x, y in E for which $x \wedge y = 0$.

The next proposition exhibits the relation between topological fullness and the preceding properties.

Proposition 1. *Let E be a Banach lattice. Consider the following statements:*

(i) $Z(E)$ is topologically full

(ii) Every principal ideal has approximate extension property

(iii) $Z(E)$ has approximate separation property.

Then (i) $\Leftrightarrow$ (ii) and (ii) $\Rightarrow$ (iii).

Proof. (i)$\Rightarrow$ (ii). Let $x \in E_+$ be arbitrary and I_x be the ideal generated by x. Let $\pi \in Z(I_x)$ be such that $0 \leq \pi \leq I$. As $0 \leq \pi x \leq x$, there exists (π_n) in $Z(E)$ with $\| \pi_n(x) - \pi(x) \| \to 0$. Thus we have

$$0 \leq |\pi_n - \pi|(y) = |\pi_n(y) - \pi(y)| \leq |\pi_n - \pi|(x) = |\pi_n(x) - \pi(x)|$$

for each $0 \leq y \leq x$. Continuity of the lattice operations imply that $\| \pi_n(y) - \pi(y) \| \to 0$ for each $0 \leq y \leq x$. That is to say $\pi_n \to \pi$ pointwise on I_x.

(ii)$\Rightarrow$(i). Let $x \in E_+$ be arbitrary. Then there exists a unique f-algebra structure on I_x where x serves as the algebra unit. For each $y \in I_x$ with $0 \leq y \leq x$, the operator, $\pi_y : I_x \to I_x$, defined as $\pi_y(z) = yz(z \in I_x)$, is in $Z(I_x)$ and $\pi_y(x) = y$. Hence there exists (π_n) in $Z(E)$ with $\pi_n \to \pi_y$ pointwise on I_x. In particular, $\| \pi_n(x) - \pi_y(x) \| = \| \pi_n(x) - y \| \to 0$ and we have (i).

(ii)$\Rightarrow$(iii). Let $x, y \in E$ be such that $x \wedge y = 0$. If $w = x + y$ then $I_w = I_x \oplus I_y$ by Theorem 17.6 in [6]. Each $z \in I_w$ has a unique decomposition $z_1 + z_2$ where $z_1 \in I_x, z_2 \in I_y$. Let $\pi : I_w \to I_w$ be the projection of I_w onto I_x. There exists (π_n) in $Z(E)$ such that $\pi_n \to \pi$ pointwise on I_w. Then $\pi_n(x) \to \pi(x) = x$ and $\pi_n(y) \to \pi(y) = 0$. □

Let E, F and G be Riesz spaces. A bilinear map $\varphi : E \times F \to G$ is called bipositive (bilattice homomorphism) if $\varphi_x : F \to G :: \varphi_x(y) = \varphi(x, y)$ and $\varphi_y : E \to G :: \varphi_y(x) = \varphi(x, y)$ are both positive (lattice homomorphisms) for each $x \in E_+$ and $y \in F_+$.

Given the bilinear map

$$Z(E) \times E \to E \tag{1}$$

defined by $(T, x) \to Tx$, consider its Arens extensions (2) and (3)

$$E^{\sim} \times E \to Z(E)' :: (x, f) \to \mu_{x,f} \tag{2}$$

where $\mu_{x,f}(\pi) = f(\pi x)$ for each $x \in E, f \in E^{\sim}$ and $\pi \in Z(E)$.

$$E^{\sim} \times Z(E)'' \to E^{\sim} :: (f, F) \to F \bullet f \tag{3}$$

where $F \bullet f(x) = F(\mu_{x,f})$ for each $x \in E, f \in E^{\sim}$ and $F \in Z(E)''$. The maps defined in (1), (2) and (3) are bipositive. (3) makes it possible to define a linear operator $m : Z(E)'' \to L_b(E^{\sim})$ where $m(F)(f) = F \bullet f$. It is easily checked that $m(T) = T^{\sim}$ whenever $T \in Z(E)$. We will call the map $m : Z(E)'' \to L_b(E^{\sim})$ the Arens homomorphism of the bidual of $Z(E)$ into $L_b(E^{\sim})$.

Proposition 2. *m is a unital algebra and order continuous lattice homomorphism such that $m(Z(E)'') \subset Z(E^{\sim})$.*

Proof. It is easy to see that m is a positive order continuous unital algebra homomorphism.

For each $F \in Z(E)''$, there is a net $\{T_\alpha\}$ in $Z(E)_+$ such that $\| T_\alpha \| \leq \| F \|$ and $T_\alpha \to |F|$ in $\sigma(Z(E)'', Z(E)')$. Let $f \in E_+^{\sim}$ and $x \in E_+$ be arbitrary. Then

$$|m(F)(f)|(x) \leq m(|F|)(f)(x) = |F|(\mu_{x,f}) = \lim_\alpha \mu_{x,f}(T_\alpha) \leq \| F \| f(x).$$

Therefore $m(F) \in Z(E^{\sim})$ for each $F \in Z(E)''$. That m is a Riesz homomorphism follows from the fact that is an algebra homomorphism by Corollary 5.5 in [4]. □

Lemma 1. *Let E be a Banach lattice. If Z(E) has approximate separation property, then the map $E \to Z(E)' :: x \to \mu_{x,f}$ is a lattice homomorphism for each $f \in E'_+$.*

Proof. It suffices to show that $\mu = \mu_{x,f} \wedge \mu_{y,f} = 0$ whenever $x \wedge y = 0$ in E. Let I be the identity operator on E. It is enough to verify that $\mu(I) = 0$. Recall that

$$\begin{aligned} 0 \leq \mu(I) &= \inf\{\mu_{x,f}(\pi_1) + \mu_{y,f}(\pi_2) : \text{ where } 0 \leq \pi_1, \pi_2 \text{ and } \pi_1 + \pi_2 = I\} \\ &= \inf\{f(\pi_1 x + \pi_2 y) : \text{ where } 0 \leq \pi_1, \pi_2 \text{ and } \pi_1 + \pi_2 = I\}. \end{aligned}$$

On the other hand the hypothesis ensures the existence of (π_n) in $Z(E), 0 \leq \pi_n \leq I$ with $\pi_n x \to x$ and $\pi_n y \to 0$ in E. Clearly, $\mu(I) \leq f((I - \pi_n)x) + f(\pi_n y)$ for each n. Hence

$$\mu(I) \leq \lim_n f((I-\pi_n)x) + \lim_n f(\pi_n y)$$

This shows $\mu(I)=0$. □

Lemma 2. *Let E be a Banach lattice with topologically full centre. The map $E' \to Z(E)' :: f \to \mu_{x,y}$ is a lattice homomorphism for each $x \in E_+$.*

Proof. Positivity of the map $f \to \mu_{x,f}$ immediately shows that $(\mu_{x,f})^+ \leq \mu_{x,f^+}$. We show $\mu_{x,f^+} \leq (\mu_{x,f})^+$.

Let $\pi \in Z(E)_+$ be arbitrary. Then recall that

$$\mu_{x,f^+}(\pi) = f^+(\pi x) = \sup\{f(y) : 0 \leq y \leq \pi x\}.$$

Choose y with $0 \leq y \leq \pi x$. There exists (π_n) in $Z(E), 0 \leq \pi_n \leq I$ such that $\pi_n(\pi(x)) \to y$ in E. This leads to the fact that $\lim_n f(\pi_n \pi(x)) = f(y)$. On the other hand,

$$f(\pi_n(\pi(x)) \leq \sup\{f(Tx) : 0 \leq T \leq \pi\} = (\mu_{x,f})^+(\pi)$$

as $0 \leq \pi_n \pi \leq \pi$. Hence $f(y) \leq (\mu_{x,f})^+(\pi)$. Since y was chosen arbitrary in $[0, \pi x]$, we conclude that $\mu_{x,f^+}(\pi) \leq (\mu_{x,f})^+(\pi)$. □

Let now E, F and G be Banach lattices. A bilinear map $\varphi : E \times F \to G$ is a bilattice homomorphism if and only if $\varphi^\sim : G' \to L_b(E, F') :: f \to \varphi^\sim(f)$ where $\varphi^\sim(f)(x)(y) = f(\varphi(x,y))$ is an interval preserving (Maharam) operator for each $f \in E'$ and $x \in E$ by Theorem 13 in [9].

Position 3. *Let E be a Banach lattice. The Arens homomorphism $m : Z(E)'' \to Z(E')$ is surjective if and only if Z(E) is topologically full.*

Proof. Suppose $Z(E)$ is topologically full. Proposition 1 and preceding lemmata show that the bilinear map $E' \times E \to Z(E)'$ of (2) is a bilattice homomorphism. Hence $m : Z(E)'' \to Z(E')$ is an interval preserving operator. Let $T \in Z(E')$ be such that $0 \leq T \leq I^\sim = m(I)$ where I is the identity operator on E. Then there exists $F \in Z(E)''$ with $m(F) = T$. Thus m is surjective.

We now suppose m is surjective. It suffices to show that $I_x \subset \overline{Z(E)x} = M$. M is a closed subspace of E. We claim M^0 is an order ideal in E'. Let $f \in M^0$ be arbitrary. Let B_f be the band generated by f and π be the projection onto B_f. Clearly we have $\pi(f) = f^+$ and $\pi \in Z(E')$. Hence there exists F in $Z(E)''$ such that $m(F) = \pi$. Choose (π_α) in $Z(E)$ with $\pi_\alpha \to F$ in $\sigma(Z(E)'', Z(E)')$ topology. Thus $\pi_\alpha(\mu_{x,g}) \to F(\mu_{x,g})$ for $x \in E$ and $g \in E'$. On the other hand order continuity of m implies that it is $\sigma(Z(E)'', Z(E)'), \sigma(E', (E')'_n)$ continuous. This observation yields

$m(\pi_\alpha)(g)(x) \to m(F)(g)(x)$ for each $x \in E, g \in E'$. As $m(\pi_\alpha) = \pi_\alpha^\sim$ for $\pi_\alpha \in Z(E)$, we have $\pi_\alpha^\sim(g)(x) = g(\pi_\alpha x) \to \pi(g)(x)$. Let $f \in M^0$ as above and $y \in M$ be arbitrary. Then $f(\pi_\alpha y) \to \pi(f)(y) = f^+(y)$. We choose (π_n) in $Z(E)$ such that $\pi_n(x) \to y$ in E. Then $\pi_\alpha(\pi_n x) \to \pi_\alpha y$ as π_α is continuous for each α. Commutativity of $Z(E)$ implies that $\pi_n(\pi_\alpha x) \to \pi_\alpha y$ so that $\pi_\alpha y \in M$ for each α. Therefore, $f^+(y) = \lim_\alpha f(\pi_\alpha y)$ imply that $f^+(y) = 0$. As y is arbitrary in M, this shows $f^+ \in M^0$. Thus M^0 is a Riesz subspace.

Let now $0 \le g \le f$ with $f \in M^0$ and $g \in E'$. There exists $\pi \in Z(E')$ such that $\pi(f) = g$ by Dedekind completeness of E'[1]. Arguing as above we obtain (π_α) in $Z(E)$ such that $\pi_\alpha^\sim(f)(y) \to \pi(f)(y) = g(y)$ for each $y \in M$. Thus $g(y) = 0$ as $\pi_\alpha^\sim(f)(y) = f(\pi_\alpha y)$ and $\pi_\alpha(y) \in M$ for each α. Thus M i s an ideal in E as it is closed and ${}^0(M^0) = M$. Thus $I_x \subset M$ as claimed. □

Let E, F be Riesz spaces. Suppose F is Dedekind complete. It is well-known that a positive operator $T : E \to F$ is a lattice homomorphism if and only if for every positive operator $S : E \to F$ with $0 \le S \le T$, there exists a positive orthomorphism $\pi \in Orth F$ satisfying $S = \pi \circ T$ [1]. We refer the reader to [11] for a further study for lattice homomorphisms between Banach lattices.

Definition. *Let E,F be Banach lattices and $T : E \to F$ be a positive operator. Then*

(i) If $S \in L(E,F)$ with $0 \le S \le T$ implies that there is $\pi \in Z(E)_+$ with $S = T \circ \pi$ then T is called Z(E)-extermal. T is called nearly Z(E)-extremal if under the same circumstances we can find a net (π_α) in $Z(E)_+$ with $T \circ \pi_\alpha \to S$ in weak operator topology.

(ii) T is called Z(F)-extremal if $S \in L(E,F)$ with $0 \le S \le T$ implies that there is $\pi \in Z(F)$ with $S = \pi \circ T$. T is called nearly Z(F)-extremal if under the same circumstances we can find a net (π_α) in Z(F) with $\pi_\alpha \circ T \to S$ in weak operator topology.

Recall that a positive operator $T : E \to F$ between Banach lattices E and F is called nearly interval preserving if $[0, Tx] \subseteq \overline{T[0,x]}$ for each $x \in E^+$.

Corollary 1. *Let E,F be Banach lattices and suppose Z(F) is topologically full. Then every Z(E)-extremal $T : E \to F$ is nearly interval preserving.*

Proof. Let $y \in [0, Tx]$ be arbitrary. There exists (π_n) in $Z(F), 0 \le \pi_n \le I$ such that $\pi_n(Tx) \to y$ since $Z(F)$ is topologically full. As $0 \le \pi_n \circ T \le T$, we can find $H_n \in Z(E)_+$ such that $T \circ H_n = \pi_n \circ T$ for each n. Let $x_n = H_n(x)$. $0 \le H_n \le I$ imply $0 \le x_n \le x$. Then $T(x_n) \to y$ implies $y \in \overline{T[0,x]}$. □

Let E be a Banach lattice. For each $\pi \in Z(E), \pi^\sim \in Z(E')$. If $Z(E)$ is topologically full, then for $T \in Z(E')$ there is $F \in Z(E)''$ with $m(F) = T$. Let us choose (π_α) in $Z(E)$ with $\pi_\alpha \to F$ in $\sigma(Z(E)'', Z(E)')$. Then $m(\pi_\alpha) = \pi_\alpha^\sim \to m(F) = T$ in the weak operator topology induced by $\sigma(E',(E')'_n)$ on E'. In particular, $(\pi_\alpha^\sim f)(x) \to (Tf)(x)$ for each $x \in E$ and $f \in E'$.

Corollary 2. *Let E,F be Banach lattices. Suppose Z(E) is topologically full. Then every nearly interval preserving $T : E \to F$ is nearly Z(E)-extremal.*

Proof. Let S be such that $0 \leq S \leq T$. Then $T^\sim$ is a lattice homomorphism and $0 \leq S^\sim \leq T^\sim$. By Dedekind completeness of $E', S^\sim = H \circ T^\sim$ for some H with $0 \leq H \leq I$ in $Z(E')$. By the preceeding paragraph there is a net (π_α) in $Z(E)$ such that $\pi_\alpha^\sim \to H$ in the weak operator topology induced by $\sigma(E',(E')'_n)$. In particular, we have $(\pi_\alpha^\sim f)(x) \to (Hf)(x)$ for each $x \in E$ and $f \in E'$. Hence $\pi_\alpha^\sim(T^\sim g)(x) \to H(T^\sim g)(x)$ for each $x \in E$ and $g \in F'$. That is,

$$g(T \circ \pi_\alpha)(x) \to (H \circ T^\sim)(g)(x) = S^\sim(g)(x) = g(Sx) \text{ for each } x \in E \text{ and } g \in F'.$$

Therefore $T \circ \pi_\alpha \to S$ in weak operator topology and T is nearly $Z(E)$-extremal. □

Corollary 3. *Let E,F be Banach lattices. Suppose Z(F) is topologically full. If $T : E \to F$ is a lattice homomorphism, then T is nearly Z(F)-extermal.*

Proof. Let $S \in L(E,F)$ be such that $0 \leq S \leq T$. $T^\sim$ is an interval preserving operator. There exists $H \in Z(F')$ such that $S^\sim = T^\sim \circ H$ by Theorem 3.1 in [7]. As observed earlier, the hypothesis $Z(F)$ is topologically full implies that there exist a net (π_α) in $Z(F)$ such that $\pi_\alpha^\sim \to H$ in the weak operator topology induced by $\sigma(E',(E')'_n)$. In particular $(\pi_\alpha^\sim f)(x) \to (Hf)(x)$ for each $x \in E$ and $f \in E'$. This immediately gives $\pi_\alpha \circ T \to S$ in weak operator topology. □

Corollary 4. *Let E,F be Banach lattices. Suppose Z(E) is topologically full. If $T : E \to F$ is Z(F)-extremal, T is a lattice homomorphism.*

Proof. We first show $[0,T^\sim f] \subset \overline{T^\sim[0,f]}$ where the closure is taken in $\sigma(E',(E')'_n)$. Let $g \in [0,T^\sim f]$. There is $H \in Z(E'), 0 \leq H \leq I$ and $H(T^\sim f) = g$ as E' is Dedekind complete [1]. On the other hand there exists (π_α) in $Z(E)$ such that $G(\pi_\alpha^\sim(f)) \to G(H(f))$ for each $G \in (E')'_n$ and $f \in E'$. In particular, $G(\pi_\alpha^\sim(T^\sim f)) \to G(H(T^\sim f))$ for each $f \in E'$ and $G \in (E')'_n$ so that $G((T \circ \pi_\alpha)^\sim f) \to G(H \circ T^\sim(f)) = G(g)$. There is (S_α) in $Z(F)$ such that $S_\alpha \circ T = T \circ \pi_\alpha$ for each α as $0 \leq T \circ \pi_\alpha \leq T$ and T is $Z(F)$-extremal. Utilizing this, we obtain $G((S_\alpha \circ T)^\sim(f)) \to G(g)$ for each $G \in (E')'_n$ and $f \in F'$ or $G(T^\sim(S_\alpha^\sim(f))) \to G(g)$ for each $G \in (E')'_n$. $0 \leq S'_\alpha f \leq f$ implies $0 \leq T^\sim(S_\alpha^\sim f) \leq T^\sim f$ for each α. Therefore we have $[0,T^\sim f] \subset \overline{T^\sim[0,f]}$ as $g \in [0,T^\sim f]$ is arbitrary.

Now we show $T^{\sim\sim} : (E')'_n \to (E')'_n$ is a lattice homomorphism. It suffices to show $T^{\sim\sim}|G| \leq |T^{\sim\sim}G|$ for $G \in (E')'_n$. Let $0 \leq h \in (E')'_n$ be arbitrary. Recall that

$$|T^{\sim\sim}G|(h) = \sup\{T^{\sim\sim}(G)(2g - h) : g \in [0,h]\} = \sup\{G(2T^{\sim}g - T^{\sim}h) : g \in [0,h]\}$$

and

$$T^{\sim\sim}|G|(h) = |G|(T^{\sim}h) = \sup\{G(2f - T^{\sim}h) : f \in [0, T^{\sim}h]\}.$$

Let $\nu = 2f - T^{\sim}h$ for some $f \in [0, T^{\sim}h]$. Then there exists (f_α) in $[0,h]$ with $T^{\sim}f_\alpha \to f$ in $\sigma(E', (E')'_n)$. Since $G(2T^{\sim}f_\alpha - T^{\sim}h) \leq |T^{\sim\sim}G|(h)$ for each α, we have $G(\nu) \leq |T^{\sim\sim}G|(h)$. We conclude that $T^{\sim\sim}|G|h \leq |T^{\sim\sim}G|(h)$ for each $h \in E'_+$ and that $T^{\sim\sim}$ is a lattice homomorphism as h is arbitrary. That T is a lattice homomorphism is immediate as E is a Riesz subspace of $(E')'_n$. □

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ARENS HOMOMORFİZMASININ ÖRTENLİĞİ ÜZERİNE

Özet

Banach örgü uzayları için yaklaşık ayırma ve genişleme özellikleri tanımlanmış, bunlar ile merkezin topolojik zenginliği arasındaki ilişki çalışılmıştır. Merkezin topolojik zenginliği $m : Z(E)'' \to Z(E')$ Arens homomorfizmasının örtenliği ile betimlenmiştir. E, F Banach örgü uzayları olmak üzere $Z(E)$ ve $Z(F)$ ekstrem operatörleri tanımlanmış ve özellikleri incelenmiştir.

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